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On the Geometry of Grassmannian Manifold

T. Venkatesh [1], Shruti Kamalakar Govekar [2]

[1]Department of Mathematics, and Director, Mathematical Sciences Institute Belagavi , Karnataka,India.

e-mail: tmathvenky@yahoo.co.in

[2]Department of Mathematics, Rani Channamma University, Belagavi, Karnataka, India

e-mail: shru16_shruti@rediffmail.com

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Abstract:
We start with basics in Differential manifold such as complex manifolds, tangent space to a manifold, complex sub manifolds and sub varieties and specified their generalizations to projective spaces with its rich topological and smooth manifold structure.

Keywords: Complex manifolds, tangent space, Sub manifold, Sub varieties.

1. Introduction

The core group of compact complex manifolds is known as the Grassmannians. They could be viewed as an extension of projective space as well. We'll define grassmannians formally here. Understanding their individual structures is crucial for both geometric and topological analyses of grassmannians.

1.1 Definition : For this purpose, we may write $G(k,n)$ for $G(k,C^n)$ and define the Grassmannians $G(k,V)$ as the set of k-dimensional linear subspaces of V. Let V be a complex vector space of size n. To represent Λ in C^n, one may use a collection of k-row vectors in C^nthat span a particular k-plane A. ie by a kxn matrix

$$\left(\frac{v_{11} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots v_{1n}}{v_{kn} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots v_{kn}}\right)$$

of rank k. If $\wedge = g\wedge'$ for some $g \in GL_k$ then A and A', two such matrices, may both represent for the same $k-n$ element in G. Any matrix of this kind clearly represents a point or an element of $G(k,n)$.

Let $I = \{i_1, \dots\dots\dots i_k\} \subset \{1, \dots\dots.., n\}$ of cardinality k, then for $V_I \subset C^n$, a $(n-k)$ – plane in C^n be spanned by the vectors $\{e_j : j \notin I\}$.

2. Complex manifolds and examples:

We define a complex manifold and provide some important examples of Complex manifolds.

2.1 Definition : A differentiable manifold is a multilayered manifold. M allows coordinate mappings $\varphi_\alpha : U_\alpha \to C^n$ and an open cover $\{U_\alpha : \alpha \epsilon \Lambda\}$, C^n such that, for all α, β such that $\varphi_\alpha o \varphi_\beta^{-1}$ is holomorphic on $\varphi_\beta(U_\alpha \cap U_\beta) \subset C^n$

$$\begin{array}{ccc} \cup\, U_\alpha \cap U_\beta \subset U_\alpha & \xrightarrow{\varphi_\alpha} & C^{n} \supset \varphi_\alpha(U_\alpha \cap U_\beta) \\ \varphi_\beta \downarrow & & \downarrow \varphi_\alpha o \varphi_\beta^{-1} \\ & C^{n} \supset \varphi_\beta(U_\alpha \cap U_\beta) & \end{array}$$

Our intension, to define holomorphic maps on M into N, where M and N smooth complex manifolds.

2.2 Definition: On an open set $U \subset M$, a function f is holomorphic if and only if $fo\varphi_\alpha^{-1}$ is holomorphic on φ_α $(U \cap U_\alpha) \subset C^{n}$ for all α. Similarly, a collection $Z = (z_1, \dots \dots .., z_n)$ of functions on $U \subset M$ is called a holomorphic coordinate system if and only if $\varphi_\alpha o z^{-1}$ and are $zo\varphi_\alpha^{-1}$ holomorphic on $z(U \cap U_\alpha)$ and φ_α $(U \cap U_\alpha)$ respectively, for α.

2.3 Definition : A complex manifold map $f: M \longrightarrow N$ is holomorphic if and only if holomorphic functions provide local holomorphic coordinates on N.

(i) A complex manifold that is one-dimensional is referred to as a Riemann surface.

