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Periodic Behaviour of General Systems

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Abstract:

A continuous function on the product of compact metric spaces to itself and back to the same space is known as a genera system. Where each element's orbit is an infinite sequence and where the first two elements are the same as given and next to it depends on the two elements prior to it form stronger conditions for the orbit. We could define an m-step dynamical system by extending the definition of a compact metric space to its m-times product. Because the system's current state frequently depends directly on the conditions of previous terms, this system appears more realistic. The basic theorems regarding periodic points and their related points, such as fixed point, limit points, recurrent points, will be proved in this paper. We also define the topological transitivity and its properties. In the end we find periodic points with periods one and two for affine maps and periodicity of tent map in dynamical system and generalised dynamical system.

Keywords: generalized dynamical systems, periods and periodic points.

1. Introduction

Topological dynamics is an intriguing field of mathematics. In that periodicity is interesting as it is related to real life situations like the path of orbit of planets etc. A number of mathematicians have explained the periodic behaviour on various dynamical systems, such as linear operators[1] and operators on Hilbert spaces[2] and periods and periodic points on linear cellular automata in[3]. The generalized systems that define X×X →X unlike X→X as described in dynamical systems. This new concept was defined[4] in the year 2008. In this instance, the author views X as a complete metric space. Dumitru[5] defined the topological version of generalized iterated functions based on this novel idea. By creating new concepts that work for generalized systems[6] explored chaos and shadowing properties in generalized dynamical systems in 2023[7] defined the Generalized function systems on metric spaces. The basic ideas of periodic points, such as fixed points, periodic points that repeat, non-wandering points, and the relationships between them, will be covered in this work. The powerful character of a point and the fact that it depends on the two or m elements (as in m step dynamical system) before it in that element's orbit present the biggest challenge in solving this system. Most of the statements that are true for dynamical systems are not valid in the present system like the fundamental elements in a periodic orbit are periodic in dynamical systems but not true in our case.

2. Preliminaries

In this paper we refer (X, d) as compact metric space throughout. The continuous map $f: X \times X \to X$ is Generalised system. In this paper we analyse the preliminary data and extending domain to,

$$f: X \times X \times \ldots . X(m\ times) \to X\ or\ f: X^m \longrightarrow X \qquad (2.1)$$

Defining as m step generalized system considering the same preliminaries. This assumption makes us calling the regular dynamical system as 1-step dynamical system and generalized system as 2-step dynamical system.

The *orbit* of any $x \in X$ is the infinite sequence $O(x) = \{x_n\}_0^\infty$ where $f: X^m \longrightarrow X$ is a continuous map and,

$$x_0 = x_1 = \cdots = x_m = x \text{ and } f(x_{n-m}, x_{n-m+1}, \ldots ., x_{n-1}) = x_n, n \geq m \qquad (2.2)$$

For $m = 2$ it becomes $x_0 = x_1 = x$ and $x_n = f(x_{n-1}, x_{n-2}), n \geq 2$

Here x_n denotes n^{th} term in the orbit $O(x) = \{x_n\}_0^\infty$.

For the convenience we consider $m = 2$ and define the preliminaries.

We say that $x \in X$ is fixed if $f(x, x) = x$. (2.3)

We say that $x \in X$ is *periodic* of periodic of *period* n if $x_{kn+i} = x_i$ for all $k \in \mathbb{N}$ and $0 \leq i < n$. We may call periodic point of period one is fixed point. (2.4)

We say that $x \in X$ is strongly periodic if every element of the orbit is periodic. Unlike in 1-step dynamical systems in general systems every element in periodic orbit need not be periodic. (2.5)

The *recurrent point* is $x \in X$ and $x \in \omega(x, f)$ that is for each $\epsilon > 0$ there is $n \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$. (2.6)

The *non-wandering* is $x \in X$ and for each > 0, $\delta > 0$ there is $n \in \mathbb{N}, z \in B_\delta(x)$ implies $d(z_n, x) < \epsilon$. (2.7)

We Also define strong non wandering if x is non-wandering and every element of the orbit is also non wandering. (2.8)

A subset $B \subseteq X$ is said to be *invariant* if $\{x_n\}_0^\infty \subseteq B$ for every $x \in B$. (2.9)

We denote the following notions for generalized system.

