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Some Exact Solutions of Unsteady Viscous Incompressible Fluid Flows with Topology Conservation of Vorticity Vector Field by Ansatz Method

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Abstract:
Navier-Stokes equations for an unstable incompressible fluid flow with conservative body forces are discussed in this article. Under topology conservation of vector fields, the governing equations are derived. It is assumed that the fluid flow velocity acts as a generating vector field. Viscous fluid flows were also generated using topology conserving vorticity vector fields. Graphs show the vectors of velocity for different fluids.

Keywords: Topological conservation, Navier-Stoke equation, vorticity equation, Non linear algebraic equations.

1.Introduction

Mathematically speaking, things that may change form without discontinuity (smooth mapping) are considered topologically comparable. This is the field of study known as topology. Economic representations of the flow are permitted by topological invariants (fewer degrees of freedom). In fluid dynamics, topological features have become increasingly important in recent times. Since the Navier-Stoke equation contains higher order nonlinear partial differential equations, finding exact solutions to it is extremely challenging. The solutions to nonlinear partial differential equations are important in mathematical physics because these equations involve many physical factors. Only in certain circumstances can the majority of these equations be solved. Therefore, a number of approximation and numerical techniques have been developed to solve these kinds of nonlinear partial differential equations. However, precise solutions are required to verify the level of accuracy of these techniques. There are situations when there are very few exact answers, making it challenging to verify the accuracy of approximation techniques. The governing differential equation solutions are used by Bakker (1991) to derive the topological features of two-dimensional continuous viscous flows.

Some of the exact solutions for the Navier-Stokes equations that are currently available are given by Wang (1991). In 1996, Ross Ethier and D. A. Steinman presented non-zero (and non-trivial) velocities in each of the three coordinate directions together with unsteady analytical solutions to the incompressible Navier-Stokes equations.

Topological kinematics associated with rod stirring of a two-dimensional fluid is described by Philip Boyland (2013). It was demonstrated that, depending on the stirring approach, the critical topological length of material lines grows either exponentially or linearly. Using periodic boundary conditions, Farazmand (2016) proposed a unique method for finding the equilibrium and traveling wave (relative equilibrium) solutions to the forced Navier Stokes equations. Daniel Paralta Salas (2016) conducted a survey to look at specific geometric properties of inviscid and incompressible fluid flows.
Suqiong et al. (2021) evaluated the accuracy of the design sensitivity by examining the lattice kinetic scheme (LKS) as a topology optimization approach for flow channel design. New precise solutions for three-dimensional incompressible steady state generalized Beltrami flows are obtained by Joseph (2021). Moffatt (2021) provided an easy introduction to various issues in fluid dynamics that pointed towards topology. Naiko and Yamada (2022) derived the vorticity equation for an incompressible fluid completely submerged in a three-dimensional Euclidean space on a two-dimensional surface with any topology.
By the late 1980s, the topology optimisation technique had evolved from size and form optimisation in the discipline of solid mechanics. Casper Schousboe Andreasen and Joe Alexandersen (2020) presented a survey of the literature on topology optimization for fluid-based challenges..
We assume specific ansatz forms for the necessary answers because there are no exact methods available to solve the highly nonlinear partial differential equations. We then solve the associated equations to find the exact solutions. In order to solve the nonlinear partial differential equations, ZhengdeDai et al. (2013) proposed the mixed exponential function ansatz approach. Yarom et al. (2014) conducted numerical simulations in an Israel-Stewart-like theory of second order viscous hydrodynamics and the heat current of a steady state connecting two asymptotic equilibrium systems with linear hydrodynamics. The finite difference technique for 3D viscous incompressible flows on non-staggered grids in deformable surfaces was extended by Xilin Xie and Chen (2016).
The current research derives the topological conserved vorticity equation and employs the potential ansatz approach to determine exact solutions for two separate families of solutions to the Navier-Stokes equation.