(ii) The set of all lines in C^{n+1} that intersect at the origin is denoted as P^n. For any $z \neq 0 \in l$ we can write, $p^{n} = \{[z] \neq 0 \in C^{n+1}\} / [z] \sim [\lambda z]$ determines a line l in C^{n+1}. A bijective map φ_i to C^{n} is given by $\varphi_i([z_0, \dots ., z_n]) = (\frac{z_0}{z_1}, \dots \dots, \frac{\widehat{z_i}}{z_i}, \dots . \frac{z_n}{z_i})$ and a subset U_i of p^{n} looks like, $U_i = \{[z]: z_i \neq 0\} \subset p^{n}$ of lines not contained in the hyper plane $(z_i = 0)$.

On $(z_j \neq 0) = \varphi_i(U_i \cap U_j) \subset C^{n}$, $\varphi_j o \varphi_i^{-1}(z_1, \dots ., z_n) = (\frac{z_1}{z_j}, \dots \dots, \frac{\widehat{z_j}}{z_j}, \dots \frac{1}{z_j}, \dots, \frac{z_n}{z_j})$ is clearly holomorphic.

As a consequence of this, n is a complicated manifold that has the structure of a complex projective space. The coordinates that are supplied by the mappings φ_i are referred to as Euclidean coordinates, whereas the "coordinates" $z = [z_1, \dots ., z_n]$ are known as homogeneous coordinates on p^{n}. Given that there is a continuous surjective map from the unit sphere in C^{n+1} to p^{n}, it is necessary to conclude that pn is compact. It should be noted that $\mathbb{P}^1$ is simply the Riemann surface $C \cup \{\alpha\}$.

Any reference to an inclusion is caused by $C^{k+1} \longrightarrow C^{n+1}$. The picture of a map such as $\mathbb{P}^k \longrightarrow \mathbb{P}^n$ is a linear subspace of p^{n}. In general, a k-plane is the image of $C^{k+1} \subset C^{n+1}$, a line is the image of a 2-plane $C^2 \subset C^{n+1}$, and a hyperplane is the image of a hyperplane in C^{n+1}again. Now, we can discuss linear relations between points in p^{n} in this particular situation.

2.4 Example : If the span of a line in p^{n} is a $(k-1)$ -plane, then the line in C^{n+1} is said to be linearly independent of k points. Assuming that the image of the subspace in C^{n+1} that the lines $\pi^{-1}(p_i)$ cover in $\mathbb{P}^n$ is the span of a collection of points p_i in $\mathbb{P}^n$, it is assumed that the span of $\mathbb{P}^n$ is the image of the subspace.

2.5 Remark : As a result, the set of hyper planes in $\mathbb{P}^n$ is a projective space in and of itself; it is known as the dual projective space and is represented by $\mathbb{P}^{n*}$. It corresponds to the set $C^{n+1} - \{0\}$, of non zero linear functionals on C^{n+1} modulo scalar multiplication.

Sometimes it is advantageous to visualize $\mathbb{P}^n$ on the compactification of C^n that results from appending the hyper plane H at infinity. Coordinates-wise, the inclusion $C^n \longrightarrow \mathbb{P}^n$ is $[z_1, \dots, z_n] \longrightarrow [1, z_1, \dots, z_n]$; here H has identification (z_0=0) and the identification $H \backsimeq \mathbb{P}^{n-1}$ is obtained by taking the hyper plane at infinity as the directions that go to infinity in C^n.

Let be a lattice $\Lambda = Z^k \subset C^n$ after that, the projective map π: $C^n \longrightarrow C^n/\Lambda$ induces a complex manifold structure in the quotient group C^n/Λ. Only if $k = 2n$ does it qualify as compact; in this instance, C^*/Λ is referred to as a complex torus.

When $\pi: M \longrightarrow N$ is a complex manifold, it is common for N to inherit the structure of M. It is possible for N to acquire the structure of a complex manifold due to the fact that it is both a complex manifold and a topological covering space. However, this is only the case if M is also a complex manifold and its deck transformations are holomorphic.

