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Generating Function Involving General Function Related to Hurwitz-Lerch Zeta Function

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Abstract:
In this paper, we have studied a general function which unifies the Hurwitz-Lerch Zeta function and Mittage-Leffler function. The integral representation of the function and certain generating functions involving this general function are established. A mild extension of this general function is also presented in this paper. The generating functions for this extended function are also studied in this paper. Certain known results involving Hurwitz-Lerch Zeta function for several parameters are shown to be obtained here.

Keywords: Hurwitz-Lerch Zeta function; Mittag-Leffler function; Generating functions; Generalized Hypergeometric function; Integral representation

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1. Introduction and Preliminaries

A general Hurwitz-Lerch Zeta function $\phi(z,s,a)$ is defined in the literature in the following manner [1]:

$$\phi(z,\text{s},a)=\sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}, \qquad \ldots(1.1)$$

$$(a\in C/Z_0^-, \text{s}\in C \text{ when} |z|<1 \text{ and } \operatorname{Re}(s)>1 \text{ when } |z|=1)$$

Where C,R,R^+,Z represents the Set of complex numbers, Set of real numbers, Set of positive real numbers and Set of integer number respectively and $Z_0^-=Z^-\cup\{0\}$, $Z^-=\{-1,-2,-3...\}$

The Hurwitz-Lerch Zeta function $\phi(z,s,a)$ is defined in (1.1) contains, its special cases, as the Riemann Zeta function $\zeta(s)$ and Hurwitz-Lerch Zeta function $\zeta(s,a)$ defined by (see, for details,[1, chapter I])

$$\zeta(s)=\sum_{n=0}^{\infty}\frac{1}{(n+1)^s}=\phi(1,s,1) \quad (\operatorname{Re}(s)>1) \qquad \ldots(1.2)$$

$$\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}=\phi(1,s,a) \quad \left(\operatorname{Re}(s)>1; a\in C/Z_0^-\right) \qquad \ldots(1.3)$$

A generalization of the Hurwitz-Lerch Zeta function $\phi(z,s,a)$ was studied by Lin and Srivastava [2, p.727, eq.(8)]

$$\phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}}\frac{z^n}{(n+a)^s} \qquad \ldots(1.4)$$

$(\mu\in C; a,\nu\in C/Z_0^-, \rho,\sigma\in R^+; \rho<\sigma$ when $s,z\in C; \rho=\sigma$ and $s\in C$ when $|z|<\delta=\rho^{-\rho}\sigma^{\sigma}; \rho=\sigma$ and $\operatorname{Re}(s-\mu-\nu)>1$ when $|z|=\delta)$

Where $(\lambda)_n$ denotes the general pochhammer symbol which is defined by in terms of the gamma function by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma\lambda}=\begin{cases}1 & \left(n=0; \lambda\in C/\{0\}\right)\\ \lambda(\lambda+1)\ldots\lambda(\lambda+n-1) & (n=N; \lambda\in C)\end{cases} \qquad \ldots(1.5)$$

The function at $\rho=\sigma=1$ and $\nu=1$ yields the Hurwitz-Lerch Zeta function defined and studied by Goyal and Loddha [3, p.100, eq.(1.5)]

$$\phi_{\mu}^{*}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_n}{n!}\frac{z^n}{(n+a)^s} \qquad \ldots(1.6)$$

$(\mu\in C; a\in C/Z_0^-, s\in C$ when $|z|<1$ and $\operatorname{Re}(s-\mu)>1$ when $|z|=1)$

Further generalization of the functions $\phi(z,s,a)$ and $\phi_{\mu}^{*}(z,s,a)$ was considered by Garg et.al [4] in the following manner [4,p.313, eq.(1.7)]:

$$\phi_{\lambda,\mu,\nu}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\lambda)_n(\mu)_n}{(\nu)_n n!}\frac{z^n}{(a+n)^s} \qquad \ldots(1.7)$$

$(\lambda,\mu\in C; \nu,a\in C/Z_0^-; s\in C$ when $|z|<1$ and $\operatorname{Re}(s+\nu-\lambda-\mu)>1$ when $|z|=1)$