2. Basic equations

Constant in motion is an essential and sufficient criterion for an arbitrary vector field S through an arbitrary surface. Therefore

$$\frac{\partial S}{\partial t} + (w.\nabla)S - (S.\nabla)w + S(\nabla.w) = 0 \tag{1.1}$$

In addition, for vector tube S to exist, the conditions that follow must be met:

$$S \times \left(\frac{\partial S}{\partial t} + (w.\nabla)S - (S.\nabla)w\right) = 0 \tag{1.2}$$

Here, w is the generating vector field.
An arbitrary vector field's topological conservation requirement is met by

$$\frac{\partial S}{\partial t} + (w.\nabla)S - (S.\nabla)w = \lambda S \tag{1.3}$$

In a smooth domain of the Euclidean space $\Omega \subseteq R^3$, consider the Navier-Stokes equations of an unsteady incompressible fluid flow under conservative body forces. The basic equations are

$$\nabla.u = 0 \tag{1.4}$$

$$\frac{\partial u}{\partial t} + (u.\nabla)u \quad = -\nabla p + \nabla\varphi + \vartheta\nabla^2 \mathrm{u} \tag{1.5}$$

Where $u(x,t)$ is a time dependent vector field (or) velocity field, $\nabla\varphi$ is the conservative body force, ϑ is the coefficient of kinematic viscosity and $p(x,t)$ is the pressure function.
Consider another time dependent vector field (i.e) vorticity. It is defined as

$$w = \nabla \times u \quad (or) \;\; curl\, u \tag{1.6}$$

By substituting the vorticity, equ(1.5) becomes

$$\frac{\partial w}{\partial t} + \nabla \times (w \times u) - \vartheta\nabla^2 w = 0 \tag{1.7}$$

It is now essential to discuss whether or not topology conserving flows of incompressible viscous fluids take place. The vorticity vector field must satisfy the equation (1.7) if such a flow is present. We obtain precise, reliable solutions to the equation (1.7) for Newtonian fluid flows, where the vorticity fields preserve topology.

3.Exact solution by ansatz method

Since the Navier-Stoke equation contains higher order nonlinear partial differential equations, establishing exact solutions to it is extremely hard. By converting them into ordinary differential equations, a number of nonlinear partial differential equations have been solved. We want to transform the partial differential equations of incompressible viscous flows into ordinary differential equations so that the precise solutions can be found by solving them. In order satisfy the equation (1.7), it is expected that there is a velocity potential. In the case of viscous flows, we apply the following ansatz form for the vector potential for the appropriate velocity field to find a solution for the topology conserving vorticity field.

$$V = e^{-A\vartheta t}\begin{pmatrix} a_1\sin(-ax+by+bz) + a_2\cos(-ax+by+bz)\,, \\ a_3\sin(-ay+bx+bz) + a_4\cos(-ay+bx+bz)\,, \\ a_5\sin(-az+bx+by) + a_6\cos\,(-az+bx+by) \end{pmatrix} \tag{2.1}$$

Where a_1, a_2, a_3, a_4, a_5, and a_6 are arbitrary real parameters.

Taking the curl of the above equation, we get

$$u = \; be^{-A\vartheta t}\begin{pmatrix} -a_4\sin(-ay+bx+bz) - a_6\sin(-az+bx+by) - a_3\cos(-ay+bx+bz) + a_5\cos(-az+bx+by) \\ a_2\sin(-ax+by+bz) + a_6\sin(-az+bx+by) + a_1\cos(-ax+by+bz) - a_5\cos(-az+bx+by) -,, \\ a_3\cos(-ay+bx+bz) + a_4\sin(-ay+bx+bz) - a_1\cos(-ax+by+bz) + a_2\sin(-ax+by+bz) \end{pmatrix}$$
(2.2)

Equation (2.2) can be substituted into equation (1.8) in order to satisfy the continuity equation.
The vorticity field can be obtained by putting equation (2.2) in equation (1.6).

$$w = \; be^{-A\vartheta t}(-2ba_1\sin(-ax+by+bz) + 2ba_2\cos(-ax+by+bz) + a(a_3\sin(-ay+bx+bz) - a_4\cos(-ay+bx+bz) - a_5\sin(-az+bx+by) + a_6\cos(-az+bx+by)), a(-a_1\sin(-ax+by+bz) - a_2\cos(-ax+by+bz) + a_4\cos(-ay+bx+bz) - a_5\sin(-az+bx+by) - a_6\cos(-az+bx+by)) + b(-2a_3\sin(-ay+bx+bz) + a_4\cos(-ay+bx+bz)), a(a_1\sin(-ax+by+bz) - a_2\cos(-ax+by+bz) + a_3\sin(-ay+bx+bz) - a_4\cos(-ay+bx+bz)) + b(2a_5\sin(-az+bx+by) + 2a_6\cos(-az+bx+by)))$$
(2.3)

