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STABILITY OF A AFFINE TYPE AQ FUNCTIONAL EQUATION IN VARIOUS BANACH SPACES

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ABSTRACT. In this article, a new affine type AQ functional equations is proposed. The generalized Ulam-Hyers stability of this equations is analyzed using the product, sum, and mixed product-sum of powers of norms, as well as the general control function. The stability analysis is carried out in Banach space and Intu-itionistic Fuzzy Banach spaces using Hyers direct method. Also, we examine the stability of same functional equation by using Radus Fixed point method in both the spaces.

1. INTRODUCTION

S.M. Ulam's question [31] in 1940 rewoke the journey of the research in the stability theory of functional equations. Many mathematicians have studied and published several novel results in the field of stability theory, such as, D.H. Hyers (1941) [14] , T. Aoki (1950) [2], Th.M. Rassias (1978) [24], J.M. Rassias (1982) [23], P. Gavruta (1994) [13], and K. Ravi, M. Arunkumar, J.M. Rassias (2008) [26].

Famous functional equations for additive and quadratic functions are

$$\mathcal{F}(w_1 + w_2) = \mathcal{F}(w_1) + \mathcal{F}(w_2), \tag{1.1}$$

and

$$\mathcal{F}(w_1 + w_2) + \mathcal{F}(w_1 - w_2) = 2\mathcal{F}(w_1) + 2\mathcal{F}(w_2). \tag{1.2}$$

S.M. Jung [15], PL. Kannappan [16], and Th.M. Rassias [25] discussed the general solution and generalized Ulam - Hyers stability of different forms of functional equations in various normed spaces. In fact, M. Arunkumar et. al., [3], M. Arunkumar, J.M. Rassis [4], M. Arunkumar et. al., [5, 6], A. Bodaghi [8], and references therein establish the general solution and generalized Hyers-Ulam stability of the several AQ functional equations.

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L. Lucht, C. Methfessel [17] proposed affine functional equations and recurrent sequences in 1993. Additionally, in 2013, L. Cadariu, L. Gavruta, and P. Gavruta [11] demonstrated the generalized Hyers-Ulam stability and obtained the general solution for an affine functional equation of the form

$$f(2x+y)+f(x+2y)+f(x)+f(y)=4f(x+y+z) \tag{1.3}$$

by using the direct method as well as the fixed point method. Infact, in 2014, M. Mursaleen, KJ. Ansari [19] considered the following affine functional equation

$$f(3x+y+z)+f(x+3y+z)+f(x+y+3z)+f(x)+f(y)+f(z)=6f(x+y+z) \tag{1.4}$$

and find its general solution and proved some stability results by using direct method as well as the fixed point method. Also in 2015, Md. Nasiruzzaman [21] provide the fuzzy version Hyers-Ulam-Rassias stability of (1.4) . Further, in 2016 M. Mursaleen, KJ. Ansari[20] prove the general solution of the following affine functional equation

$$\begin{aligned} f(kx_1+x_2+\cdots+x_k)+f(x_1+kx_2+\cdots+x_k)+\cdots+f(x_1+x_2+\cdots+kx_k)& \\ +f(x_1)+f(x_2)+\cdots+f(x_k)=2kf(x_1+x_2+\cdots+x_k),k\geq 2.& \end{aligned} \tag{1.5}$$

and established the Hyers-Ulam-Rassias stability of the above functional equation in the fuzzy normed spaces which as an generalized version of (1.4).

Recently, C. Benzarouala et.al., [9, 10] proved the general Ulam stability result for the functional equation

$$\sum_{i=1}^{m} A_i f\left(\sum_{j=1}^{n} a_{ij}x_j\right) = D(x_1,\cdots,x_n), \tag{1.6}$$

in the class of functions f mapping a module X, over a commutative ring $\mathbb{K}$, into a Banach space Y, where m and n are fixed positive integers, $a_{ij}\in\mathbb{K}$ for every $i\in\{1,\cdots,m\}$ and $j\in\{1,\cdots n\}$, $A_1,\cdots,A_m$ are scalars, and the function $D: X^n\to Y$ is fixed.

Numerous important functional equations are particular cases of $A_i's$ the homogeneous version of (1.6) are Cauchy, Jensen, Jordanvon Neumann, Drygas, Frechet, Popoviciu, Wright and many others. Also, for particular cases of $A_i's$ in (1.6), we get functional equations , like equation in a single variable, cohomological equation, Schroder equation, Abel equation and many others. The stability of (1.6) in random normed spaces has been studied by C. Benzarouala et.al., [10].

Inspired by the aforementioned information and study findings, in this paper we present a novel affine type additive quadratic mixed functional equation of the form

$$\begin{aligned} &\mathcal{F}(3w_1+w_2+w_3)+\mathcal{F}(w_1+3w_2+w_3)+\mathcal{F}(w_1+w_2+3w_3)\\ &=6\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)+\frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)+\mathcal{F}\left(-\sum_{\psi=1}^{3}w_\psi\right)\right\}-\sum_{\psi=1}^{3}\left\{\mathcal{F}(w_\psi)-\frac{5}{2}\Big[\mathcal{F}(w_\psi)+\mathcal{F}(-w_\psi)\Big]\right\}. \end{aligned} \tag{1.7}$$

We analyze the stability in the sense of Ulam, Hyers, Rassias's, Gavruta and Radu of the above affine type AQ Functional Equation in Banach Space and Intuitionistic Fuzzy Banach Space using Direct and Fixed Methods.

Remark 1.1. *The homogeneous version of (1.6) for $m=n=3$, $A_1=A_2=A_3=1$, $a_{11}=a_{22}=a_{33}=3$ and $a_{12}=a_{13}=a_{21}=a_{23}=a_{31}=a_{32}=1$ is*

$$f\left(3x_1+x_2+x_3\right)+f\left(x_1+3x_2+x_3\right)+f\left(x_1+x_2+3x_3\right)=0. \tag{1.8}$$

So, we cant get our functional equation (1.7) from (1.6).

Remark 1.2. *In the functional equation (1.6), for $n = 3$, $m = 15$, $D = 0$,*

$$A_1 = A_2 = A_3 = 1, a_{11} = a_{22} = a_{33} = 3, a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 1,$$
$$A_4 = -6, a_{41} = a_{42} = a_{43} = 1, A_5 = A_6 = -\frac{1}{2}, a_{51} = a_{52} = a_{53} = 1, a_{61} = a_{62} = a_{63} = -1$$
$$A_7 = A_8 = A_9 = 1, a_{71} = 1, a_{72} = a_{73} = 0, a_{81} = a_{83} = 0, a_{82} = 1, a_{91} = a_{92} = 0, a_{93} = 1,$$
$$A_{10} = A_{11} = A_{12} = A_{13} = A_{14} = A_{15} = -\frac{5}{2},$$
$$a_{10\,1} = 1, a_{10\,2} = a_{10\,3} = 0, a_{11\,1} = a_{11\,3} = 0, a_{11\,2} = 1, a_{12\,1} = a_{12\,2} = 0, a_{12\,3} = 1,$$
$$a_{13\,1} = -1, a_{13\,2} = a_{13\,3} = 0, a_{14\,1} = a_{14\,3} = 0, a_{14\,2} = -1, a_{15\,1} = a_{15\,2} = 0, a_{15\,3} = -1.$$

So, after giving particular values to $A_i's$ and $a_i's$, we get our functional equation (1.7) from (1.6).

Moreover, the results in the manuscript under review complement of the results in [9, 10].

Lemma 1.3. [21] *Let A and B be real vector spaces. Suppose $\mathcal{F} : A \to B$ be an odd mapping satisfying (1.7). Then $\mathcal{F}$ is additive.*

Lemma 1.4. [12] *Let A and B be real vector spaces. Suppose $\mathcal{F} : A \to B$ be an even mapping satisfying (1.7). Then $\mathcal{F}$ is quadratic.*

Now, we present the result due to Margolis, Diaz [18] and Radu [22] for fixed point theory.