$fix(f)$ = Fixed points in the space X.

$Per(f)$=Periodic points in X.

$\mathcal{R}(f)$ = Recurrent points in X.

$\Omega(f)$ =Non-Wandering points in X.

3. Results

Since the results do not hold true for generalized systems, we must demonstrate all of the results that hold true for 1-step dynamical systems.

Theorem 3.1. For any generalized system defined in Eq.(1)

$$fix(f) \subseteq Per(f) \subseteq \mathcal{R}(f) \subseteq \Omega(f)$$

Proof:

1. The first two proofs are trivial. For any $x \in fix(f)$ it is clear that fixed point are of period one.

2. For any $x \in Per(f)$ then for some $n, x_n = x$.

So for each $\epsilon > 0$ for we get $d(x_n, x) < \epsilon$ that implies $x \in \mathcal{R}(f)$.

3. For any $x \in \mathcal{R}(f)$ then from Eq. (1.6), for any $\epsilon > 0$ there is $n \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$. Let $z \in B_\delta(x)$ for $\delta > 0$.

Then $d(x, z) < \delta$ implies $d(x_n, z_n) < \epsilon$ as the function as the function f defined on the compact metric space.

So, $d(z_n, x) \leq d(z_n, x_n) + d(x_n, x) < \epsilon + \epsilon = 2\epsilon$.

Then $x \in \Omega(f)$.

Lemma 3.1. If an orbit of an element converges then its converges to fixed point.

Proof: Let $x \in X$ be any point as orbit of x and $O(x) = \{x_n\}_0^\infty$ converges to for some p.

That is for any $\epsilon > 0$ there is $m \in \mathbb{N}$ such that $d(x_n, p) < \epsilon, n \geq m$.

By Eq (1.1), we can write $d(f(x_{n-2}, x_{n-1}), p) < \epsilon$

$$\Longrightarrow d(f(p,p), p) < \epsilon$$

We can say $f(p,p) = p$. That is p is a fixed point.

Theorem 3.2. $fix(f)$ closed in X.

Proof: Let $p \in \overline{fix(f)}$ is the closure of the set. There exist an orbit $O(x) = \{x_n\}_0^\infty$ converges to p.

p is a fixed point By the lemma 3.1. Which means set of all fixed point is a closed set as every limit point p is in that set.

Theorem 3.3. The set of Strong non-wandering points is closed set in X.

Proof: As we defined S is the set of Strong non-wandering points in X.

Let p is a limit point of S.

Then There exist an orbit $O(x) = \{x_n\}_0^\infty$ converges to p.

So for every $\delta > 0$ we get $B_\delta(p)$ which contains large number of non wandering points x_n. For any $y \in B_\delta(p)$ which is non-wandering then there exist $z \in B_\delta(p)$ and for some natural number n,$d(z_n, y) < \delta$. This is true for every $\in B_\delta(p)$.

So p is a strong non-wandering point.

The strong version of the theorem is taken into consideration because every element of the non-wandering orbit is non-wandering.

Lemma 3.2. Set of fixed points, strong periodic points and strong non-wanderings points is invariant in X.

Proof: The proof is direct application of the definition of invariant set and consideration strong form.

The Lemma 3.2 is true for set of periodic points and non-wandering points in m-step dynamical system.

4. Transitivity

Function $f: X \times X \to X$ is *topological transitive* if for every pair of non empty open sets $U, V \subseteq X$ then for some $x \in U$ then for some $\in \mathbb{N}, x_n \in V$.

$x \in X$ is a *transitive point* if it has a dense orbit.

Every element of a dense orbit is also a dense orbit in a *strong dense orbit.*

A non empty, closed, invariant subset Y of X is minimal set if it is not contained in any proper invariant closed set, Which is equivalent to orbit of every element of Y is dense.