A further generalization of a family of Hurwitz-Lerch Zeta function defined and studied by Srivastava et al [5, p.491, eq.(1.20)] in the following form:

$$\phi_{\lambda,\mu;\nu}^{(\rho,\sigma,k)}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{kn}\,n!}\frac{z^n}{(n+a)^s} \quad \ldots(1.8)$$

$(\lambda,\mu\in C;\nu,a\in {C}/{Z_0^-};\rho,\sigma,\mathrm{k}\in\mathrm{R}^+;k-\rho-\sigma>-1$ when $s,z\in C;k-\rho-\sigma=-1$ and $s\in C$ when $|z|<\delta^*=\rho^{-\rho}\sigma^{-\sigma}\mathrm{k}^k;k-\rho-\sigma=-1$ and $\mathrm{Re}(s+\nu-\lambda-\mu)>1$ when $|z|=\delta^*)$

The well-known Mittage-Leffler function $E_\alpha(z)$ in defined by following manner [13](See also [14,15])

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)},\ \left(\mathrm{z}\in C,\mathrm{Re}(\alpha)>0\right) \quad \ldots(1.9)$$

This function further generalized by Prabhakar[16] in the following form [See also [17]]:

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)n!} \quad \ldots(1.10)$$

$$\left(\alpha,\beta,\gamma\in C;\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(\gamma)>0\right)$$

In view of extension of parameters, Mittage-Leffler function was also studied by Khan et al [6, p.2, eq.(1.9)]

$$E_{\alpha,\beta,\nu,\sigma,\delta,p}^{\mu,\rho,\eta,\mathrm{q}}(z)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\eta)_{qn}}{(\nu)_{\sigma n}(\delta)_{pn}}\frac{z^n}{\Gamma(\alpha n+\beta)} \quad \ldots(1.11)$$

Where $\alpha,\beta,\eta,\delta,\mu,\nu,\rho,\sigma\in C;p,q>0$ and $\mathrm{q}\le\mathrm{Re}(\alpha)+p$ and,
$\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\eta),\mathrm{Re}(\delta),\mathrm{Re}(\mu),\mathrm{Re}(\nu),\mathrm{Re}(\rho),\mathrm{Re}(\sigma)\}>0$ if $\delta=p=1$, it takes the following form

$$E_{\alpha,\beta,\nu,\sigma}^{\mu,\rho,\eta,\mathrm{q}}(z)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\eta)_{qn}}{(\nu)_{\sigma n}\Gamma(\alpha n+\beta)}\frac{z^n}{n!} \quad \ldots(1.12)$$

Where $\alpha,\beta,\eta,\mu,\nu,\rho,\sigma\in C;p,q>0$ and $\mathrm{q}\le\mathrm{Re}(\alpha)$ and,
$\min\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\eta),\mathrm{Re}(\mu),\mathrm{Re}(\nu),\mathrm{Re}(\rho),\mathrm{Re}(\sigma)\}>0$

Wright Introduced and studied the Taylor-Maclaurin Series [7, Pg.:424]:

$$\in_{\alpha,\beta}(\phi;z)=\sum_{n=0}^{\infty}\frac{\phi(n)z^n}{\Gamma(\alpha n+\beta)}\quad\left(\alpha,\beta\in C;\mathrm{Re}(\alpha)>0\right) \quad \ldots(1.13)$$

Where $\phi(n)$ is a function satisfying suitable conditions Motivated by the work of Wright[7] E.W. Barnes [23] considered the asymptotic expansion of functions in the class which is defined as follows:

$$E_{\alpha,\beta}^{(k)}(\mathrm{s};z)=\sum_{n=0}^{\infty}\frac{z^n}{(n+k)^s\,\Gamma(\alpha n+\beta)} \quad \left(\alpha,\beta\in C;\mathrm{Re}(\alpha)>0\right) \qquad \ldots(1.14)$$

For suitable restricted parameters k and s

Recently for suitably restricted sequence $\{\phi(n)\}_{n=0}^{\infty}$, Srivastava [8] introduced and studied a class of function [See also [9][10]]:

$$\in_{\alpha,\beta}(\phi;z,s,a)=\sum_{n=0}^{\infty}\frac{\phi(n)\,z^n}{(a+n)^s\,\Gamma(\alpha n+\beta)} \quad \text{where}\left(\alpha,\beta\in C;\mathrm{Re}(\alpha)>0\right) \qquad \ldots(1.15)$$