3. Tangent space to a manifold M

We offer a complex manifold M as well as a holomorphic coordinate system $z = (z_1, \dots, z_n)$ that revolves around p. A tangent space to M at p is defined in this article in three distinct ways.

a) $T_{R,p}(M)$ We define M to be a real manifold of size $2n$. (M) is the normal real tangent space at M at p.

If we write $z_i = x_i + iy_i$, $T_{R,p}$ (M) $=R\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right\}$ then $T_{R,p}$ (M) may be expressed as the set of all R−linear derivations on the set of all C^∞ functions with real values that are close to p.

b) $T_{c,p}(M) = T_{R,p}(M) \otimes_R C$ We refer to the complexified tangent space to M as at p. The space of C −linear derivations in the ring of Complex−valued C^∞ functions on M around p is one way to realise it. We may write

$$T_{c,p}(M) = C\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right\}$$

$$= C\left\{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\right\}$$

Where as before, $\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right)$

$$\frac{\partial}{\partial \underline{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right)$$

c) The space of holomorphic tangents to M at point p is represented by an T_p(M) = $C\left\{\frac{\partial}{\partial z_i}\right\} \subset T_{c,p}$(M).

Due to the fact that the subspace of $T_{C,p}(M)$ is composed of derivations that vanish on anti-holomorphic functions or functions such that f is holomorphic, the coordinate system that was selected $(z_1, \ldots, z_n)$ is not valid. The subspace is shown by the anti holomorphic tangent space to M at p is T''_p(M) $=C\left\{\frac{\partial}{\partial \bar{z}_i}\right\}$.

Clearly,

$T_{C,p}(M)$n= T'_p(M)⊕T''_p(M) observe that for M,N complex manifolds any C^∞ map F:M⟶N induces a linear map F_*:$T_{R,p}$(M) ⟶ $T_{R,f(p)}$(N)

$T_{C,p}(M)$n= T'_p(M)⊕T''_p(M) observe that any C^∞ map F:M⟶N induces a linear map for M, N complex manifolds. F_*:$T_{R,p}$(M) ⟶ $T_{R,f(p)}$(N) . And hence a map F_*:$T_{C,P}$(M)⟶ $T_{C,f(P)}$(N).

Actually, a holomorphic map $f: M \longrightarrow N$ exists for each p in M only if $F_*(T'_p$(M)) ⊂ $T'_{f(p)}$(N) exists.

Observe that, because, $T_{C,p}(M)$ is provided naturally as the real vector space , $T_{R,p}(M)$ tensored with, the statement conjugation sending $\frac{\partial}{\partial z_i} to \frac{\partial}{\partial \bar{z}_i}$ is well defined and $T''_p(M)$ =$T'_p(M)$.

The projection $T_{R,p}$(M) ⟶ $T_{C,p}$(M)⟶ T'_p(M) is therefore an R-linear isomorphism of this type. Therefore, this meant that the only place where we could "do geometry" was in the holomorphic tangent space.

3.1 Example: Assume that $z(t)$, where $0 \le t \le 1$, is a smooth arc in the complex z-plane. Then

$z(t) = x(t) + \sqrt{-1}y(t)$ and The tangent to the arc may be calculated as follows: x'(t)$\frac{\partial}{\partial x}$ + y'(t)$\frac{\partial}{\partial y}$ in $T_R(C)$ Or z'(t) $\frac{\partial}{\partial z}$ in $T'(C)$ and these two coincide under the projection $T_R(C)$⟶ $T'(C)$.

Suppose that M and N are Complex Manifolds. The set of holomorphic coordinates centred at p∈M is denoted as $z = (z_1, \ldots, z_n)$. A set of holomorphic coordinates centred at $q \in N$ is denoted as w =$(w_1, \ldots, w_n)$. We have different thoughts on the Jacobian of f ,we refer to $[G, H]$. With

$f(p) = q$, the holomorphic map $f: M \longrightarrow N$ corresponds to the different tangent spaces to M and N at p and q, respectively.