Theorem 1.5. [18, 22] *Suppose that for a complete generalized metric space (Ω, δ) and a strictly contractive mapping $T : \Omega \longrightarrow \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1} x) = \infty \; \forall \; n \geq 0,$$

or there exists a natural number n_0 such that
(FPC1) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;
(FPC2) The sequence $(T^n x)$ is convergent to a fixed point y^* of T
(FPC3) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0} x, y) < \infty\}$;
(FPC4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

2. Stability of (1.7) In Banach Spaces

In this section, we explore the generalized Ulam - Hyers stability of the functional equation (1.7) in Banach space. To prove the stability results, let us take $\mathcal{W}_1$ be a normed space and $\mathcal{W}_2$ be a Banach space. Suppose that $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ and $\Psi : \mathcal{W}_1^3 \to [0, \infty)$ satisfying the following functional inequalities

$$\Bigg\| \mathcal{F}(3w_1 + w_2 + w_3) + \mathcal{F}(w_1 + 3w_2 + w_3) + \mathcal{F}(w_1 + w_2 + 3w_3) - 6\mathcal{F}\left(\sum_{\psi=1}^{3} w_\psi\right) - \frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3} w_\psi\right) + \mathcal{F}\left(-\sum_{\psi=1}^{3} w_\psi\right)\right\} + \sum_{\psi=1}^{3}\left\{\mathcal{F}(w_\psi) - \frac{5}{2}\Big[\mathcal{F}(w_\psi) + \mathcal{F}(-w_\psi)\Big]\right\} \Bigg\| \leq \Psi(w_1, w_2, w_3), \tag{2.1}$$

and

$$\left\| \mathcal{F}(3w_1 + w_2 + w_3) + \mathcal{F}(w_1 + 3w_2 + w_3) + \mathcal{F}(w_1 + w_2 + 3w_3) - 6\mathcal{F}\left(\sum_{\psi=1}^{3} w_\psi\right) - \frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3} w_\psi\right) + \mathcal{F}\left(-\sum_{\psi=1}^{3} w_\psi\right)\right\} + \sum_{\psi=1}^{3}\left\{\mathcal{F}(w_\psi) - \frac{5}{2}\Big[\mathcal{F}(w_\psi) + \mathcal{F}(-w_\psi)\Big]\right\} \right\|$$

$$\leq \begin{cases} \delta, \\ \delta \sum_{\psi=1}^{3} |w_\psi|^\varphi, \\ \delta \sum_{\psi=1}^{3} |w_\psi|^{\varphi_\psi}, \\ \delta \prod_{\psi=1}^{3} |w_\psi|^\varphi, \\ \delta \prod_{\psi=1}^{3} |w_\psi|^{\varphi_\psi}, \\ \delta \left\{ \sum_{\psi=1}^{3} |w_\psi|^{3\varphi} + \prod_{\psi=1}^{3} |w_\psi|^\varphi \right\}, \end{cases} \tag{2.2}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$, δ be a positive constant and φ be any real number.

2.1. Oddness of $\mathcal{F}$: Additive Case Stability Results : Direct Method.

Theorem 2.1. *Suppose that an odd function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (2.1) where $\Psi : \mathcal{W}_1^3 \to [0, \infty)$ with the condition*

$$\lim_{\ell\to\infty} \frac{\Psi\left(5^{\ell m}w_1, 5^{\ell m}w_2, 5^{\ell m}w_3\right)}{5^{\ell m}} = 0; \quad \mu = \pm 1, \tag{2.3}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality

$$\|\mathcal{F}(w_1) - \mathcal{A}(w_1)\| \leq \frac{1}{5} \sum_{\eta=\frac{1-\mu}{2}}^{\infty} \frac{1}{5^{\eta\mu}} \Psi_A\left(5^{\eta\mu}w_1\right) \tag{2.4}$$

$$= \frac{1}{5} \sum_{\eta=\frac{1-\mu}{2}}^{\infty} \frac{1}{5^{\eta\mu}} \left\{ \frac{1}{3}\Big\{\Psi\left(5^{\eta\mu}w_1, 5^{\eta\mu}w_1, 5^{\eta\mu}w_1\right) + 3\Psi\left(5^{\eta\mu}w_1, 5^{\eta\mu}w_1, -5^{\eta\mu}w_1\right)\Big\}\right\}, \tag{2.5}$$

and the mapping $\mathcal{A}(w_1)$ is obtained by

$$\mathcal{A}(w_1) = \lim_{\ell\to\infty} \frac{1}{5^{\ell m}} \mathcal{F}\left(5^{\ell m}w_1\right), \tag{2.6}$$

for all $w_1 \in \mathcal{W}_1$.

Proof. Using oddness of $\mathcal{F}$ in (2.1), we get

$$\left\| \mathcal{F}(3w_1 + w_2 + w_3) + \mathcal{F}(w_1 + 3w_2 + w_3) + \mathcal{F}(w_1 + w_2 + 3w_3) - 6\mathcal{F}\left(\sum_{\psi=1}^{3} w_\psi\right) + \sum_{\psi=1}^{3} \mathcal{F}(w_\psi) \right\| \leq \Psi(w_1, w_2, w_3), \forall\, w_1, w_2, w_3 \in \mathcal{W}_1. \tag{2.7}$$

Interchanging (w_1, w_2, w_3) by (w_1, w_1, w_1) in (2.7), we obtain

$$\left\| 3\mathcal{F}(5w_1) - 6\mathcal{F}(3w_1) + 3\mathcal{F}(w_1) \right\| \leq \Psi(w_1, w_1, w_1), \forall\, w_1 \in \mathcal{W}_1. \tag{2.8}$$

Again interchanging (w_1, w_2, w_3) by $(w_1, w_1, -w_1)$ in (2.7), we have

$$\left\|2\mathcal{F}(3w_1) - 6\mathcal{F}(w_1)\right\| \leq \Psi(w_1, w_1, -w_1)$$
$$\Rightarrow \left\|6\mathcal{F}(3w_1) - 18\mathcal{F}(w_1)\right\| \leq 3\Psi(w_1, w_1, -w_1), \forall\, w_1 \in \mathcal{W}_1. \tag{2.9}$$

Combining (2.8) and (2.9), we arrive

$$\begin{aligned}\left\|3\mathcal{F}(5w_1) - 15\mathcal{F}(w_1)\right\| &\leq \left\|3\mathcal{F}(5w_1) - 6\mathcal{F}(3w_1) + 3\mathcal{F}(w_1)\right\| + \left\|6\mathcal{F}(3w_1) - 18\mathcal{F}(w_1)\right\| \\ &\leq \Psi(w_1, w_1, w_1) + 3\Psi(w_1, w_1, -w_1), \forall\, w_1 \in \mathcal{W}_1.\end{aligned} \tag{2.10}$$

One can see from (2.10) that

$$\left\|\mathcal{F}(5w_1) - 5\mathcal{F}(w_1)\right\| \leq \frac{1}{3}\Big\{\Psi(w_1, w_1, w_1) + 3\Psi(w_1, w_1, -w_1)\Big\} = \Psi_A(w_1), \forall\, w_1 \in \mathcal{W}_1. \tag{2.11}$$

It follows from (2.11) that

$$\left\|\frac{1}{5}\mathcal{F}(5w_1) - \mathcal{F}(w_1)\right\| \leq \frac{1}{5}\Psi_A(w_1), \forall\, w_1 \in \mathcal{W}_1. \tag{2.12}$$

Generalizing for a positive integer ℓ, we get

$$\left\|\frac{1}{5^\ell}\mathcal{F}(5^\ell w_1) - \mathcal{F}(w_1)\right\| \leq \frac{1}{5}\sum_{\eta=0}^{\ell}\frac{1}{5^\eta}\,\Psi_A(5^\eta w_1), \forall\, w_1 \in \mathcal{W}_1. \tag{2.13}$$

Now, changing w_1 by $5^{\ell_1}w_1$ in (2.13), we obtain

$$\begin{aligned}\left\|\frac{1}{5^{\ell+\ell_1}}\mathcal{F}(5^{\ell+\ell_1}w_1) - \frac{1}{5^{\ell_1}}\mathcal{F}(5^{\ell_1}w_1)\right\| &= \frac{1}{5^{\ell_1}}\left\|\frac{1}{5^\ell}\mathcal{F}(5^{\ell+\ell_1}w_1) - \mathcal{F}(5^{\ell_1}w_1)\right\| \\ &\leq \frac{1}{5}\sum_{\eta=0}^{\ell}\frac{1}{5^{\eta+\ell_1}}\,\Psi_A\left(5^{\eta+\ell_1}w_1\right) \\ &\rightarrow\ 0\ as\ \ell_1\ \rightarrow\ \infty, \forall\, w_1 \in \mathcal{W}_1.\end{aligned} \tag{2.14}$$