By substituting equation (2.3) in the vorticity equation (1.7), we obtain the set of following nonlinear algebraic equations

$$a^2 - A + 2b^2 = 0$$

$$(a_1a_4 - a_2a_3)(a^2 - ab - 2b^2) = 0$$

$$(a_1a_3 + a_2a_4)(a^2 - ab - 2b^2) = 0$$

$$(a_2a_5 + a_1a_6)(a^2 - ab - 2b^2) = 0$$

$$(a_1a_5 - a_2a_6)(a^2 - ab - 2b^2) = 0$$

$$(a_3a_6 - a_4a_5)(a^2 - ab - 2b^2) = 0$$

$$(a_3a_5 + a_4a_6)(a^2 - ab - 2b^2) = 0 \tag{2.4}$$

3.1 First solution

A solution to the above system of nonlinear equation (2.4) is given by $A = 3a^2$ and $b = -a$.
The specific solution of the Navier-Stokes equation, which satisfies equations (1.4) and (1.5), can be obtained by

$$u = ae^{-3a^2\vartheta t}\begin{pmatrix}(a_3 - a_5)\cos(a(x+y+z)) - (a_4 + a_6)\sin(a(x+y+z)),\\ (a_5 - a_1)\cos(a(x+y+z)) + (a_2 + a_6)\sin(a(x+y+z)),\\ (a_1 - a_3)\cos(a(x+y+z)) + (a_2 + a_4)\sin(a(x+y+z))\end{pmatrix} \tag{3.1}$$

Equation (3.1) satisfies equation (1.4) and (1.5). Hence, the above is the exact solution of the Navier-Stokes equation.
Corresponding vorticity vector is

$$w = be^{-3a^2\vartheta t}(-2ba_1\sin(-a(x+y+z)) + 2ba_2\cos(-a(x+y+z))$$

$$+a(a_3\sin(-a(x+y+z)) - a_4\cos(-a(x+y+z)) - a_5\sin(-a(x+y+z)) + a_6\cos(-a(x+y+z)),$$

$$a(-a_1\sin(-a(x+y+z)) - a_2\cos(-a(x+y+z)) + a_4\cos(-a(x+y+z)) - a_5\sin(-a(x+y+z)) - a_6\cos(-a(x+y+z))$$

$$+b(-2a_3\sin(-a(x+y+z)) + a_4\cos(-a(x+y+z)),$$

$$a(a_1\sin(-a(x+y+z)) - a_2\cos(-a(x+y+z)) + a_3\sin(-a(x+y+z)) - a_4\cos(-a(x+y+z))$$

$$+b(2a_5\sin(-a(x+y+z)) + 2a_6\cos(-a(x+y+z)))) \tag{3.2}$$

Velocity vector field is shown through figures (1-4) for various kinematic viscosity.

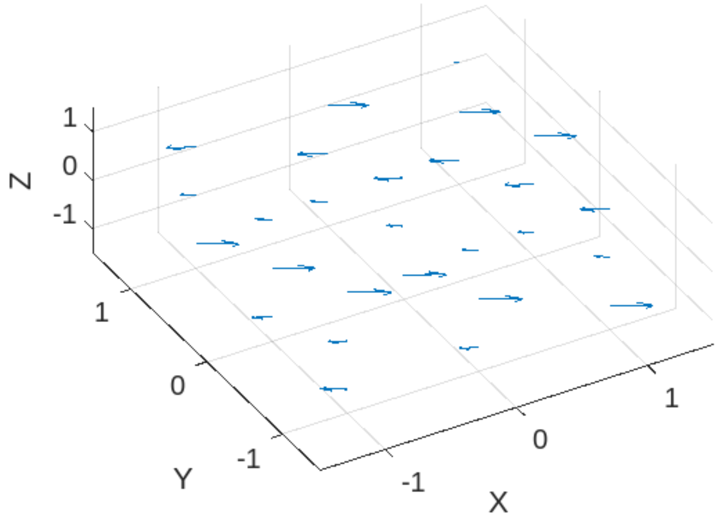

Fig 1. Plot of velocity vector field for air (a_1=2.0, a_2=2.0, a_3=1.0, a_4=2.0, a_5=2.0, a_6=1.0, a=2.0, b=1.0)