For $\phi(n)=\frac{(\mu)_n}{n!}$ the following function recently studied by Virendra [11, p.13, Eq.(1.9)]:

$$\in_{\alpha,\beta,\delta}^{\mu}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_n}{(\delta n+a)^s\,\Gamma(\alpha n+\beta)}\frac{z^n}{n!} \qquad \ldots(1.16)$$

Where $\mathrm{Re}(\mu)>0;\mathrm{Re}(\delta)>0;\mathrm{Re}(a)>0;\mathrm{Re}(s)\geq 0;\mathrm{Re}(\alpha)>0;\mathrm{Re}(\beta)>0$ and $|z|\leq 1$

To give more extension in view of the parameters, we consider here the following more generalized function defined and represented as below.

$$\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{\nu n}\,z^n}{(\sigma)_{\xi n}(a+n)^s\,\Gamma(\alpha n+\beta)\,n!} \qquad \ldots(1.17)$$

Where $(\alpha,\beta,\mu,\lambda\in C;a,\sigma\in C/Z_0^-;\delta,\nu,\xi\in R^+;\xi-\delta-\nu>-1$ when $s,z\in C;\xi-\delta-\nu=-1$ and $s\in C$ when $|z|<\rho^*=\delta^{-\delta}\nu^{-\nu}\xi^{\xi};\ \xi-\delta-\nu=-1$ and $\mathrm{Re}(s+\sigma-\mu-\lambda)>1$ when $|z|=\rho^*)$

The integral representation of function defined in (1.17) is also obtained here and given as the following theorem.

Theorem-1.1

For $\min\{\mathrm{Re}(a),\mathrm{Re}(s)\}>0;\alpha,\beta,\mu,\lambda\in C;a,\sigma\in C/Z_0^-;\delta,\nu,\xi\in R^+;\xi-\delta-\nu\geq -1$ with $|z|<\delta^*=\delta^{-\delta}\nu^{-\nu}\xi^{\xi}$; integral representation of the function defined in (1.17) $\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a)$ holds true

$$\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a)=\frac{1}{\Gamma s}\int_0^{\infty}e^{-at}t^{s-1}\left\{E_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}\left(ze^{-t}\right)\right\}dt \qquad \ldots(1.18)$$

Where $E_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z)$ is the Mittage-Leffler function defined in (1.12)

Proof: The function defined in (1.17) can be written straight in the following form

$$\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}\left(z,s,a\right)=\frac{1}{\Gamma s}\sum_{n=0}^{\infty}\frac{\left(\mu\right)_{\delta n}\left(\lambda\right)_{\nu n}}{\left(\sigma\right)_{\xi n}n!\Gamma\left(\alpha n+\beta\right)}\left\{\frac{\Gamma s}{\left(a+n\right)^{s}}\right\}$$

Now using the following elementary Gamma integral

$$\frac{\Gamma s}{\left(a+n\right)^{s}}=\int_{0}^{\infty}t^{s-1}e^{-(a+n)t}dt \qquad \left(\min\left\{\mathrm{Re}\left(s\right),\mathrm{Re}\left(a\right)\right\}>0;\mathrm{n}\in\mathrm{N}_{0}\right) \qquad \ldots(1.19)$$

and then changing the order of integration and summation, we have

$$\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}\left(z,s,a\right)=\frac{1}{\Gamma s}\int_{0}^{\infty}e^{-at}t^{s-1}\left\{\sum_{n=0}^{\infty}\frac{\left(\mu\right)_{\delta n}\left(\lambda\right)_{\nu n}}{\left(\sigma\right)_{\xi n}\Gamma\left(\alpha n+\beta\right)}\frac{\left(ze^{-t}\right)^{n}}{n!}\right\}dt$$

On interpreting the inner series in view of (1.10) we at once arrive at the desired result it completes the proof of theorem (1.1).

Throughout the paper $\left\{\begin{matrix}m+l-1\\ m\end{matrix}\right\}=\frac{\left(m\right)_{l}}{l!}$ where $\left(m\right)_{l}$ is the pochhammer's notation

2. Generating Functions

The four generating functions involving the general function define in (1.17) are establish here.