Therefore, the sequence

$$\left\{\frac{1}{5^\ell}\mathcal{F}(5^\ell w_1)\right\},$$

is a Cauchy sequence and it converges to $\mathcal{A}(w_1)$ in $\mathcal{W}_2$. So, we define

$$\mathcal{A}(w_1) = \lim_{\ell\to\infty}\frac{1}{5^\ell}\mathcal{F}\left(5^\ell w_1\right), \forall\, w_1 \in \mathcal{W}_1. \tag{2.15}$$

Taking limit $\ell \to \infty$ in (2.13), we have

$$\left\|\mathcal{A}(w_1) - \mathcal{F}(w_1)\right\| \leq \frac{1}{5}\sum_{\eta=0}^{\infty}\frac{1}{5^\eta}\,\Psi_A(5^\eta w_1), \forall\, w_1 \in \mathcal{W}_1. \tag{2.16}$$

Thus, (2.4) and (2.5) holds for $\mu = 1$. Interchanging

$$(w_1, w_2, w_3) = \left(5^\ell w_1, 5^\ell w_2, 5^\ell w_3\right),$$

we arrive

$$\begin{aligned}\frac{1}{5^\ell}\,\Bigg\|\mathcal{F}(5^\ell(3w_1 + w_2 + w_3)) + \mathcal{F}(5^\ell(w_1 + 3w_2 + w_3)) + \mathcal{F}(5^\ell(w_1 + w_2 + 3w_3)) - 6\mathcal{F}\left(\sum_{\psi=1}^{3}5^\ell w_\psi\right) \\ - \frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3}5^\ell w_\psi\right) + \mathcal{F}\left(-\sum_{\psi=1}^{3}5^\ell w_\psi\right)\right\} + \sum_{\psi=1}^{3}\left\{\mathcal{F}(5^\ell w_\psi) - \frac{5}{2}\Big[\mathcal{F}(5^\ell w_\psi) + \mathcal{F}(-5^\ell w_\psi)\Big]\right\}\Bigg\| \\ \leq \frac{1}{5^\ell}\Psi\left(5^\ell w_1, 5^\ell w_2, 5^\ell w_3\right), \forall w_1, w_2, w_3 \in \mathcal{W}_1.\end{aligned} \tag{2.17}$$

Taking limit $\ell \to \infty$ in (2.17), using (2.15) and (2.3), we get

$$\begin{aligned}&\mathcal{A}(3w_1+w_2+w_3)+\mathcal{A}(w_1+3w_2+w_3)+\mathcal{A}(w_1+w_2+3w_3)\\&=6\mathcal{A}\left(\sum_{\psi=1}^{3} w_\psi\right)+\frac{1}{2}\left\{\mathcal{A}\left(\sum_{\psi=1}^{3} w_\psi\right)+\mathcal{A}\left(-\sum_{\psi=1}^{3} w_\psi\right)\right\}-\sum_{\psi=1}^{3}\left\{\mathcal{A}(w_\psi)-\frac{5}{2}\Big[\mathcal{A}(w_\psi)+\mathcal{A}(-w_\psi)\Big]\right\},\end{aligned}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$. So, $\mathcal{A}(w_1)$ satisfies (1.7). In order to confirm that $\mathcal{A}(w_1)$ is unique, suppose $\mathcal{B}(w_1)$ be another mapping (1.7), (2.15) and (2.16), we obtain

$$\begin{aligned}\Big\|\mathcal{A}(w_1)-\mathcal{B}(w_1)\Big\| &= \Big\|\frac{1}{5^\ell}\mathcal{A}\left(5^\ell w_1\right)-\frac{1}{5^\ell}\mathcal{B}\left(5^\ell w_1\right)\Big\|\\&\le \frac{1}{5^\ell}\Big\|\mathcal{A}\left(5^\ell w_1\right)-\mathcal{F}\left(5^\ell w_1\right)\Big\|+\frac{1}{5^\ell}\Big\|\mathcal{F}\left(5^\ell w_1\right)-\mathcal{B}\left(5^\ell w_1\right)\Big\|\\&\le \frac{2}{5}\sum_{\eta=0}^{\infty}\frac{1}{5^{\eta+\ell}}\,\Psi_A\left(5^{\eta+\ell}w_1\right)\to\ 0\ as\ \ell_1\ \to\ \infty,\end{aligned}$$

for all $w_1 \in \mathcal{W}_1$. Therefore $\mathcal{A}(w_1)$ is unique. So, the Theorem holds for $\mu = 1$.
Changing $w_1 = \frac{w_1}{5}$ in (2.11), we have

$$\Big\|\mathcal{F}(w_1)-5\mathcal{F}\left(\frac{w_1}{5}\right)\Big\| \le \Psi_A\left(\frac{w_1}{5}\right), \forall\, w_1 \in \mathcal{W}_1. \tag{2.18}$$

Generalizing for a positive integer ℓ, we get

$$\Big\|\mathcal{F}(w_1)-5^\ell\mathcal{F}\left(\frac{w_1}{5^\eta}\right)\Big\| \le \frac{1}{5}\sum_{\eta=1}^{\ell}5^\eta\,\Psi_A\left(\frac{w_1}{5^\eta}\right), \forall\, w_1 \in \mathcal{W}_1. \tag{2.19}$$

The rest of the proof is similar to that of above case. So, the Theorem holds for $\mu = -1$. Hence the proof is complete □

Corollary 2.2. *Suppose that an odd function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (2.2) for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\|\mathcal{F}(w_1)-\mathcal{A}(w_1)\| \le \begin{cases}\frac{\delta}{|3|}, & \\ \frac{4\delta|w_1|^\varphi}{|5-5^\varphi|}; & \varphi \neq 1,\\ \frac{4\delta}{3}\sum_{\psi=1}^{3}\frac{|w_\psi|^{\varphi_\psi}}{|5-5^{\varphi_\psi}|}; & \varphi_1,\varphi_2,\varphi_3 \neq 1,\\ \frac{4\delta|w_1|^{3\varphi}}{3|5-5^{3\varphi}|}; & 3\varphi \neq 1,\\ \frac{4\delta|w_\psi|^{\sum_{\psi=1}^{3}\varphi_\psi}}{3\left|5-5^{\sum_{\psi=1}^{3}\varphi_\psi}\right|}; & \sum_{\psi=1}^{3}\varphi_\psi \neq 1,\\ \frac{16\delta|w_1|^{3\varphi}}{3|5-5^{3\varphi}|}; & 3\varphi \neq 1,\end{cases} \tag{2.20}$$

for all $w_1 \in \mathcal{W}_1$.

2.2. Evenness of $\mathcal{F}$: Quadratic Case Stability Results : Direct Method.

Theorem 2.3. *Suppose that an even function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (2.1) where $\Psi : \mathcal{W}_1^3 \to [0,\infty)$ with the condition*

$$\lim_{\ell\to\infty}\frac{\Psi\left(5^{\ell m}w_1, 5^{\ell m}w_2, 5^{\ell m}w_3\right)}{25^{\ell m}} = 0; \quad \mu = \pm 1, \tag{2.21}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality

$$\|\mathcal{F}(w_1) - \mathcal{Q}(w_1)\| \leq \frac{1}{25} \sum_{\eta=\frac{1-\mu}{2}}^{\infty} \frac{1}{25^{\eta\mu}} \Psi_Q\left(5^{\eta\mu} w_1\right) \tag{2.22}$$

$$= \frac{1}{25} \sum_{\eta=\frac{1-\mu}{2}}^{\infty} \frac{1}{25^{\eta\mu}} \left\{\frac{1}{3}\left\{\Psi\left(5^{\eta\mu} w_1, 5^{\eta\mu} w_1, 5^{\eta\mu} w_1\right) + \frac{7}{2}\Psi\left(5^{\eta\mu} w_1, 5^{\eta\mu} w_1, -5^{\eta\mu} w_1\right)\right\}\right\}, \tag{2.23}$$

and the mapping $\mathcal{Q}(w_1)$ is obtained by

$$\mathcal{Q}(w_1) = \lim_{\ell\to\infty} \frac{1}{25^{\ell m}} \mathcal{F}\left(5^{\ell m} w_1\right), \tag{2.24}$$

for all $w_1 \in \mathcal{W}_1$.

