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Strong Regular Domination in Litact Graphs

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Abstract:

Strong regular domination in a litact graph is a novel domination parameter that has been introduced in this paper. A dominating set $D \subseteq V(G)$ is known as Strong regular dominating set of G, if for each point $x \in V(G) - D$ there is a vertex $y \in D$ with an edge $xy \in E(G)$ and $\deg(x) \leq \deg(y)$ and all vertices of $\langle D \rangle$ holds the equal degree. The lowest cardinality of such vertices of D is known as strong regular domination number of G which is represented by $\gamma_{str}(G)$. The current study aims by taking strong regular domination on a litact graph $m(G)$ denoted by $\gamma_{str}[m(G)]$ and to obtain some bounds on $\gamma_{str}[m(G)]$ in terms of various parameters of G such as vertices, edges, maximum degree, diameter and so on and also in terms of various domination parameters of G such as total domination of G, connected domination of G and so on. Furthermore, outcomes resembling those of Nordhaus-Gaddum were also obtained.

Keywords: litact graph, strong domination number, regular domination number, strong regular domination number.

2020 AMS Subject Classification: 05C72.

1. Introduction

One area of graph theory that has been studied in great detail is domination. The theory of domination has multiple origins. According to historical accounts, the first domination-type problem originated with a chess board problem in 1850 that C.F. De Jaenisch mathematically explained in 1862. Domination in graphs has implications beyond the chess board problem, including facility location problems, electric networks, power grids, land surveying, and more. The topic of domination was first introduced by C. Berge in 'The Theory of Graphs and its Application', and it was formalized mathematically by Ore in 1962. Berge referred to the number of domination as the coefficient of external stability and the domination as the external stability. The term "domination" was first used by Ore in his well-known 1962 book "Theory of Graphs".

Certain real-world scenarios naturally give rise to both strong and weak domination. Take a road network that connects several places, for instance, the degree of such a network is determined by 'the number of roads that meet at a vertex v'. Let $\deg(u) \geq \deg(v)$. It seems that, 'traffic at u appears to be heavier than at v'. Vehicles travelling from u to v should be given preference when it comes to traffic between u and v. Therefore, 'u strongly dominates v' and 'v weakly dominates u' in a certain sense.

In 1996, 'Sampathkumar and Pushpa Latha' presented the ideas of 'strong and weak domination'. It was E. Sampath kumar who first proposed the concept of regular domination number. Numerous scholars have investigated these ideas, including [1,2,3,6,7].

2. Notations and Definitions

Definition-2.1: If each point in $V - X$ is adjacent to a point in X, then X is considered a ***'dominating set'*** of $G(V, E)$. The ***domination number*** of G, represented by $\gamma(G)$, is the 'minimal cardinality of such a set'.

Definition-2.2: Suppose G is a connected graph with an edge $xy \epsilon E(G)$. Then, 'x strongly dominates y' ('y weakly dominates x') if 'deg $(x) \geq$ deg (y)'. It is clear that each point in $V(G)$ has the ability to strongly dominate itself. A set $D \subseteq V(G)$ is a ***'strong dominating set'*** of G, if for each point $y \epsilon V(G) - D$ there is another point $x \epsilon D$ such that $xy \epsilon E(G)$ and deg $(x) \geq$ deg (y). The 'minimum cardinality' of such set is known as ***'strong domination number'*** of G and it is represented by $\gamma_{st}(G)$.

Definition-2.3: A 'graph' G is called a ***'regular graph'*** if each point in it holds the equal degree. A 'dominating set D' is called a ***'regular dominating set'*** of G if $\langle D \rangle$ is regular. The ***'regular domination number'*** of G, represented by $\gamma_r(G)$, is the 'minimal cardinality' of such a set.