Theorem-2.1

For $\min\left\{\mathrm{Re}\left(a\right),\mathrm{Re}\left(s\right)\right\}>0;\alpha,\beta,\mu,\lambda\in C;a,\sigma\in C/Z_{0}^{-};\delta,\nu,\xi\in R^{+};\xi-\delta-\nu\geq-1$ with $\left|z\right|<\delta^{*}=\delta^{-\delta}\nu^{-\nu}\xi^{\xi}$ we have

$$\sum_{l=0}^{\infty}\binom{s+l-1}{l}\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}\left(z,s+l,a\right)t^{l}=\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}\left(z,s,a-t\right) \qquad \ldots(2.1)$$

Theorem-2.2

For $\min\left\{\mathrm{Re}\left(a\right),\mathrm{Re}\left(s\right)\right\}>0;\alpha,\beta,\mu,\lambda\in C;a,\sigma\in C/Z_{0}^{-};\delta,\nu,\xi\in R^{+};\xi-\delta-\nu\geq-1$ with $\left|z\right|<\delta^{*}=\delta^{-\delta}\nu^{-\nu}\xi^{\xi}$ we have

$$\sum_{l=0}^{\infty}\binom{s+2l-1}{2l}\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}\left(z,s+2l,a\right)t^{2l}=\frac{1}{2}\left[\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}\left(z,s,a+t\right)+\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}\left(z,s,a-t\right)\right] \qquad \ldots(2.2)$$

Theorem-2.3

For $\min\left\{\operatorname{Re}(a),\operatorname{Re}(s)\right\}>0;\alpha,\beta,\mu,\lambda\in C;a,\sigma\in C/Z_0^-;\delta,\nu,\xi\in R^+;\xi-\delta-\nu\geq-1$ with $|z|<\delta^*=\delta^{-\delta}\nu^{-\nu}\xi^{\xi}$ we have

$$\sum_{l=0}^{\infty}\binom{s+2l}{2l+1}\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s+2l+1,a)t^{2l+1}=\frac{1}{2}\left[\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a-t)-\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s,a+t)\right] \quad \text{...(2.3)}$$

Theorem-2.4

For $\operatorname{Re}\left(\sum_{i=1}^{p}\theta_i\right)>\operatorname{Re}\left(\sum_{j=1}^{q}\beta_j\right)>0$; $\left|\frac{t}{a}\right|<1$

$$\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{p}(\theta_i)_l}{\prod_{j=1}^{q}(\beta_j)_l}\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}\left(z,\sum_{i=1}^{p}\theta_i-\sum_{j=1}^{q}\beta_j+l,a\right)\frac{t^l}{l!}$$

$$=\sum\frac{(\mu)_{\delta n}(\lambda)_{\nu n}}{(\sigma)_{\xi n}\Gamma(\alpha n+\beta)}\frac{z^n}{n!}\frac{1}{(a+n)^{\sum_{i=1}^{p}\theta_i-\sum_{j=1}^{q}\beta_j}}\,{}_pF_q\left[\begin{matrix}(\theta_p);\\(\beta_q);\end{matrix}\frac{t}{a+n}\right] \quad \text{...(2.4)}$$

Where ${}_pF_q$ is a generalized hypergeometric function defined by

$${}_pF_q\left[\begin{matrix}(a_p);\\(b_q);\end{matrix}\ z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{z^n}{n!};\qquad(|z|<1) \quad \text{...(2.5)}$$

Outlines Of Proofs

Proof of Theorem 2.1:

Let left hand side of (2.1) is denoted by Δ_1 i.e.

$$\Delta_1=\sum_{l=0}^{\infty}\binom{s+l-1}{l}\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu}(z,s+l,a)t^l$$

On using the definition (1.17) and changing the order of summation we have

$$\Delta_1=\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{\nu n}z^n}{(\sigma)_{\xi n}(a+n)^s\Gamma(\alpha n+\beta)n!}\left\{\sum_{l=0}^{\infty}\frac{(s)_l}{l!}\left(\frac{t}{a+n}\right)^l\right\}$$