Proof. Using evenness of $\mathcal{F}$ in (2.1), we get

$$\left\|\mathcal{F}(3w_1 + w_2 + w_3) + \mathcal{F}(w_1 + 3w_2 + w_3) + \mathcal{F}(w_1 + w_2 + 3w_3) - 7\mathcal{F}\left(\sum_{\psi=1}^{3} w_\psi\right) - 4\sum_{\psi=1}^{3} \mathcal{F}(w_\psi)\right\| \leq \Psi\left(w_1, w_2, w_3\right), \forall\, w_1, w_2, w_3 \in \mathcal{W}_1. \tag{2.25}$$

Interchanging (w_1, w_2, w_3) by (w_1, w_1, w_1) in (2.25), we obtain

$$\left\|3\mathcal{F}(5w_1) - 7\mathcal{F}\left(3w_1\right) - 12\mathcal{F}(w_1)\right\| \leq \Psi\left(w_1, w_1, w_1\right), \forall\, w_1 \in \mathcal{W}_1. \tag{2.26}$$

Again interchanging (w_1, w_2, w_3) by $(w_1, w_1, -w_1)$ in (2.25), we have

$$\left\|2\mathcal{F}(3w_1) - 18\mathcal{F}\left(w_1\right)\right\| \leq \Psi\left(w_1, w_1, -w_1\right)$$
$$\Rightarrow \left\|7\mathcal{F}(3w_1) - 63\mathcal{F}\left(w_1\right)\right\| \leq \frac{7}{2}\Psi\left(w_1, w_1, -w_1\right), \forall\, w_1 \in \mathcal{W}_1. \tag{2.27}$$

Combining (2.26) and (2.27), we arrive

$$\left\|3\mathcal{F}(5w_1) - 75\mathcal{F}(w_1)\right\| \leq \left\|3\mathcal{F}(5w_1) - 7\mathcal{F}\left(3w_1\right) - 12\mathcal{F}(w_1)\right\| + \left\|7\mathcal{F}(3w_1) - 63\mathcal{F}\left(w_1\right)\right\|$$
$$\leq \Psi\left(w_1, w_1, w_1\right) + \frac{7}{2}\Psi\left(w_1, w_1, -w_1\right), \forall\, w_1 \in \mathcal{W}_1. \tag{2.28}$$

One can see from (2.28) that

$$\left\|\mathcal{F}(5w_1) - 25\mathcal{F}(w_1)\right\| \leq \frac{1}{3}\left\{\Psi\left(w_1, w_1, w_1\right) + \frac{7}{2}\Psi\left(w_1, w_1, -w_1\right)\right\} = \Psi_Q\left(w_1\right), \forall\, w_1 \in \mathcal{W}_1. \tag{2.29}$$

It follows from (2.29) that

$$\left\|\frac{1}{25}\mathcal{F}(5w_1) - \mathcal{F}(w_1)\right\| \leq \frac{1}{25}\Psi_Q\left(w_1\right), \forall\, w_1 \in \mathcal{W}_1. \tag{2.30}$$

The rest of the proof is similar to that of Theorem 2.1. Hence the proof is complete. □

Corollary 2.4. *Suppose that an even function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (2.2) for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and* *the functional inequality*

$$\|\mathcal{F}(w_1) - \mathcal{Q}(w_1)\| \leq \begin{cases} \frac{3\delta}{2|24|}, & \\ \frac{27\delta|w_1|^{\varphi}}{6|25-5^{\varphi}|}; & \varphi \neq 2, \\ \frac{9\delta}{6}\sum_{\psi=1}^{3}\frac{|w_\psi|^{\varphi_\psi}}{|25-5^{\varphi_\psi}|}; & \varphi_1, \varphi_2, \varphi_3 \neq 2, \\ \frac{9\delta|w_1|^{3\varphi}}{6|25-5^{3\varphi}|}; & 3\varphi \neq 2, \\ \frac{9\delta|w_\psi|^{\sum_{\psi=1}^{3}\varphi_\psi}}{6\left|25-5^{\sum_{\psi=1}^{3}\varphi_\psi}\right|}; & \sum_{\psi=1}^{3}\varphi_\psi \neq 2, \\ \frac{36\delta|w_1|^{3\varphi}}{6|25-5^{3\varphi}|}; & 3\varphi \neq 2, \end{cases} \tag{2.31}$$

for all $w_1 \in \mathcal{W}_1$.

2.3. Oddness and Evenness of $\mathcal{F}$: Additive Quadratic Case Stability Results : Direct Method.

Theorem 2.5. *Suppose that a function* $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ *satisfy the functional inequality (2.1) where* $\Psi : \mathcal{W}_1^3 \to [0, \infty)$ *with the conditions (2.3) and (2.21) for all* $w_1, w_2, w_3 \in \mathcal{W}_1$*. Then there exists a unique additive mapping* $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ *and a unique quadratic mapping* $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ *which satisfies (1.7) and the functional inequality*

$$\begin{aligned}
&\|\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1)\| \\
&\leq \frac{1}{2}\left\{\frac{1}{5}\sum_{\eta=\frac{1-\mu}{2}}^{\infty}\frac{1}{5^{\eta\mu}}\left\{\Psi_A\left(5^{\eta\mu}w_1\right) + \Psi_A\left(-5^{\eta\mu}w_1\right)\right\} + \frac{1}{25}\sum_{\eta=\frac{1-\mu}{2}}^{\infty}\frac{1}{25^{\eta\mu}}\left\{\Psi_Q\left(5^{\eta\mu}w_1\right) + \Psi_Q\left(-5^{\eta\mu}w_1\right)\right\}\right\} \\
&\leq \frac{1}{2}\left\{\frac{1}{5}\sum_{\eta=\frac{1-\mu}{2}}^{\infty}\frac{1}{5^{\eta\mu}}\left\{\frac{1}{3}\left\{\Psi\left(5^{\eta\mu}w_1, 5^{\eta\mu}w_1, 5^{\eta\mu}w_1\right) + 3\Psi\left(5^{\eta\mu}w_1, 5^{\eta\mu}w_1, -5^{\eta\mu}w_1\right)\right\}\right.\right. \\
&\qquad\left. + \frac{1}{3}\left\{\Psi\left(-5^{\eta\mu}w_1, -5^{\eta\mu}w_1, -5^{\eta\mu}w_1\right) + 3\Psi\left(-5^{\eta\mu}w_1, -5^{\eta\mu}w_1, 5^{\eta\mu}w_1\right)\right\}\right\} \\
&\quad + \frac{1}{25}\sum_{\eta=\frac{1-\mu}{2}}^{\infty}\frac{1}{25^{\eta\mu}}\left\{\frac{1}{3}\left\{\Psi\left(5^{\eta\mu}w_1, 5^{\eta\mu}w_1, 5^{\eta\mu}w_1\right) + \frac{7}{2}\Psi\left(5^{\eta\mu}w_1, 5^{\eta\mu}w_1, -5^{\eta\mu}w_1\right)\right\}\right. \\
&\qquad\left.\left. + \frac{1}{3}\left\{\Psi\left(-5^{\eta\mu}w_1, -5^{\eta\mu}w_1, -5^{\eta\mu}w_1\right) + \frac{7}{2}\Psi\left(-5^{\eta\mu}w_1, -5^{\eta\mu}w_1, 5^{\eta\mu}w_1\right)\right\}\right\}\right\},
\end{aligned} \tag{2.32}$$

and the mapping $\mathcal{A}(w_1)$ *and* $\mathcal{Q}(w_1)$ *are given in (2.6) and (2.24) for all* $w_1 \in \mathcal{W}_1$.