Definition-2.4: A 'dominating set D' of a 'graph G' is known as ***'strong regular dominating set'*** of G if (i) for each point $y \epsilon V(G) - D$ there is another point $x \epsilon D$ such that $xy \epsilon E(G)$ and deg $(x) \geq$ deg (y) and (ii) an induced subgraph $\langle D \rangle$ is a 'regular dominating set of G'. The smallest cardinality of such a set is known as ***'strong regular domination number'*** of G and which is represented by $\gamma_{str}(G)$.

Definition-2.5: The point set of a ***lit act graph*** $m(G)$ is made up of the edges and cut vertices of a graph G. If edges and cut vertices are incident or adjacent in G, then the two vertices in $m(G)$ are adjacent.

Example.

Figure 1 below shows a graph G alongwith it's litact graph $m(G)$.

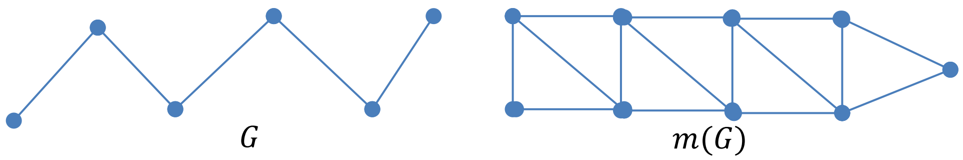

Fig. 1: A Graph G and it's Litact Graph $m(G)$ with $\gamma_r(m(G)) = \gamma_{st}(m(G)) = \gamma_{str}(m(G)) = 2$

3. Main Results

The current study exclusively considers simple, finite, non-trivial, connected and undirected graphs. All the following results are hold for any graph on which strong regular domination is found.

The following section evaluates a graph G's and it's litact graph $m(G)$'s strong regular domination number for few standard graphs, such as a 'cycle graph', 'star graph', 'wheel graph', 'complete graph' and so on.

Theorem-3.1: For any 'cycle graph' C_n , $\gamma_{str}(C_n) = \gamma_{str}(m(C_n))$.

Theorem-3.2: For any 'wheel graph' W_n, $\gamma_{str}(W_n) = 1$.

Theorem-3.3: For any 'complete graph' K_n, $\gamma_{str}(K_n) = 1$.

Theorem-3.4: For any 'complete bipartite graph' $K_{m,n}$, $\gamma_{str}(K_{m,n}) = m$ where $m \leq n$.

Theorem-3.5: For any 'star graph' $K_{1,n}$, $\gamma_{str}(K_{1,n}) = \gamma_{str}(m(K_{1,n})) = 1$.

An upper bound for $\gamma_{str}(m(G))$ in terms of order of G has been determined in the following theorem.

Theorem-3.6: If G is any 'graph G', then $\gamma_{str}(m(G)) < p$.

Proof: Suppose V represent G's vertex set, E represent G's edge set and C represent G's set of cut vertices. As stated in the definition of a litact graph $m(G)$, "$V(m(G)) = E(G) \cup C(G)$". If $X \subseteq V(m(G))$ is the smallest dominating set of $m(G)$ and deg $(x_i) \leq$ deg (x_j) where $x_i \epsilon V(m(G)) - X$ and X is k-regular then X itself forms a strong regular dominating set. Then $\gamma_{str}(m(G)) = |X|$. Otherwise, there exists $x_i \in Y \subseteq V(m(G))$ such that $\langle X \cup Y \rangle$ forms a minimal strong regular dominating set in $m(G)$. Therefore, $|X \cup Y| = \gamma_{str}(m(G))$. Then, $V(m(G)) < 2p$. Also,

$$
\begin{gathered}
|X \cup Y| \leq V(m(G)) - |X \cup Y| \\
\Rightarrow |X \cup Y| + |X \cup Y| \leq V(m(G)) \\
\Rightarrow 2|X \cup Y| \leq V(m(G)) < 2p \\
\Rightarrow 2\gamma_{str}(m(G)) < 2p \\
\Rightarrow \gamma_{str}(m(G)) < p
\end{gathered}
$$

The relationship between $\gamma_{str}(m(G))$, $\gamma_{str}(G)$, $\deg(x)$ and deg (y) is given below.