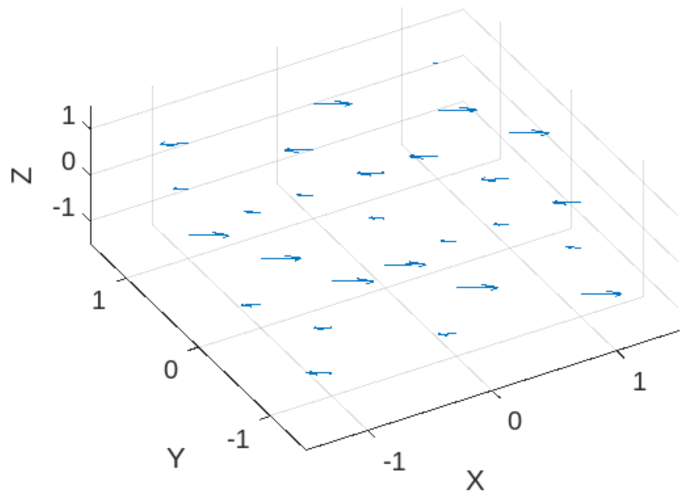

Fig 2. Plot of velocity vector field for water (a_1=2.0, a_2=2.0, a_3=1.0, a_4=2.0, a_5=2.0, a_6=1.0, a=2.0, b=1.0)

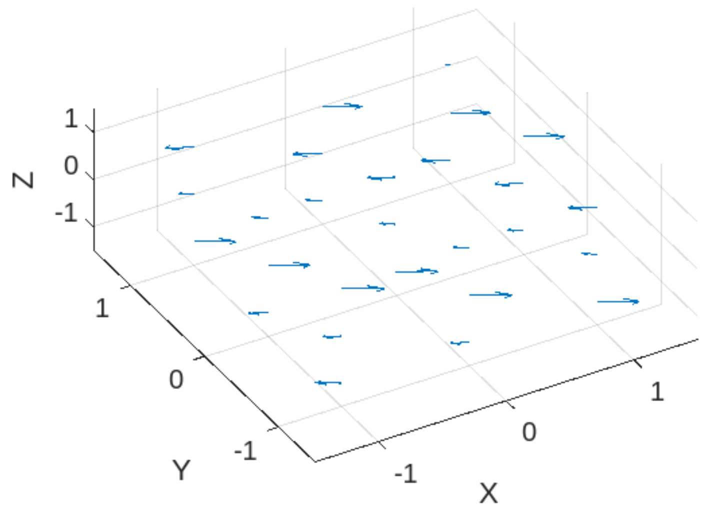

Fig 3. Plot of velocity vector field for carbondioxide (a_1=2.0, a_2=2.0, a_3=1.0, a_4=2.0, a_5=2.0, a_6=1.0, a=2.0, b=1.0)

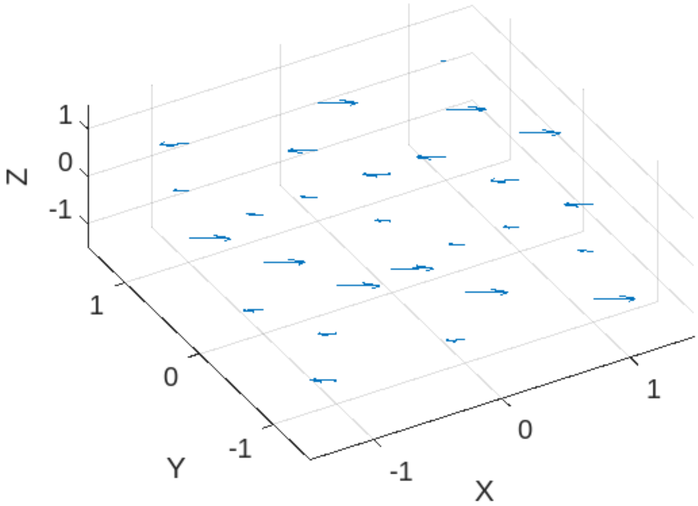

Fig 4. Plot of velocity vector field for Mercury (a_1=2.0, a_2=2.0, a_3=1.0, a_4=2.0, a_5=2.0, a_6=1.0, a=2.0, b=1.0)