Proof. Consider a function $\mathcal{F}_{odd}(w_1)$ by

$$\mathcal{F}_{odd}(w_1) = \frac{1}{2}\left\{\mathcal{F}(w_1) - \mathcal{F}(-w_1)\right\}, \forall\, w_1 \in \mathcal{W}_1, \tag{2.33}$$

which gives

$$\mathcal{F}_{odd}(0) = 0; \qquad \mathcal{F}_{odd}(-w_1) = -\mathcal{F}_{odd}(w_1), \forall\, w_1 \in \mathcal{W}_1. \tag{2.34}$$

By Theorem 2.1, it follows from (2.33), (2.1), (2.5) and (2.6), we arrive

$$\begin{aligned}
&\|\mathcal{F}_{odd}(w_1) - \mathcal{A}(w_1)\| \\
&\leq \frac{1}{2}\cdot\frac{1}{5}\sum_{\eta=\frac{1-\mu}{2}}^{\infty}\frac{1}{5^{\eta\mu}}\Big\{\Psi_A\left(5^{\eta\mu}w_1\right) + \Psi_A\left(-5^{\eta\mu}w_1\right)\Big\} \qquad (2.35)\\
&= \frac{1}{2}\cdot\frac{1}{5}\sum_{\eta=\frac{1-\mu}{2}}^{\infty}\frac{1}{5^{\eta\mu}}\ \Bigg\{\frac{1}{3}\Big\{\Psi\left(5^{\eta\mu}w_1, 5^{\eta\mu}w_1, 5^{\eta\mu}w_1\right) + 3\Psi\left(5^{\eta\mu}w_1, 5^{\eta\mu}w_1, -5^{\eta\mu}w_1\right)\Big\}\\
&\qquad + \frac{1}{3}\Big\{\Psi\left(-5^{\eta\mu}w_1, -5^{\eta\mu}w_1, -5^{\eta\mu}w_1\right) + 3\Psi\left(-5^{\eta\mu}w_1, -5^{\eta\mu}w_1, 5^{\eta\mu}w_1\right)\Big\}\Bigg\}, \qquad (2.36)
\end{aligned}$$

for all $w_1 \in \mathcal{W}_1$. Consider a function $\mathcal{F}_{even}(w_1)$ by

$$\mathcal{F}_{even}(w_1) = \frac{1}{2}\Big\{\mathcal{F}(w_1) + \mathcal{F}(-w_1)\Big\}, \forall\, w_1 \in \mathcal{W}_1, \qquad (2.37)$$

which gives

$$\mathcal{F}_{even}(0) = 0; \qquad \mathcal{F}_{even}(-w_1) = \mathcal{F}_{even}(w_1), \forall\, w_1 \in \mathcal{W}_1. \qquad (2.38)$$

By Theorem 2.3, it follows from (2.37), (2.1), (2.22) and (2.23), we see

$$\begin{aligned}
&\|\mathcal{F}_{even}(w_1) - \mathcal{Q}(w_1)\| \\
&\leq \frac{1}{2}\cdot\frac{1}{25}\sum_{\eta=\frac{1-\mu}{2}}^{\infty}\frac{1}{25^{\eta\mu}}\Big\{\Psi_Q\left(5^{\eta\mu}w_1\right) + \Psi_Q\left(-5^{\eta\mu}w_1\right)\Big\} \qquad (2.39)\\
&= \frac{1}{2}\cdot\frac{1}{25}\sum_{\eta=\frac{1-\mu}{2}}^{\infty}\frac{1}{25^{\eta\mu}}\ \Bigg\{\frac{1}{3}\Big\{\Psi\left(5^{\eta\mu}w_1, 5^{\eta\mu}w_1, 5^{\eta\mu}w_1\right) + \frac{7}{2}\Psi\left(5^{\eta\mu}w_1, 5^{\eta\mu}w_1, -5^{\eta\mu}w_1\right)\Big\}\\
&\qquad + \frac{1}{3}\Big\{\Psi\left(-5^{\eta\mu}w_1, -5^{\eta\mu}w_1, -5^{\eta\mu}w_1\right) + \frac{7}{2}\Psi\left(-5^{\eta\mu}w_1, -5^{\eta\mu}w_1, 5^{\eta\mu}w_1\right)\Big\}\Bigg\}, \qquad (2.40)
\end{aligned}$$

for all $w_1 \in \mathcal{W}_1$. Assume a function $\mathcal{F}(w_1)$ by

$$\mathcal{F}(w_1) = \mathcal{F}_{odd}(w_1) + \mathcal{F}_{even}(w_1), \forall\, w_1 \in \mathcal{W}_1. \qquad (2.41)$$

Now, it follows from (2.35), (2.36), (2.39), (2.40) and (2.41), we have

$$\begin{aligned}
&\|\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1)\| \\
&\leq \|\mathcal{F}_{odd}(w_1) - \mathcal{A}(w_1)\| + \|\mathcal{F}_{even}(w_1) - \mathcal{Q}(w_1)\| \\
&\leq \frac{1}{2}\Bigg\{\frac{1}{5}\sum_{\eta=\frac{1-\mu}{2}}^{\infty}\frac{1}{5^{\eta\mu}}\Big\{\Psi_A\left(5^{\eta\mu}w_1\right) + \Psi_A\left(-5^{\eta\mu}w_1\right)\Big\} + \frac{1}{25}\sum_{\eta=\frac{1-\mu}{2}}^{\infty}\frac{1}{25^{\eta\mu}}\Big\{\Psi_Q\left(5^{\eta\mu}w_1\right) + \Psi_Q\left(-5^{\eta\mu}w_1\right)\Big\}\Bigg\}\\
&\leq \frac{1}{2}\Bigg\{\frac{1}{5}\sum_{\eta=\frac{1-\mu}{2}}^{\infty}\frac{1}{5^{\eta\mu}}\ \Bigg\{\frac{1}{3}\Big\{\Psi\left(5^{\eta\mu}w_1, 5^{\eta\mu}w_1, 5^{\eta\mu}w_1\right) + 3\Psi\left(5^{\eta\mu}w_1, 5^{\eta\mu}w_1, -5^{\eta\mu}w_1\right)\Big\}\\
&\qquad + \frac{1}{3}\Big\{\Psi\left(-5^{\eta\mu}w_1, -5^{\eta\mu}w_1, -5^{\eta\mu}w_1\right) + 3\Psi\left(-5^{\eta\mu}w_1, -5^{\eta\mu}w_1, 5^{\eta\mu}w_1\right)\Big\}\Bigg\}\\
&\quad + \frac{1}{25}\sum_{\eta=\frac{1-\mu}{2}}^{\infty}\frac{1}{25^{\eta\mu}}\ \Bigg\{\frac{1}{3}\Big\{\Psi\left(5^{\eta\mu}w_1, 5^{\eta\mu}w_1, 5^{\eta\mu}w_1\right) + \frac{7}{2}\Psi\left(5^{\eta\mu}w_1, 5^{\eta\mu}w_1, -5^{\eta\mu}w_1\right)\Big\}\\
&\qquad + \frac{1}{3}\Big\{\Psi\left(-5^{\eta\mu}w_1, -5^{\eta\mu}w_1, -5^{\eta\mu}w_1\right) + \frac{7}{2}\Psi\left(-5^{\eta\mu}w_1, -5^{\eta\mu}w_1, 5^{\eta\mu}w_1\right)\Big\}\Bigg\}\Bigg\},
\end{aligned}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$. □

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Corollary 2.6. *Suppose that a function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (2.2) for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\|\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1)\| \leq \begin{cases} \frac{\delta}{|3|} + \frac{3\delta}{2|24|}, & \\ \frac{4\delta|w_1|^{\varphi}}{|5-5^{\varphi}|} + \frac{27\delta|w_1|^{\varphi}}{6|25-5^{\varphi}|}; & \varphi \neq 1,2, \\ \frac{4\delta}{3} \sum_{\psi=1}^{3} \frac{|w_\psi|^{\varphi_\psi}}{|5-5^{\varphi_\psi}|} + \frac{9\delta}{6} \sum_{\psi=1}^{3} \frac{|w_\psi|^{\varphi_\psi}}{|25-5^{\varphi_\psi}|}; & \varphi_1, \varphi_2, \varphi_3 \neq 1,2, \\ \frac{4\delta|w_1|^{3\varphi}}{3|5-5^{3\varphi}|} + \frac{9\delta|w_1|^{3\varphi}}{6|25-5^{3\varphi}|}; & 3\varphi \neq 1,2, \\ \frac{4\delta|w_\psi|^{\sum_{\psi=1}^{3}\varphi_\psi}}{3\left|5-5^{\sum_{\psi=1}^{3}\varphi_\psi}\right|} + \frac{9\delta|w_\psi|^{\sum_{\psi=1}^{3}\varphi_\psi}}{6\left|25-5^{\sum_{\psi=1}^{3}\varphi_\psi}\right|}; & \sum_{\psi=1}^{3}\varphi_\psi \neq 1,2, \\ \frac{16\delta|w_1|^{3\varphi}}{3|5-5^{3\varphi}|} + \frac{36\delta|w_1|^{3\varphi}}{6|25-5^{3\varphi}|}; & 3\varphi \neq 1,2, \end{cases} \quad (2.42)$$

for all $w_1 \in \mathcal{W}_1$.