Theorem-3.7: For any 'graph G', $\gamma_{str}(m(G)) \leq \gamma_{str}(G) + \deg(x) + \deg(y)$ where x, y are the elements of a 'strong regular dominating set of $m(G)$'.

Proof: Let $m(G)$ be the litact graph of G with V points and E edges. Suppose D is a 'strong regular dominating set' of $m(G)$. Assume that the points $x, y \in D$ and x has the same degree as some of its neighbours (except y) and strongly dominates them, and that y does the same. Let v_1 is adjacent to $x, v_1 \neq y$ such that $\deg(x) = \deg(v_1)$, and v_1 is strongly dominated only by x. Hence, 'D is a strong regular dominating set', so that $|D| = \gamma_{str}(m(G))$. Let S be the strong regular dominating set in G such that $|S| = \gamma_{str}(G)$. From the above, it is easy to verify that $|D| \subseteq |S| \cup |N(x)| \cup |N(y)| \Rightarrow \gamma_{str}(m(G)) \leq \gamma_{str}(G) + \deg(x) + \deg(y)$.

The relationship between $\gamma_{str}\left(m(H_p)\right)$, 'edges' and 'cut vertices' of the 'helm graph' has been determined in the subsequent theorem.

Theorem-3.8: For any (p,q) −Helm graph $H_p, \gamma_{str}\left(m(H_p)\right) < q + l$ where l is the number of cut vertices of H_p.

Proof: Suppose $G(V,E)$ be a Helm graph with $V = \{v_1, v_2, v_3, \ldots, v_{p-1}, v_1', v_2', v_3', \ldots, v_{p-1}'\}$ vertices and $E = \{v_1 v_i, 2 \le i \le p\} \cup \{v_i v_{i+1}, v_i v_{i-1}', 2 \le i \le p-1\} \cup \{v_2 v_p\} \cup \{v_1 v_i', 1 \le i \le p-1\}$ edges. Since the helm graph is formed by attaching one vertex by an edge to each of the $p-1$ points in the wheel graph's 'outer circuit', the vertex v_1 comes after each of the $p-1$ vertices in the wheel graph. Every $p-1$ vertex in the wheel network is adjacent to v_1. This indicates that the strong regular domination for a wheel graph is 1 such that $\gamma_{str}(W_p) = 1$. But the vertex v_1 strongly dominates only the wheel graph; the helm graph is not dominated by it. Given that the helm network contains $p-1$ pendent vertices, and for each pendant vertex $v_j, 2 \le j \le p$ is connected by an edge. The $p-1$ vertices in the outer circuit are therefore superior to both of the $p-1$ vertices in the helm graph. Thus, the 'strong regular domination number' of the helm graph is $p-1$, that is t $\gamma_{str}(C) = p-1$. Since the litact graph of helm graph is formed by the cut vertices and edges of the helm graph, and it is clear that if D is the strong regular domination number of the helm graph, then $|D| < q(H_p) + l(H_p)$. Hence, $\gamma_{str}\left(m(H_p)\right) < q + l$.

The next theorem connects $\gamma_{str}(m(G)), \gamma(G), p(G)$ and $\alpha_0(G)$.

Theorem-3.9: For any connected (p,q) graph G, $\gamma_{str}(m(G)) + \gamma(G) \le p + \left\lceil \frac{\alpha_0}{2} \right\rceil$.