Now on using the binomial expansion $(1-x)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} x^n$, we have

$$\Delta_1 = \sum_{n=0}^{\infty} \frac{(\mu)_{\delta n} (\lambda)_{\nu n}}{(\sigma)_{\xi n}\, n!\Gamma(\alpha n+\beta)} \frac{1}{(a-t+n)^s}$$

Now on interpreting the resulting series in view of (1.17) arrive at the desired result in (2.1)

Proof of Theorem 2.2:

Let left hand side of (2.2) is denoted by Δ_2 i.e

$$\Delta_2 = \sum_{l=0}^{\infty} \binom{s+2l-1}{2l} \in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu} (z, s+2l, a) t^{2l}$$

On using the definition (1.17) and changing the order of summation we have

$$\Delta_2 = \sum_{n=0}^{\infty} \frac{(\mu)_{\delta n} (\lambda)_{\nu n} z^n}{(\sigma)_{\xi n} (a+n)^s \Gamma(\alpha+n\beta) n!} \left\{ \sum_{l=0}^{\infty} \frac{(s)_{2l}}{(2l)!} \left(\frac{t}{a+n} \right)^{2l} \right\}$$

Now in view of the binomial expansion

$$\sum_{n=0}^{\infty} \frac{(a)_{2n}}{(2n)!} x^{2n} = \frac{1}{2}\left[(1-x)^{-a} + (1+x)^{-a} \right]$$

It takes the following form

$$\Delta_2 = \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{(\mu)_{\delta n} (\lambda)_{\nu n} z^n}{(\sigma)_{\xi n}\, n!\Gamma(\alpha n+\beta)(a-t+n)^s} + \sum_{n=0}^{\infty} \frac{(\mu)_{\delta n} (\lambda)_{\nu n} z^n}{(\sigma)_{\xi n}\, n!\Gamma(\alpha n+\beta)(a+t+n)^s} \right]$$

On interpreting the resulting series in view of definition (1.17), we at once arrive at the desired result in (2.2)

Proof of Theorem 2.3:

Let left hand side of (2.3) is denoted by Δ_3 i.e

$$\Delta_3 = \sum_{l=0}^{\infty} \binom{s+2l}{2l+1} \in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,\nu} (z, s+2l+1, a) t^{2l+1}$$

On using the definition (1.17) and changing the order of summation we have

$$\Delta_3 = \sum_{n=0}^{\infty} \frac{(\mu)_{\delta n} (\lambda)_{\nu n} z^n}{(\sigma)_{\xi n}\, n!\Gamma(\alpha n+\beta)(a+n)^s} \left\{ \sum_{l=0}^{\infty} \frac{(s)_{2l+1}}{(2l+1)!} \left(\frac{t}{a+n} \right)^{2l+1} \right\}$$

Now in view of the binomial expansion

$$\sum_{n=0}^{\infty}\frac{(a)_{2n+1}}{(2n+1)!}x^{2n+1}=\frac{1}{2}\left[(1-x)^{-a}-(1+x)^{-a}\right]$$

It takes the following form

$$\Delta_3=\frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{vn}\,z^n}{(\sigma)_{\xi n}\,\Gamma(\alpha n+\beta)(a-t+n)^s\,n!}-\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{vn}\,z^n}{(\sigma)_{\xi n}\,\Gamma(\alpha n+\beta)(a+t+n)^s\,n!}\right]$$

On interpreting the inner series in view of definition (1.17), we at once arrive at the desired result in (2.3)

Proof of Theorem 2.4:

Let left hand side of (2.4) is denoted by Δ_4 i.e,

$$\Delta_4=\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{p}(\theta_i)_l}{\prod_{j=1}^{q}(\beta_j)_l}\in_{\alpha,\beta,\sigma,\xi}^{\mu,\delta,\lambda,v}\left(z,\sum_{i=1}^{p}\theta_i-\sum_{j=1}^{q}\beta_j+l,a\right)\frac{t^l}{l!}$$