2.4. **Oddness of $\mathcal{F}$: Additive Case Stability Results : Fixed Point Method.**

Theorem 2.7. *Suppose that an odd function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (2.1) where $\Psi : \mathcal{W}_1^3 \to [0, \infty)$ with the condition*

$$\lim_{\ell\to\infty} \frac{\Psi\left(\tau_v^\ell w_1, \tau_v^\ell w_2, \tau_v^\ell w_3\right)}{\tau_v^\ell} = 0; \quad \tau_v = \begin{cases} 5; \nu = 0 \\ \frac{1}{5}; \nu = 1 \end{cases}, \forall\, w_1, w_2, w_3 \in \mathcal{W}_1. \quad (2.43)$$

If there exists $L = L(\nu)$ be a function have the property

$$\Psi_A(w_1) = \Psi_A\left(\frac{w_1}{5}\right) \qquad and \qquad \frac{1}{\tau_v}\Psi_A(\tau_v w_1) = L\,\Psi_A(w_1),\ \forall\, w_1 \in \mathcal{W}_1. \quad (2.44)$$

Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality

$$\|\mathcal{F}(w_1) - \mathcal{A}(w_1)\| \leq \frac{L^{1-\nu}}{1-L}\, \Psi_A(w_1) \quad (2.45)$$

$$= \frac{L^{1-\nu}}{1-L} \left\{ \frac{1}{3} \Big\{ \Psi(w_1, w_1, w_1) + 3\Psi(w_1, w_1, -w_1) \Big\} \right\}, \quad (2.46)$$

and the mapping $\mathcal{A}(w_1)$ is obtained by

$$\mathcal{A}(w_1) = \lim_{\ell\to\infty} \frac{1}{\tau_v^\ell} \mathcal{F}\left(\tau_v^\ell w_1\right), \quad (2.47)$$

for all $w_1 \in \mathcal{W}_1$.

Proof. Assume a set

$$\mathcal{G} = \{\mathcal{F}/\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2,\ \mathcal{F}(0) = 0\}, \quad (2.48)$$

and introduce the generalized metric on the above set $\mathcal{G}$ as

$$d(\mathcal{F}, \mathcal{F}_1) = \inf\{K \in (0, \infty) : \|\mathcal{F}(w_1) - \mathcal{F}_1(w_1)\| \leq K\, \Psi(w_1, w_1, w_1), w_1 \in \mathcal{W}_1\}. \quad (2.49)$$

It is easy to see that $(\mathcal{G}, d)$ is complete. Define a function $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ by

$$\mathcal{H}\mathcal{F}(w_1) = \frac{1}{\tau_v}\mathcal{F}(\tau_v\, w_1), \quad for\ all\ w_1 \in \mathcal{W}_1. \quad (2.50)$$

Now $\mathcal{F}, \mathcal{F}_1 \in \mathcal{G}$ and $w_1 \in \mathcal{W}_1$, we see

$$\begin{aligned} d(\mathcal{F},\mathcal{F}_1) \leq K \Rightarrow & \parallel \mathcal{F}(w_1) - \mathcal{F}_1(w_1) \parallel \leq K\,\Psi(w_1,w_1,w_1), \\ & \Rightarrow \left\| \frac{1}{\tau_v}\mathcal{F}(\tau_v w_1) - \frac{1}{\tau_v}\mathcal{F}_1(\tau_v w_1) \right\| \leq \frac{1}{\tau_v} K\,\Psi(\tau_v w_1, \tau_v w_1, \tau_v w_1), \\ & \Rightarrow \parallel \mathcal{H}\mathcal{F}(w_1) - \mathcal{H}\mathcal{F}_1(\ w_1\ ) \parallel \leq L\,K\,\Psi(w_1,w_1,w_1), \\ & \Rightarrow d(\mathcal{H}\mathcal{F}, \mathcal{H}\mathcal{F}_1) \leq L\,K, \end{aligned}$$

i.e., $\mathcal{H}$ is a strictly contractive mapping on $\mathcal{G}$ with Lipschitz constant L (see [18]).
For the case $\nu = 0$, it follows from (2.12) and with the help of (2.44), (2.50), (2.49), we get

$$\left\| \frac{1}{5}\mathcal{F}(5w_1) - \mathcal{F}(w_1) \right\| \leq \frac{1}{5}\Psi_A\left(w_1\right), \Rightarrow d(\mathcal{H}\mathcal{F}, \mathcal{F}) \leq L = L^{1-\nu},\ \forall\, w_1 \in \mathcal{W}_1. \tag{2.51}$$

For the case $\nu = 1$, it follows from (2.18) and with the help of (2.44), (2.50), (2.49), we obtain

$$\left\| \mathcal{F}(w_1) - 5\mathcal{F}\left(\frac{w_1}{5}\right) \right\| \leq \Psi_A\left(\frac{w_1}{5}\right), \Rightarrow d(\mathcal{F}, \mathcal{H}\mathcal{F}) \leq 1 = L^{1-\nu},\ \forall\, w_1 \in \mathcal{W}_1. \tag{2.52}$$

Combining (2.51) and (2.52), we have

$$d(\mathcal{F}, \mathcal{H}\mathcal{F}) \leq 1 = L^{1-\nu}. \tag{2.53}$$

Therefore $(FPC1)$ of Theorem 1.5 holds. The rest of the proof follows by Theorem 1.5. Hence the proof is complete. □

Corollary 2.8. *Suppose that an odd function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (2.2) for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality (2.20) for all $w_1 \in \mathcal{W}_1$.*

Proof. If we take

$$\Psi\left(w_1, w_2, w_3\right) = \begin{cases} \delta, \\ \delta \sum_{\psi=1}^{3} \left|w_\psi\right|^{\varphi}, \\ \delta \sum_{\psi=1}^{3} \left|w_\psi\right|^{\varphi_\psi}, \\ \delta \prod_{\psi=1}^{3} \left|w_\psi\right|^{\varphi}, \\ \delta \prod_{\psi=1}^{3} \left|w_\psi\right|^{\varphi_\psi}, \\ \delta \left\{ \sum_{\psi=1}^{3} \left|w_\psi\right|^{3\varphi} + \prod_{\psi=1}^{3} \left|w_\psi\right|^{\varphi} \right\}, \end{cases} \tag{2.54}$$

in Theorem 2.7 and changing (w_1, w_2, w_3) by $\left(\tau_v^\ell w_1, \tau_v^\ell w_2, \tau_v^\ell w_3\right)$ and dividing by τ_v^ℓ in (2.54), one can see

$$\frac{1}{\tau_v^\ell}\,\Psi\left(\tau_v^\ell w_1, \tau_v w_2^\ell, \tau_v^\ell w_3\right) = \begin{cases} \frac{\delta}{\tau_v^\ell} & \to 0 \text{ as } \ell \text{ to } \infty, \\ \frac{\delta}{\tau_v^\ell} \sum_{\psi=1}^{3} \left|\tau_v^\ell w_\psi\right|^{\varphi}, & \to 0 \text{ as } \ell \text{ to } \infty, \\ \frac{\delta}{\tau_v^\ell} \sum_{\psi=1}^{3} \left|\tau_v^\ell w_\psi\right|^{\varphi_\psi}, & \to 0 \text{ as } \ell \text{ to } \infty, \\ \frac{\delta}{\tau_v^\ell} \prod_{\psi=1}^{3} \left|\tau_v^\ell w_\psi\right|^{\varphi}, & \to 0 \text{ as } \ell \text{ to } \infty, \\ \frac{\delta}{\tau_v^\ell} \prod_{\psi=1}^{3} \left|\tau_v^\ell w_\psi\right|^{\varphi_\psi}, & \to 0 \text{ as } \ell \text{ to } \infty, \\ \frac{\delta}{\tau_v^\ell} \left\{ \sum_{\psi=1}^{3} \left|\tau_v^\ell w_\psi\right|^{3\varphi} + \prod_{\psi=1}^{3} \left|\tau_v^\ell w_\psi\right|^{\varphi} \right\}, & \to 0 \text{ as } \ell \text{ to } \infty. \end{cases}$$