4. Sub manifolds and sub varieties

4.1 Definition: Specified sub manifold S of a complicated manifold M with complex characteristics A finite collection of holomorphic functions $f_1, f_2, \ldots, f_k$ with rank $g(f) = k$ is represented locally by the subset M, which is known as the zeros. The sign V^* is used to denote the location of smooth points in the curve V. On the other hand, a singular point of V is defined as $p \in V$ - V^*, and the singular locus of V is denoted by V_s. The only conditions under which V is deemed smooth or non-singular are those in which it is a sub-manifold of M or if V is equal to V^*.

Specifically, for every point p on an analytic hyper surface $V \subset M$ defined in terms of local coordinates Z by the function f, we say that the multiplicity $mulp(V)$ is the degree to which f eliminates at p, or the greatest number m for which all partial derivatives are identical

$$\frac{\partial^k f}{\partial z_{i_1,\ldots\ldots} \partial z_{i_k}}(P) = 0 , k \le m - 1.$$

When dealing with a family of objects that are parametrized locally by a complex manifold or an analytic subvariety of a complex manifold, the statement "a generic relative has a certain property" indicates that the set of objects in the family that do not possess the property is contained in a sub variety of strictly smaller dimension. This is the case when dealing with a complex manifold.

In most cases, the proper way to parametrize items in our family will be obvious. In $\mathbb{P}^n$, the generic k-plane is an exception(this one will be again revisited in the sections on Grassmannians in the work that follows later).

4.2 Remark: Grassmannians are a family of compact complex manifolds. To be specific are generalizations to Projective spaces. They have rich topological and smooth manifold structure. Before we define them formally some background of affine and projective vector spaces are defined. The setting is complex.

5. Vector spaces:

These are rich algebraic structures under additive binary operation they form abelian group and multiplication/ scalar distribution operation ensures distribution ensures distribution properties. One main thing about them is that they appear as pairs of fields. Thus every field is a vector space over itself. The best examples are the real field R is a vector space projected onto itself. C, the self-describing complex field vector space. Next, is their higher analogues R^n over R, C^n over C. If we choose a vector space with n dimensions that is isomorphic to R^{2n}, then n is its dimension.

5.1 Proposition: F^n is isomorphic to any finite dimensional vector space of size n, where F is a scalar field. The real field $F = R$ is defined as a real vector space $V\ '^s$of size n over R.

Proof: Each vector space has a basis, which is not unique; two bases of V can be equivalent, but the dimension of n is unique.

We take these vital facts for granted, despite their importance. An instance of an inner product on R^n is the bilinear map $f\colon RXR \to R$, which is positive definite, linear in the slot, and compatible with scalar multiplication.

In reality, the only differences are that the inner product is Hermitian and complex numbers have conjugates.

A metric induces an inner product space, which in turn is a metric space. Grassmannians are obviously of importance to us. We briefly touch on affine projective geometry as a digression before moving on to the primary topic.

Observe that R^n can be decomposed by writing it as $R^{k(n-k)}$. If it were C then $C^{k(n-k)}$.

k rows and $(n-k)$ column matrices are imaginable in the language of matrices $k.(n-k)$. When a metric is represented by a k-row vector and vice versa, two matrices A and B are declared to have the same mean if there is a scalar g such that $A = g.B$. This connection splits the set into equivalent classes and is an equivalent relation.

Since x and y in the real plane R^2 are equivalent if and only if $y = ax$ for some constant a, we can assert that x and y are equivalent. The equivalence class determined by X is indicated by $[X]$, allowing R^2 to be divided.

In other words, the line passing through the origin sets a constant slope. Thus, one can set, $y = ax$.

How to realise non trivial examples where projectivization crop up.

The stereographic projections of the sphere and circle are the best examples.

6. Conclusion

Imagine a line passing through origin in R^2 and R^3 with vectors originating from the origin of unit length then we are done. Differential Geometry methods vividly capture them as one and two dimensional smooth compact manifolds. A natural inclusive tower of subspaces of R^n is R^1contained in R^2………$R^{(n-1)}$ and contained in R^n. Therefore R^k with k row vectors and $k \times n$ matrix and totality of them would be aright frame work for imagining Grassmannians.

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