On using the definition (1.17) and changing the order of summation we have

$$=\sum_{n=0}^{\infty}\frac{(\mu)_{\delta n}(\lambda)_{vn}\,z^n}{(\sigma)_{\xi n}\,n!\,\Gamma(\alpha n+\beta)(a+n)^{\sum_{i=1}^{p}\theta_i-\sum_{j=1}^{q}\beta_j}}\left\{\sum_{l=0}^{\infty}\frac{\prod_{i=1}^{p}(\theta_i)_l}{\prod_{j=1}^{q}(\beta_j)_l}\left(\frac{t}{a+n}\right)^l\frac{1}{l!}\right\}$$

Now interpreting the inner series into generalized hypergeometric function ${}_pF_q$ in view of (2.5) at once arrive at the desired result in (2.4)

3. Mild Extension And Associated Generating Functions

A mild extension is also considered here, it is defined and represented in the following manner:

$$\in_{\alpha,\beta,(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,s,a)=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\mu_i)_{\delta_i n}}{\prod_{j=1}^{q}(\sigma_j)_{\rho_j n}(a+n)^s\,\Gamma(\alpha n+\beta)}\frac{z^n}{n!}\qquad\ldots(3.1)$$

Where $(p,q\in N_0;\ s,z,\mu_i,\alpha,\beta\in C\ (i=1,2,.......\text{p});\ a,\sigma_j\in C/Z_0^-\ (j=1,2,.......\text{q});\ \delta_i,\rho_j\in R^+$
$(i=1,2,.......\text{p};\ j=1,2,.......\text{q}))$

The integral representation for the above function defined in (3.1) is given in the following theorem.

Theorem 3.1:

For $p,q\in N_0; s,z,\mu_i,\alpha,\beta\in C\,(i=1,2,.......\text{p});\ a,\sigma_j\in C/Z_0^-\ (j=1,2,.......\text{q}); \delta_i,\rho_j\in R^+$ $(i=1,2,.......\text{p}; j=1,2,.......\text{q})$

$$\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,s,a)=\frac{1}{\Gamma s}\int_0^\infty e^{-at}t^{s-1}E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(ze^{-t}\right)dt \qquad \ldots(3.2)$$

Where $E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z)$ is the Mittage-Leffler function defined as follows:

$$E_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z)=\sum_{n=0}^\infty\frac{\prod_{i=1}^p(\mu_i)_{\delta_i n}}{\prod_{j=1}^q(\sigma_j)_{\rho_j n}\Gamma(\alpha n+\beta)}\frac{z^n}{n!} \qquad \ldots(3.3)$$

Proof: Theorem 3.1 can be easily proved following the similar lines as to prove Theorem 1.1

The four generating functions involving general function defined in (3.1) are establish here and given in the following theorem (3.2) to (3.5).

Theorem 3.2:

For $p,q\in N_0; s,z,\mu_i,\alpha,\beta\in C\,(i=1,2,.......\text{p}); a,\sigma_j\in C/Z_0^-\ (j=1,2,.......\text{q}); \delta_i,\rho_j\in R^+$ $(i=1,2,.......\text{p}; j=1,2,.......\text{q})$, we have

$$\sum_{l=0}^\infty\binom{s+l-1}{l}\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,\text{s}+1,a)t^l=\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,\text{s},a-\text{t}) \qquad \ldots(3.4)$$

Theorem 3.3:

For $p,q\in N_0; s,z,\mu_i,\alpha,\beta\in C\,(i=1,2,.......\text{p}); a,\sigma_j\in C/Z_0^-\ (j=1,2,.......\text{q}); \delta_i,\rho_j\in R^+$ $(i=1,2,.......\text{p}; j=1,2,.......\text{q})$, we have

$$\sum_{l=0}^\infty\binom{s+2l-1}{2l}\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,\text{s}+21,a)t^{2l}=\frac{1}{2}\left[\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,\text{s},a+\text{t})+\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}(z,\text{s},a-\text{t})\right]\ldots(3.5)$$

Theorem 3.4:

For $p,q\in N_0;s,z,\mu_i,\alpha,\beta\in C\left(i=1,2,.......\text{p}\right);a,\sigma_j\in C/Z_0^-\ \left(j=1,2,.......\text{q}\right);\delta_i,\rho_j\in R^+$
$\left(i=1,2,.......\text{p};j=1,2,.......\text{q}\right)$

$$\sum_{l=0}^{\infty}\binom{s+2l}{2l+1}\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z,\text{s}+2\text{l}+1,a\right)t^{2l+1}=\frac{1}{2}\left[\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z,\text{s},a-\text{t}\right)-\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z,\text{s},a+\text{t}\right)\right]\ldots(3.6)$$