Proof: Let $V = \{v_1, v_2, v_3, \ldots, v_i\}$ be the smallest set of vertices that encompasses every edge in G, ensuring that $|V| = \alpha_0$. Let D be the 'dominating set of G' going forward so that $|D| = \gamma(G)$. Assume that the 'set of edges' in G that form a minimal connected edge dominating set in G is represented by the set $E = \{e_1, e_2, e_3, \ldots, e_k\}$. Let $C \subseteq V(G)$ represent G's set of cut vertices. The minimal strong dominating set in $m(G)$ is then the set $X \epsilon V(m(G))$ where $X \subseteq F \cup C$ and $N(X) = V(m(G)) - X$. Further, let $Y \subseteq V(m(G)) - X$ and $Y \in N(X)$, then take a set $Y' \subset Y$ so that $\langle X \cup Y' \rangle$ is the minimal strong k-regular sub graph of $m(G)$, then $X \cup Y'$ is the minimal strong regular dominating set of $m(G)$. Then,

$$|X \cup Y'| \cup |D| \le p + \left\lceil \frac{\alpha_0}{2} \right\rceil \Rightarrow \gamma_{str}(m(G)) + \gamma(G) \le p + \left\lceil \frac{\alpha_0}{2} \right\rceil.$$

The next outcome involving $\gamma_{str}(m(G)), \gamma_c(G), \beta_0(G)$ and $\gamma(G)$ has been determined.

Theorem-3.10: For any 'graph G', $\gamma_{str}(m(G)) + \gamma_c(G) \le \beta_0(G) + \gamma(G)$.

Proof: Let $X \subseteq V(G)$ be the set of points that cover all of the edges in G at a minimum distance of two and have $\deg(v_i) \ge 2, \forall v_i \in X, 1 \le i \le k$. Additionally, if $N(x) = V(G) - X$ for every vertex $x \in X$, then X is an independent vertex collection. Otherwise, $|X_1 \cup X_2| = \beta_0(G)$ creates a maximum independent set of G where $X_1 \subseteq X$ and $X_2 \subseteq V(G) - X$. Let $Y = X' \cup X''$ represent the minimal set of points that covers all points in G, where $X' \subseteq X$ and $X'' \subseteq V(G) - X$. Y undoubtedly makes up G's minimum dominating set. Assume that there is just one element in the sub graph $\langle Y \rangle$. Then, Y is a 'connected dominating set of G' on its own.

Otherwise, attach the smallest number of vertices $\{u_j\} \in V(G) - Y$ where $\deg(u_j) \geq 2$ that are between the vertices of Y such that $Y_1 = Y \cup \{u_j\}$ forms precisely one component in the subgraph $\langle Y_1 \rangle$. Y_1 undoubtedly constitutes a minimal connected dominating set within G. Let $D \subseteq C$, where C is the set of points in G that correspond to the edges that meet Y's vertices. A minimal set of vertices where $\forall v_k \in D$ and $N(D) = V(m(G)) - \{v_k\}$ are found. D undoubtedly creates a strong regular dominating set in $m(G)$.

Therefore, $|D| \cup |Y_1| \leq |X_1 \cup X_2| \cup |Y| \Rightarrow \gamma_{str}(m(G)) + \gamma_c(G) \leq \beta_0(G) + \gamma(G)$. Hence, the proof.

Using $\gamma_{str}(m(G)$ in terms of q and $\Delta'(G)$, the following theorem has been discovered.

Theorem-3.11: For any 'graph G', $\left\lceil \frac{q}{2\Delta'(G)+1} \right\rceil < \gamma_{str}(m(G) + 1$.

Proof: Let 'G be a graph' with the set of edges $E = \{e_1, e_2, e_3, \dots \dots, e_n\}$, such that $|E(G)| = q$ and $\Delta'(G)$ be the highest degree of an edge in G. Suppose $X \subseteq V(m(G))$ is the smallest dominating set in $m(G)$, then X creates a strong regular dominating set by itself. Next, it is clear that, according to the litact graph definition $V(m(G)) = E(G) \cup C(G)$, we get,

$$(2\Delta'(G) + 1)|X| \geq |E(G)|$$
$$\Rightarrow (2\Delta'(G) + 1)|X| \geq q$$
$$\Rightarrow |X| \geq \frac{q}{2\Delta'(G) + 1}$$
$$\Rightarrow \left\lceil \frac{q}{2\Delta'(G) + 1} \right\rceil \leq \frac{q}{2\Delta'(G) + 1} < |X| + 1$$
$$\Rightarrow \left\lceil \frac{q}{2\Delta'(G) + 1} \right\rceil < \gamma_{str}(m(G) + 1$$

We derive the lower bound for $\gamma_{str}(m(G))$ in terms of diameter of G in the subsequent theorem.