For conservation of vorticity, equation (3.1) must satisfy equation (1.6). Equation (1.3) can be solved by modifying the velocity field along with the vorticity field to get

$$3a^2\vartheta + \lambda = 0 \tag{3.3}$$

$$\Rightarrow \lambda = -3a^2\vartheta$$

Thus, we can conclude that the vorticity field's topology is conserved. Therefore, the exact solution for incompressible viscous fluid flow is given by equation (3.1), and the corresponding vorticity field is topology conserving.

3.2 Second solution

Another solution to the nonlinear equation (2.4) system can be given by

$$A = a^2 + 2b^2 \text{ and } a_1 = a_2 = a_3 = a_4 = 0 \tag{4.1}$$

This solution yields an exact solution to the Navier-Stokes equation that satisfies equations (1.4) and (1.7). It is given by

$$u = be^{-(a^2+2b^2)\nu t}\begin{pmatrix} a_5\cos(b(x+y)-az) - a_6\sin\,(b(x+y)-az), \\ a_6\sin(b(x+y)-az) - a_5\cos(b(x+y)-az)\,, \\ 0 \end{pmatrix} \tag{4.2}$$

Velocity vector for second family of solution is expressed through figures (5-8)

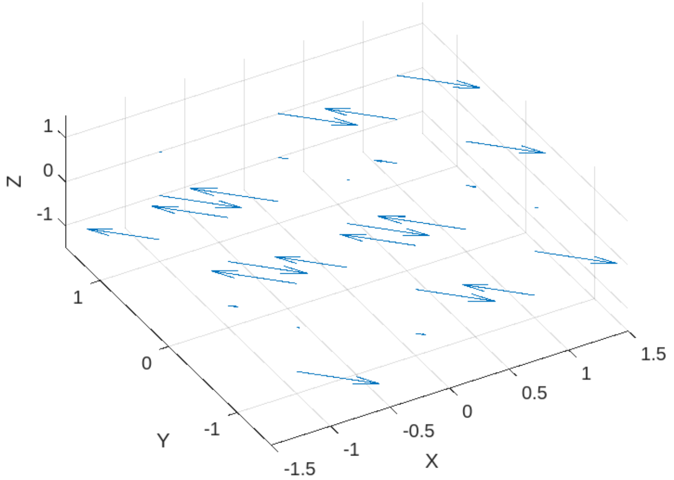

Fig 5. Plot of velocity vector field for air a_5=2.0, a_6=1.0, a=2.0, b=1.0)

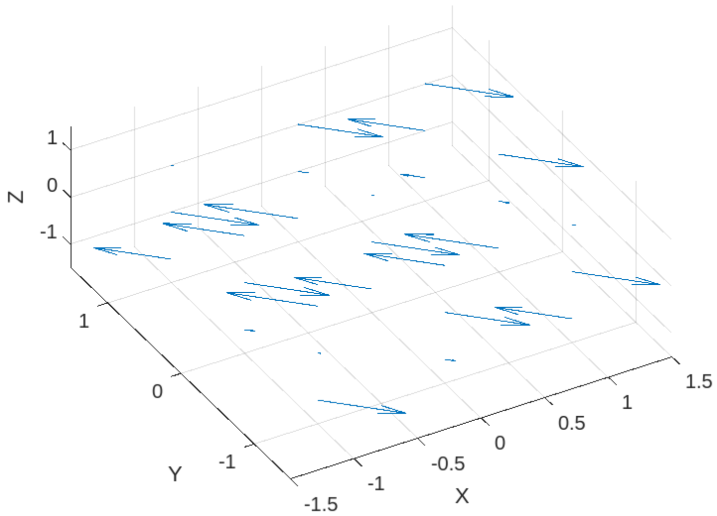

Fig 6. Plot of velocity vector field for water (a_5=2.0, a_6=1.0, a=2.0, b=1.0)