Theorem 3.5:

For $p,q\in N_0;s,z,\mu_i,\alpha,\beta\in C\left(i=1,2,.......\text{p}\right);a,\sigma_j\in C/Z_0^-\ \left(j=1,2,.......\text{q}\right);\delta_i,\rho_j\in R^+$

$\left(i=1,2,.......\text{p};j=1,2,.......\text{q}\right);\text{Re}\left(\sum_{u=1}^{r}\theta_u\right)>\text{Re}\left(\sum_{v=1}^{k}\lambda_v\right)>0;\left|\frac{t}{a}\right|<1$ where ${}_pF_q$ in the generalized hypergeometric function defined in (2.5)

$$\sum_{l=0}^{\infty}\frac{\prod_{u=1}^{r}(\theta_u)_l}{\prod_{v=1}^{k}(\lambda_v)_l}\in_{\alpha,\beta;(\sigma_q),(\rho_q)}^{(\mu_p),(\delta_p)}\left(z,\sum_{u=1}^{r}\theta_u-\sum_{v=1}^{k}\lambda_v+l,a\right)\frac{t^l}{l!}$$

$$=\sum_{i=0}^{\infty}\frac{\prod_{i=1}^{p}(\mu_i)_{\delta_i n}}{\prod_{j=1}^{q}(\sigma_j)_{\rho_j n}\Gamma(\alpha n+\beta)}\frac{z^n}{n!}\frac{1}{(a+n)^{\sum_{u=1}^{r}\theta_u-\sum_{v=1}^{k}\lambda_v}}\,{}_uF_v\left[\begin{matrix}(\theta_u);\ \frac{t}{}\\(\lambda_v);\ a+n\end{matrix}\right]\qquad\ldots(3.7)$$

The proof of Theorem (3.2),(3.3),(3.4),(3.5) are developed following the similar lines as to proof of (2.1),(2.2),(2.3) and(2.4) respectively in view of the definition of general function defined in (3.1)

4. Applications

The functions studied in this paper are very general in nature and unify all the functions related to Hurwitz-Lerch Zeta function, therefore many known and New results can be obtained from our main results. As an application, some of these have been shown to derive as follows.

(i) If in the result (1.17), we consider $Lim\ \alpha\to 0$, then it reduces to the known result due to Srivastava et al [5, p. 494,eq. (2.4)]
(ii) If in result (2.1) to (2.4), we apply $Lim\ \alpha\to 0$, then these results reduce to the known result studied by Gupta [12, pp. 132-133, eq. (2.9.1) to (2.9.4)]respectively.
(iii) If in result (3.2) we take $Lim\ \alpha\to 0$, then it reduces to the integral representation for Hurwitz-Lerch Zeta function studied by Srivastava et al [5, pp.504, eq. (6.4)]
(iv) If we take $\alpha\to 0$, the result (3.4) to (3.7) reduce to known results for Hurwitz-Zeta function due to Gupta [12, pp. 134-135, eq. (2.9.7) to (2.9.10)]
(v) If we take $\mu=\sigma,\delta=\xi,v=1$ in (2.1), it reduces to the known generating function due to Virendra [11, p. 17, eq. (3.1)] at $\delta=1$.

(vi) For $p = 2, q = 1, \mu = \sigma, v = 1, \delta = \xi$ the theorem 2.4 provides the known results due to Virendra [11, p. 17, Theorem (3.2)].

5. Conclusion

The general function and its Mild generalization studied in this paper unify all the function related to Hurwitz-Lerch Zeta function, its extension as well as the Mittag-Leffler function and their extension scattered in the literature. Many properties involving these general functions like The derivatives, the fractional calculus and fractional order differential equations and more results related to finite and infinite integrals can be established. The present study is worthwhile as the properties of Mittage-Leffler function and their generalizations are being study [18,19,20]. The Mittag-Leffler function is widely used to solve differential equation of fractional order [21,22]. So our proposed study will be very useful to the young researchers, who are engaged in the field of study.