Theorem-3.12: For any 'graph G', $diam(G) - 1 \leq 3\gamma_{str}(m(G))$.

Proof: Let the collection of edges in G that make up the diameteral path in G be denoted by $E_1 = \{e_i; 1 \leq i \leq n\}$. Therefore, $|E_1| = diam(G)$. Let $E_2 \subseteq E(G), \forall e_i \in E_2$ has the maximal edge degree in G. E_2 becomes a minimum dominating set of $m(G)$ since $E_2 \subseteq V(m(G))$, it is the smallest set of points which covers all of the vertices in $m(G)$. Moreover, E_2 is a smallest strong regular dominating set in $m(G)$ if $\deg(x) \leq \deg(y), \forall x \epsilon V(m(G)) - E_2$ and E_2 is k-regular. Consequently, $|E_1| - 1 \leq 3|E_2| \Rightarrow diam(G) - 1 \leq 3\gamma_{str}(m(G)$.

Hence, $diam(G) - 1 \leq 3\gamma_{str}(m(G)$.

The subsequent theorem relates $\gamma_{str}(m(G)), \gamma_c(G), diam(G), \alpha_0(G)$ and $\gamma(G)$.

Theorem-3.13: For any connected graph G, $\gamma_{str}(m(G)) + \gamma_c(G) < diam(G) + \alpha_0(G) + \gamma(G)$.

Proof: Define $A \subseteq V(G)$ as the smallest set of vertices that encompasses all edges in G, ensuring that $|A|$ equals $\alpha_0(G)$. Moreover, an edge set $E \subseteq E'$ exists, where E' is the collection of edges that coincide with the vertices of V to form the longest path in G, with $|E|$ equal to $diam(G)$. To build a 'minimal dominating set of G', let $S = \{v_i; 1 \leq i \leq n\} \subseteq V(G)$. If $\langle S' \rangle$ is a connected sub graph of

S, then 'S is a connected dominating set'. In the absence of this, a 'minimal connected dominating set of G' is formed by at least one vertex, $x \in V(G) - S'$ and $S" = S' \cup \{x\}$.

Now, in $m(G)$, let $U = \{u_j; 1 \leq j \leq n\} \subseteq V(m(G))$ such that $\{u_j\} = \{e_j\} \in E(G), 1 \leq j \leq n$, where $\{e_j\}$ are incident with the vertices of S'. Additionally, assume that D represents the dominant set of $m(G)$ and that $\forall u_k \in \langle V(m(G)) - D \rangle, \deg(u_k) \leq \deg(u_j), \forall u_j \in D$ and D is k-regular. So, D creates a strong regular dominating set in $m(G)$. Otherwise, $\deg(u_k) > \deg(u_j), \forall u_j \in D$, exists for at least one vertex $\{u\} \in V(m(G)) - D$. It is evident that $D \cup \{u\}$ constitutes a minimal strong regular dominating set within $m(G)$. Consequently,

$$|D \cup \{u\}| \cup |S"| < |E| + |S'| + |A| \Rightarrow \gamma_{str}(m(G) + \gamma_c(G) < diam(G) + \alpha_0(G) + \gamma(G)$$

The following outcome has been ascertained. $\gamma_{str}(m(T))$ concerning $\gamma(T)$ and $\gamma_c(T)$.

Theorem-3.14: For any tree T, $\gamma_{str}(m(T)) \leq \gamma(T) + \gamma_c(T)$.