Theorem 4.1.
The Following are equivalent. (These also hold true for 1-step dynamical systems, but additional proof is required in m-step dynamical systems case.)

1. Topological transitivity.
2. The set $V^* = \{x \in X: x_n \in V for\ some\ n \in \mathbb{N}\}$ is dense for every non empty open subset V of X.
3. If K is any invariant subset of X then either K is dense or K is nowhere dense.

Proof: 1⇒2

Assume f is a topological transitive. Then for every non empty pair of open subsets $W, V \subseteq X$, there exist $x \in W$ and $x_n \in V$ for some n. So$V^*(\neq \emptyset) \subseteq X$. And it is true for every pair of W and fixed V. Then for every $x \in X$ we get a non empty open set which intersects the given V^*. We can conclude V^* is dense in X.

2⇒ 3

Assume K be non empty invariant subset of X.We have K is invariant so compliment of K in X doesn't contain any element in the orbit of element of K.

Define $(K^c)^* = \{x \in X: x_n \in K^c\ for\ some\ n\}$ is K^c only.

If K^c has an interior point then from consequence of (2) it is dense and K has empty interior. In other case K^c has empty interior. In both cases K or K^c has empty interior. Which means K or K^c is dense. Which is same as either K is dense or K^c is dense.

Theorem 4.2: f is topological transitive then set of transitive points are residual.

Proof: x is a transitive point then $O(x)$ is dense in X. Let $\{U_i\}$ be the arbitary collection of open sets whose union is X.

Define $D(U_i) = \{x \in X: O(x) \cap U_i \neq \emptyset\}$ which is non empty as f is topological transitive and is open subset of X as f is continuous [6]. $D(U_i)$ is open dense set for each i.

As X is Compact so complete and we have $\cap U_i \neq \emptyset$ for each i and $x \in (\cap U_i) \neq \emptyset$. Which means x is the residual point.

Theorem 4.3: (X, d) is a compact metric space and $f: X^m \longrightarrow X$ is continuous. If $\overline{O(x)}$ is minimal for some $x \in X$ then for any non empty open subset U of X the cardinality of the set $Z = \{n: (x_n)_p \in U \; for \; every \; n \in \mathbb{N}, p \in \mathbb{N}\}$ is finite.

Proof: Assume $Y = \overline{O(x)}$ is minimal. Assume U be open subset of Y.

Given Y is minimal then orbit of every element is dense in Y. So for every $y \in Y$ there is some n such that $y_n \in U$. We Define $O^{-n}(U) = \{y \in X: y_n \in U \; for \; some \; n\}$. Which is open as f is continuous.

So $\mathcal{M} = \{O^{-n}(U): for \; every \; n\}$ is open cover of X. Being X is compact we have a finite subcover for $\mathcal{M}$.Then the set $\{U, O^{-1}(U), \ldots O^{-(p-1)}(U)\}$ is a finite sub cover for $\mathcal{M}$. Then for every $n \in \mathbb{N}$ there exist $k < p$ and $x_n \in O^{-k}(U)$.

Which is equivalent to $(x_n)_k \in U, 0 \leq k < p$

That is the set $Z = \{n: (x_n)_p \in U \; for \; every \; n \in \mathbb{N}, p \in \mathbb{N}\}$ is finite.

5. Periods and Periodic Points

In this section we find period and periodic points for the general system in the form of,

$$f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

Defined as,

$$f(x, y) = sx + ty + c \text{ (Eq 5.1)}$$

Where $s, t \; and \; c$ are constants.

Case I (If $c \neq 0$)

Orbit of any point $O(x) = \{x_0 = x, x_1 = x, x_2 = f(x, x), \ldots\}$

We examine fixed points for this map by $x_2 = x \Rightarrow f(x, x) = x$

$$\Longrightarrow sx + tx + c = x$$
$$\Rightarrow (s + t)x - x = -c$$
$$\Rightarrow x(s + t - 1) = -c$$
$$\Rightarrow x = \frac{c}{1 - s - t} \qquad \text{(Eq 5.2)}$$

If any map defined as in (Eq 5.1) then $x = \frac{c}{1-s-t}, 1 - s - t \neq 0$ is a fixed point.