References

[1] Erd´elyi A., Magnus W., Oberhettinger F. and Tricomi F. G., Higher Transcendental Functions, Vol.I, McGraw-Hill Book Company, New York, Toronto and London, (1953).

[2] Lin S.-D. and Srivastava H.M., Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations, Appl. Math. Comput. 154 (2004),pp. 725–733.

[3] Goyal S.P. and Laddha R.K., On the generalized Zeta function and the generalized Lambert function, Ganita Sandesh 11 (1997), pp. 99–108.

[4] Garg, M., JainK., and KallaS.L., A further study of general Hurwitz-Lerch zeta function, Algebras Groups Geom. 25 (2008), pp. 311–319.

[5] Srivastava H. M. and Pogany T.K. Integral and computational representations of the extended Hurwitz-Lerch Zeta function, Integral Transforms and Special Functions Vol. 22, No. 7,(2011) ,pp. 487–506

[6] Khan and Ahmed, On some properties of the generalized Mittag-Leffler function., Springer Plus.2013, 2: 337.

[7] Wright E. M., The asymptotic expansion of integral functions defined by Taylor series. I, Philos. Trans. Roy. Soc. London Ser. A Math. Phys. Sci. 238 (1940),pp. 423–451.

[8] Srivastava H. M., Some families of hybrid -type fractional order Kinetic equations based upon the hilfer type and other related operators of fractional derivatives, J. of non-linear and Convex Anal, volume 25, (2024) ,pp. 2647- 2669.

[9] Srivastava H.M., An introductory overview of fractional-calculus operators based upon the Fox- Wright and related higher transcendental functions, J. Adv. Engineering and computation, 5(3), (2021), pp. 135-166.

[10] Srivastava H.M., Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations, J. Nonlinear Convex Anal., 22(8), (2021), pp. 1501-1520 .

[11] Kumar V., On a Generalized Mittag-Leffler Function and Fractional Integrals Fundamental Journal of Math.andApp.7(1) (2024),pp. 12-25.

[12] Gupta J., Investigation in generalized special functions and the fractional calculus with it's application to univalent and multivalent functions, Ph.D.*thesis, University of Kota, Kota, India, 2011.
[13] Mittag-Leffler GM., Sur la nouvelle fonction E (x).C R Acad. Sci. Paris., 137, (1903), pp. 554-558.
[14] Mittag-Leffler GM., Sur la repr'esentation analytiqued ;une branche uniformed' une function monogène: cinquième note. Acta. Math., 29, (1905), pp.101-181.
[15] Wiman A., ÜBer den fundamental satz in der theorie der funcktionen E (x). Acta Math. 29, (1905),pp. 191-201.
[16] Prabhakar T.R., A singular integral equation with a generalized Mittag-Leffler function in the kernel. Yokohama Math J.,19, (1971), pp.7-15.
[17] Srivastava H.M., On an extension of the Mittag-Leffler function. Yokohama Math J., 16, (1968), pp. 77-88.
[18] Srivastava H.M, TomovskiŽ., Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel. Appl Math Comput., 211, (2009), pp.198-210.
[19] Shukla A.K., Prajapati J.C., On a generalization of Mittag-Leffler function and its properties. J Math Anal. Appl. 336, (2007), pp.797-811.
[20] Tariq O. Salim, Ahmad W. Faraj, A generalization of mittag-leffler function and integral operator associated with fractional calculus, Journal of Fract. Calc. and Appl., Vol. 3. (2012), No. 5, pp. 1 - 13.
[21] .Jaimini. B. B. and Gupta J., On certain fractional differential equations involving generalized multivariable Mittag-Leffler function, Note di Matematica, 32 (2013), pp.141-156.
[22] Jaimini B. B., Sharma M., Suthar D. L. and Purohit S. D., On multi-index Mittag-Leffler function of several variables and fractional differential equations, Journal of Mathematics, (2021), Art. ID 5458037, 8 pp.
[23] Barnes E.W., The asymptotic expansion of integral functions defined by Taylor’s series, Philos.Trans. Roy. Soc. London Ser.A Math. Phys. Sci. 206 (1906), pp.249–297.