Proof: Assume that $X = \{v_i; 1 \leq i \leq n\} \subseteq V(T)$ is the smallest set of vertices that encompasses every edge in T. It is evident that X makes up T's minimal dominating set, so that $|X| = \gamma(T)$. Assume that $\langle X' \rangle$ is a connected sub graph of X. Then X' is a connected dominating set in and of itself, so that $|X'| = \gamma_c(T)$. In the absence of this, $X" = X' \cup \{x\}$ forms a 'minimal connected dominating set' of T, and there exists at least one vertex $x \in V(T) - X'$. According to this, D creates a strong regular dominating set in $m(T)$, so that $|D| = \gamma_{str}(m(T))$. Consequently,

$$|D| \subseteq |X| \cup |X'| \Rightarrow \gamma_{str}(m(T)) \leq \gamma(T) + \gamma_c(T).$$

We require the following theorems in order to demonstrate our subsequent findings.

Theorem-A[4]: For each graph G, $\gamma_m^{-1}(G) < \gamma_c(G) + \gamma_t(G)$.

Theorem-E[5]: In each graph G, $\gamma_m'(G) + diam(G) \leq p + \gamma(G)$.

Corollary-3.15: For each graph G, $\gamma_{str}(m(G)) + \gamma_m^{-1}(G) < p + \gamma_c(G) + \gamma_t(G)$.

Proof: Theorems 3.6 and A make it simple to demonstrate the aforementioned result.

Corollary-3.16: For any 'graph G', $\gamma_{str}(m(G)) + \gamma_c(G) + \gamma_m'(G) < 2\alpha_0(G) + \beta_0(G) + 2\gamma(G)$.

Proof: Since $p(G) = \alpha_0(G) + \beta_0(G)$, Theorem 3.13 and Theorem E lead to the above outcome.

Finally, Nordhaus-Gaddum type outcomes are found at the end.

Theorem-3.17: For any (p, q) connected graphs G and $\bar{G}$,

i) $\gamma_{str}(m(G)) + \gamma_{str}(m(\bar{G})) \leq 2p$
ii) $\gamma_{str}(m(G)).\gamma_{str}(m(\bar{G})) \leq pq$

4. Conclusion

In the current study some results have been obtained on $\gamma_{str}(m(G))$ in terms of several parameters of G, such as its vertices, edges, diameter and so on, as well as several domination parameters of G, such as edge domination, connected domination and total domination and many more.

References

[1] Sri, P.A., Thamaraikannan, N., Loganathan, K., & Chaudhary, D.K. (2022). Double Domination and Regular Domination in Intuitionistic Fuzzy Hypergraph. Journal of Mathematics, 1436194. https://doi.org/10.1155/2022/1436194.

[2] Agustin, I.H., Retnowardani, D.A., & Kurniawati, E.Y. (2021). On the resolving strong domination number of graphs: a new notion. In Journal of Physics: Conference Series, 1836(1). https://doi.org/10.1088/1742-6596/1836/1/012019.

[3] Elavarasan, K., Gunasekar, T., Cepova, L., & Cep, R. (2022). Study on a Strong and Weak n-Connected Total Perfect k-Dominating set in Fuzzy Graphs. Mathematics, 10(17), 3178. https://doi.org/10.3390/math10173178.

[4] Lakshmi, A.R., Vani, M., & Majeed, A. (2022). Inverse litact domination in graphs. Materials Today: Proceedings, 65(8), 3552-3557. https://doi.org/10.1016/j.matpr.2022.06.147.

[5] Vani, M., Abdul, M., & Vasundhara, D.J. (2020). Edge Litact Domination in graphs. International Journal of Future Generation Communication and Networking, 13(3), 3636-3641. http://sersc.org/journals/index.php/IJFGCN/article/view/30753.

[6] Shi, X., Akhoundi, M., Talebi, A.A., & Mojahedfar, M. (2023). A Study on Regular Domination in Vague Graphs with Application. Advances in Mathematical Physics, 2023. https://doi.org/10.1155/2023/7098134

[7] Zaherifar, H., Alikhani, S., & Ghanbari, N. (2023). On the strong dominating sets of graphs. Journal of Algebraic Systems, 11(1), 65-76. https://doi.org/10.22044/JAS.2022.11646.1595.