Therefore (2.43) holds for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Now, from (2.44), we have

$$\Psi_A(w_1) = \Psi_A\left(\frac{w_1}{5}\right) = \frac{1}{3}\Big\{\Psi\left(\frac{w_1}{5}, \frac{w_1}{5}, \frac{w_1}{5}\right) + 3\Psi\left(\frac{w_1}{5}, \frac{w_1}{5}, -\frac{w_1}{5}\right)\Big\} = \begin{cases} \frac{4\delta}{3}, \\ \frac{12|\frac{w_1}{5}|^{\varphi}}{3}, \\ \frac{4\delta}{3}\sum_{\psi=1}^{3}\left|\frac{w_1}{5}\right|^{\varphi_\psi}, \\ \frac{4\delta|\frac{w_1}{5}|^{3\varphi}}{3}, \\ \frac{4\delta|\frac{w_1}{5}|^{\sum_{\psi=1}^{3}\varphi_\psi}}{3}, \\ \frac{16\delta|\frac{w_1}{5}|^{3\varphi}}{3}, \end{cases} \tag{2.55}$$

$$\begin{aligned} \frac{1}{\tau_v}\Psi_A(\tau_v w_1) &= \frac{1}{\tau_v}\frac{1}{3}\Big\{\Psi(\tau_v w_1, \tau_v w_1, \tau_v w_1) + 3\Psi(\tau_v w_1, \tau_v w_1, -\tau_v w_1)\Big\} \\ &= \begin{cases} \frac{4\delta}{\tau_v\cdot 3}, \\ \frac{12\delta|\tau_v w_1|^{\varphi}}{\tau_v\cdot 3}, \\ \frac{4\delta}{\tau_v\cdot 3}\sum_{\psi=1}^{3}\left|\tau_v w_\psi\right|^{\varphi_\psi}, \\ \frac{4\delta|\tau_v w_1|^{3\varphi}}{\tau_v\cdot 3}, \\ \frac{4\delta|\tau_v w_1|^{\sum_{\psi=1}^{3}\varphi_\psi}}{\tau_v\cdot 3}, \\ \frac{16\delta|\tau_v w_1|^{3\varphi}}{\tau_v\cdot 3}, \end{cases} = \begin{cases} \tau_v^{-1}\Psi_A(w_1), \\ \tau_v^{\varphi-1}\Psi_A(w_1), \\ \sum_{\psi=1}^{3}\tau_v^{\varphi_\psi-1}\Psi_A(w_1), \\ \tau_v^{3\varphi-1}\Psi_A(w_1), \\ \tau_v^{\sum_{\psi=1}^{3}\varphi_\psi-1}\Psi_A(w_1), \\ \tau_v^{3\varphi-1}\Psi_A(w_1), \end{cases} = \begin{cases} L\,\Psi_A(w_1), \\ L\,\Psi_A(w_1), \\ L\,\Psi_A(w_1), \\ L\,\Psi_A(w_1), \\ L\,\Psi_A(w_1), \\ L\,\Psi_A(w_1), \end{cases} \end{aligned} \tag{2.56}$$

for all $w_1 \in \mathcal{W}_1$.
For the case $\nu = 0$, we have $L = \tau_0^{-1} = 5^{-1}$ and from (2.46), we arrive

$$\begin{aligned} \|\mathcal{F}(w_1) - \mathcal{A}(w_1)\| \le \frac{L^{1-\nu}}{1-L}\,\Psi_A(w_1) &= \frac{L^{1-\nu}}{1-L}\,\left\{\frac{1}{3}\Big\{\Psi(w_1, w_1, w_1) + 3\Psi(w_1, w_1, -w_1)\Big\}\right\} \\ &= \frac{(5^{-1})^{1-0}}{1-5^{-1}}\,\left\{\frac{4\delta}{3}\right\} = \frac{\delta}{3}. \end{aligned}$$

For the case $\nu = 1$, we have $L = \tau_1^{-1} = (\frac{1}{5})^{-1} = 5$ and from (2.46), we obtain

$$\begin{aligned} \|\mathcal{F}(w_1) - \mathcal{A}(w_1)\| \le \frac{L^{1-\nu}}{1-L}\,\Psi_A(w_1) &= \frac{L^{1-\nu}}{1-L}\,\left\{\frac{1}{3}\Big\{\Psi(w_1, w_1, w_1) + 3\Psi(w_1, w_1, -w_1)\Big\}\right\} \\ &= \frac{(5)^{1-1}}{1-5}\,\left\{\frac{4\delta}{3}\right\} = \frac{\delta}{-3}. \end{aligned}$$

For the case $\nu = 0$, we have $L = \tau_0^{\varphi-1} = 5^{\varphi-1}$ and from (2.46), we arrive

$$\begin{aligned} \|\mathcal{F}(w_1) - \mathcal{A}(w_1)\| \le \frac{L^{1-\nu}}{1-L}\,\Psi_A(w_1) &= \frac{L^{1-\nu}}{1-L}\,\left\{\frac{1}{3}\Big\{\Psi(w_1, w_1, w_1) + 3\Psi(w_1, w_1, -w_1)\Big\}\right\} \\ &= \frac{(5^{\varphi-1})^{1-0}}{1-5^{\varphi-1}}\,\left\{\frac{12\delta|\frac{w_1}{5}|^{\varphi}}{3}\right\} = \frac{4\delta}{5-5^{\varphi}}. \end{aligned}$$

For the case $\nu = 1$, we have $L = \tau_1^{\varphi-1} = (\frac{1}{5})^{\varphi-1} = 5^{1-\varphi}$ and from (2.46), we get

$$\begin{aligned} \|\mathcal{F}(w_1) - \mathcal{A}(w_1)\| \le \frac{L^{1-\nu}}{1-L}\,\Psi_A(w_1) &= \frac{L^{1-\nu}}{1-L}\,\left\{\frac{1}{3}\Big\{\Psi(w_1, w_1, w_1) + 3\Psi(w_1, w_1, -w_1)\Big\}\right\} \\ &= \frac{(5^{1-\varphi})^{1-1}}{1-5^{1-\varphi}}\,\left\{\frac{12\delta|\frac{w_1}{5}|^{\varphi}}{3}\right\} = \frac{4\delta}{5^{\varphi}-5}. \end{aligned}$$

For the case $\nu = 0$, we have $L = \tau_0^{3\varphi-1} = 5^{3\varphi-1}$ and from (2.46), we arrive

$$\|\mathcal{F}(w_1) - \mathcal{A}(w_1)\| \leq \frac{L^{1-\nu}}{1-L}\,\Psi_A\,(w_1) = \frac{L^{1-\nu}}{1-L}\,\left\{\frac{1}{3}\Big\{\Psi\,(w_1, w_1, w_1) + 3\Psi\,(w_1, w_1, -w_1)\Big\}\right\}$$
$$= \frac{(5^{3\varphi-1})^{1-0}}{1-5^{3\varphi-1}}\,\left\{\frac{4\delta|\frac{w_1}{5}|^{3\varphi}}{3}\right\} = \frac{4\delta|w_1|^{3\varphi}}{3(5-5^{3\varphi})}.$$

For the case $\nu = 1$, we have $L = \tau_1^{3\varphi-1} = (\frac{1}{5})^{3\varphi-1} = 5^{1-3\varphi}$ and from (2.46), we obtain

$$\|\mathcal{F}(w_1) - \mathcal{A}(w_1)\| \leq \frac{L^{1-\nu}}{1-L}\,\Psi_A\,(w_1) = \frac{L^{1-\nu}}{1-L}\,\left\{\frac{1}{3}\Big\{\Psi\,(w_1, w_1, w_1) + 3\Psi\,(w_1, w_1, -w_1)\Big\}\right\}$$
$$= \frac{(5^{1-3\varphi})^{1-1}}{1-5^{1-3\varphi}}\,\left\{\frac{4\delta|\frac{w_1}{5}|^{3\varphi}}{3}\right\} = \frac{4\delta|w_1|^{3\varphi}}{3(5^{3\varphi}-5)}.$$

Similarly, we can prove the rest cases. □

2.5. **Evenness of $\mathcal{F}$: Quadratic Case Stability Results : Fixed Point Method.**