Now we find the periodic points of period 2.

$$O(x) = \{x_0 = x, x_1 = x, x_2 = sx + tx + c, x_3 = sx_1 + tx_2 + c, \ldots\}$$

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For period 2 points $x_3 = x$,

$$\Rightarrow sx + t(sx + tx + c) + c = x$$
$$\Rightarrow x(s + st + t^2 - 1) = -c(1 + t)$$
$$\Rightarrow x\big(s(1 + t) + (1 + t)(t - 1)\big) = -c(1 + t)$$
$$\Rightarrow if\ (1 + t) \neq 0, x(s + t - 1) = -c$$
$$\Rightarrow x = \frac{c}{1 - s - t}$$

Which is same as fixed point so for $(1 + t) \neq 0$, then,

The map in (Eq 5.1) doesn't have any periodic point with period 2.

If $t + 1 = 0 \Rightarrow t = -1$ then the orbit will be,

$$O(x) = \{x_0 = x, x_1 = x, x_2 = sx - x + c, x_3 = sx_1 - x_2 + c, \ldots\}$$
$$\Rightarrow x_3 = sx - (sx - x + c) + c$$
$$\Rightarrow x_3 = sx - sx + x - c - c$$
$$\Rightarrow x_3 = x \qquad \text{(Eq 5.3)}$$

We can conclude if $(1 + t) = 0$ then every point is of period 2.

Case II (If $c = 0$)
Then (Eq 5.1) becomes,

$$f(x, y) = sx + ty \qquad \text{(Eq 5.4)}$$

Orbit of any point $O(x) = \{x_0 = x, x_1 = x, x_2 = f(x, x), x_3 = f(x_1, x_2) \ldots\}$

Fixed points are $x_2 = x \ \Rightarrow\ f(x, x) = x$

$$\Longrightarrow sx + tx = x$$
$$\Rightarrow (s + t)x = x$$
$$\Rightarrow (s + t) = 1$$

If $s + t = 1$ then every point is fixed point. (Eq 5.5)

For period 2 points we have $x_3 = x \ \Rightarrow f(x, sx + tx) = x$

$$\Rightarrow sx + t(sx + tx) = x$$
$$\Rightarrow x \neq 0, s + t \neq 1, s + st + t^2 = 1$$
$$\Rightarrow s(1 + t) + (1 + t)(t - 1) = 0$$
$$\Rightarrow (1 + t)(s + t - 1) = 0$$

As we have assumed $s + t - 1 \neq 0 \ \Rightarrow 1 + t = 0$

$$\Rightarrow t = -1$$

For $s + t - 1 \neq 0$ and $t = -1$ every point is of period 2. (Eq 5.6).

6. Problems

Problem 6.1 For the map $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$f(x, y) = 2x - 3y + 4$ which is continuous so it is a general system.

By (Eq 5.2) Fixed points for the map is $x = \frac{4}{1-2-(-3)} = 2$.

By (Eq 5.3) It doesn't have any periodic points of period 2 as $1 + t \neq 0$.

Problem 6.2 For the map $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$f(x, y) = x - y + 9$ which is continuous so it is a general system.

By (Eq 5.2) Fixed points for the map is $x = \frac{9}{1-1-(-1)} = 9$.

By (Eq 5.3) $1 + t = 0$ so every point is of period 2 except fixed points.

Problem 6.3 For the map $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$f(x, y) = 2x - y$ which is continuous so it is a general system.

By (Eq 5.5) Every point is fixed point as $s + t = 1$.

By (Eq 5.3) Every point is of period 2 except fixed points as $t = -1$.

7. Conclusion

The major part of the fundamental dynamical system is still unknown in the generalized system and m-step systems, which are novel concepts. We developed relationships between periodic concepts such as fixed points, recurrent points, and non-wandering points in this study. Additionally, we looked at the theorems connected to transitivity and, finally, were provided with examples.

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