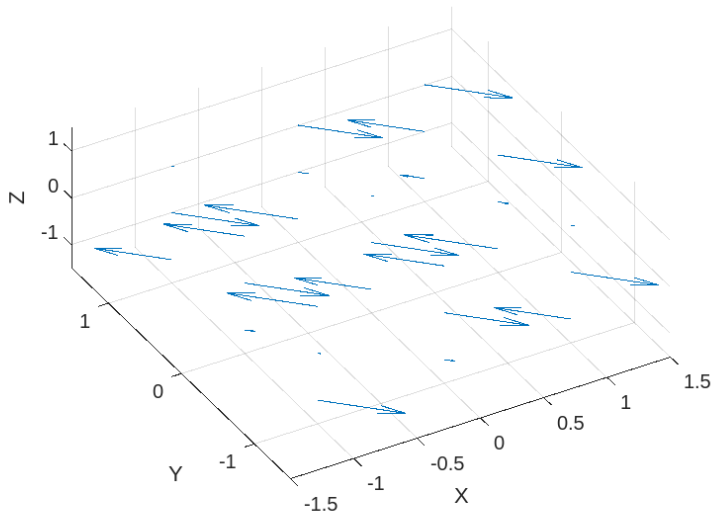

Fig 7. Plot of velocity vector field for carbondioxide (a_5=2.0, a_6=1.0, a=2.0, b=1.0)

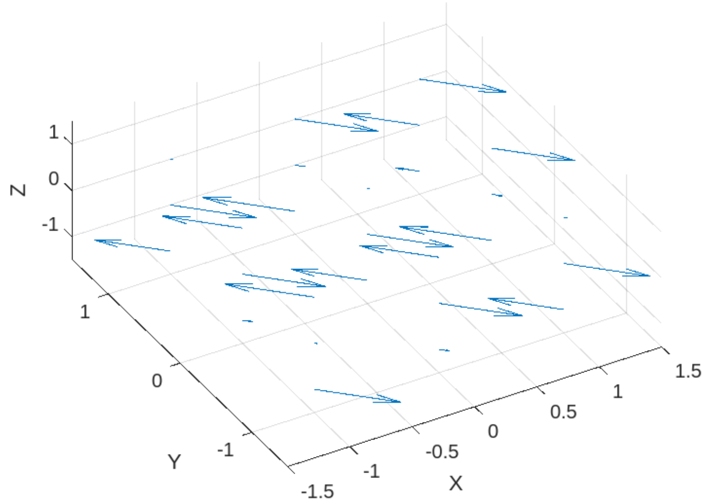

Fig 8. Plot of velocity vector field for Mercury (a_5=2.0, a_6=1.0, a=2.0, b=1.0)

Corresponding vorticity vector is

$$w = be^{-\left(a^2+2b^2\right)vt}\begin{pmatrix} aa_5\sin(b(x+y)-az)+aa_6\cos(b(x+y)-az)\,, \\ aa_5\sin(b(x+y)-az)+aa_6\cos(b(x+y)-az)\,, \\ 2b(a_5\sin(b(x+y)-az)+a_6\cos(b(x+y)-az)) \end{pmatrix} \tag{4.3}$$

Equation (1.3) must be fulfilled by the flow to preserve vorticity. Equation (1.3) will be satisfied if the velocity field and vorticity field are substituted when

$$\lambda = -(a^2+2b^2)\nu \tag{4.4}$$

Therefore, we can conclude that the vorticity field's topology is conserved. For the viscous incompressible Navier-Stokes equation, for which the vorticity field is topology preserved, we have identified the second family of exact solutions.

4.Conclusion

Over the last two decades, there has been a revival of interest in applying topological ideas in classical fluid mechanics, which has given rise to a novel branch of research termed topological fluid mechanics. The aim of this article is to find the specific solutions for the topological conservation and Navier-Stokes equation. We found the exact solutions for two different families of incompressible viscous flows. The vorticity vector potential ansatz approach is used to obtain the precise answer. Graphs are utilized to demonstrate the velocity field.

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