Theorem 2.9. *Suppose that an even function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (2.1) where $\Psi : \mathcal{W}_1^3 \to [0, \infty)$ with the condition*

$$\lim_{\ell\to\infty} \frac{\Psi\left(\tau_v^\ell w_1, \tau_v^\ell w_2, \tau_v^\ell w_3\right)}{\tau_v^{2\ell}} = 0; \quad \tau_v = \begin{cases} 5; \nu = 0 \\ \frac{1}{5}; \nu = 1 \end{cases}, \forall\, w_1, w_2, w_3 \in \mathcal{W}_1. \tag{2.57}$$

If there exists $L = L(\nu)$ be a function have the property

$$\Psi_Q(w_1) = \Psi_Q\left(\frac{w_1}{5}\right) \qquad \text{and} \qquad \frac{1}{\tau_v^2}\Psi_Q\,(\tau_v w_1) = L\,\Psi_Q(w_1),\ \forall\, w_1 \in \mathcal{W}_1. \tag{2.58}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality

$$\|\mathcal{F}(w_1) - \mathcal{Q}(w_1)\| \leq \frac{L^{1-\nu}}{1-L}\,\Psi_Q\,(w_1) \tag{2.59}$$
$$= \frac{L^{1-\nu}}{1-L}\,\left\{\frac{1}{3}\Big\{\Psi\,(w_1, w_1, w_1) + \frac{7}{2}\Psi\,(w_1, w_1, -w_1)\Big\}\right\}, \tag{2.60}$$

and the mapping $\mathcal{Q}(w_1)$ is obtained by

$$\mathcal{Q}(w_1) = \lim_{\ell\to\infty} \frac{1}{\tau_v^{2\ell}}\mathcal{F}\left(\tau_v^\ell w_1\right), \tag{2.61}$$

for all $w_1 \in \mathcal{W}_1$.

Proof. By Theorem 2.7, define a function $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ by

$$\mathcal{H}\mathcal{F}(w_1) = \frac{1}{\tau_v^2}\mathcal{F}(\tau_v\, w_1), \quad for\ all\ w_1 \in \mathcal{W}_1. \tag{2.62}$$

Now $\mathcal{F}, \mathcal{F}_1 \in \mathcal{G}$ and $w_1 \in \mathcal{W}_1$, we see

$$\begin{aligned} d(\mathcal{F}, \mathcal{F}_1) \leq K &\Rightarrow \|\ \mathcal{F}(w_1) - \mathcal{F}_1(w_1)\ \| \leq K\,\Psi(w_1, w_1, w_1), \\ &\Rightarrow \left\|\frac{1}{\tau_v^2}\mathcal{F}(\tau_v w_1) - \frac{1}{\tau_v^2}\mathcal{F}_1(\tau_v w_1)\right\| \leq \frac{1}{\tau_v^2}K\,\Psi(\tau_v w_1, \tau_v w_1, \tau_v w_1), \\ &\Rightarrow \|\ \mathcal{H}\mathcal{F}(w_1) - \mathcal{H}\mathcal{F}_1(\ w_1\ )\ \| \leq L\,K\,\Psi(w_1, w_1, w_1), \\ &\Rightarrow d(\mathcal{H}\mathcal{F}, \mathcal{H}\mathcal{F}_1) \leq L\,K, \end{aligned}$$

i.e., $\mathcal{H}$ is a strictly contractive mapping on $\mathcal{G}$ with Lipschitz constant L (see [18]). The rest of the proof is similar to that of Theorem 2.7. Hence the proof is complete. □

Corollary 2.10. *Suppose that an even function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (2.2) for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality (2.31) for all $w_1 \in \mathcal{W}_1$.*

2.6. Oddness and Evenness of $\mathcal{F}$: Additive Quadratic Case Stability Results : Fixed Point Method.

Theorem 2.11. *Suppose that a function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (2.1) where $\Psi : \mathcal{W}_1^3 \to [0, \infty)$ with the conditions (2.43) and (2.57) for all $w_1, w_2, w_3 \in \mathcal{W}_1$. If there exists $L = L(\nu)$ be function have the properties (2.44) and (2.58) Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\begin{aligned} &\|\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1)\| \\ &\leq \frac{1}{2} \cdot \frac{L^{1-\nu}}{1-L} \Big\{ \Psi_A(w_1) + \Psi_A(-w_1) + \Psi_Q(w_1) + \Psi_Q(-w_1) \Big\} \qquad (2.63) \\ &= \frac{1}{2} \cdot \frac{L^{1-\nu}}{1-L} \ \Bigg\{ \frac{1}{3} \Big\{ \Psi(w_1, w_1, w_1) + 3\Psi(w_1, w_1, -w_1) + \Psi(-w_1, -w_1, -w_1) + 3\Psi(-w_1, -w_1, w_1) \\ &\qquad + \Psi(w_1, w_1, w_1) + \frac{7}{2}\Psi(w_1, w_1, -w_1) + \Psi(-w_1, -w_1, -w_1) + \frac{7}{2}\Psi(-w_1, -w_1, w_1) \Big\} \Bigg\}, \qquad (2.64) \end{aligned}$$

and the mapping $\mathcal{A}(w_1)$ and $\mathcal{Q}(w_1)$ are given in (2.47) and (2.61) for all $w_1 \in \mathcal{W}_1$.

Proof. The proof is similar ideas to that of Theorem 2.5. □

Corollary 2.12. *Suppose that a function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (2.2) for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality (2.42) for all $w_1 \in \mathcal{W}_1$.*

3. Stability In Intuitionistic Fuzzy Banach Space of (1.7)

In this section, we explore the generalized Ulam - Hyers stability of the functional equation (1.7) in Intuitionistic Fuzzy Banach Space.

In order to prove stability results, assume $(\mathcal{W}_1, \mu, \nu)$ and $(\mathcal{W}_2, \mu', \nu')$ are Intuitionistic Fuzzy normed space and Intuitionistic Fuzzy Banach space respectively. Suppose that $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ and $\Psi : \mathcal{W}_1^3 \to [0, \infty)$ satisfy the following functional inequalities

$$\left.\begin{aligned} &\mu\Big(\mathcal{F}(3w_1 + w_2 + w_3) + \mathcal{F}(w_1 + 3w_2 + w_3) + \mathcal{F}(w_1 + w_2 + 3w_3) - 6\mathcal{F}\Big(\textstyle\sum_{\psi=1}^3 w_\psi\Big) \\ &\quad - \tfrac{1}{2}\Big\{\mathcal{F}\Big(\textstyle\sum_{\psi=1}^3 w_\psi\Big) + \mathcal{F}\Big(-\textstyle\sum_{\psi=1}^3 w_\psi\Big)\Big\} + \textstyle\sum_{\psi=1}^3 \Big\{\mathcal{F}(w_\psi) - \tfrac{5}{2}\Big[\mathcal{F}(w_\psi) + \mathcal{F}(-w_\psi)\Big]\Big\}, \Lambda\Big) \\ &\qquad\qquad \geq \mu'\left(\Psi(w_1, w_2, w_3), \Lambda\right) \\ &\nu\Big(\mathcal{F}(3w_1 + w_2 + w_3) + \mathcal{F}(w_1 + 3w_2 + w_3) + \mathcal{F}(w_1 + w_2 + 3w_3) - 6\mathcal{F}\Big(\textstyle\sum_{\psi=1}^3 w_\psi\Big) \\ &\quad - \tfrac{1}{2}\Big\{\mathcal{F}\Big(\textstyle\sum_{\psi=1}^3 w_\psi\Big) + \mathcal{F}\Big(-\textstyle\sum_{\psi=1}^3 w_\psi\Big)\Big\} + \textstyle\sum_{\psi=1}^3 \Big\{\mathcal{F}(w_\psi) - \tfrac{5}{2}\Big[\mathcal{F}(w_\psi) + \mathcal{F}(-w_\psi)\Big]\Big\}, \Lambda\Big) \\ &\qquad\qquad \leq \nu'\left(\Psi(w_1, w_2, w_3), \Lambda\right) \end{aligned}\right\} \qquad (3.1)$$

$$\left.\begin{aligned} &\mu\Big(\mathcal{F}(3w_1 + w_2 + w_3) + \mathcal{F}(w_1 + 3w_2 + w_3) + \mathcal{F}(w_1 + w_2 + 3w_3) - 6\mathcal{F}\Big(\textstyle\sum_{\psi=1}^3 w_\psi\Big) \\ &\quad - \tfrac{1}{2}\Big\{\mathcal{F}\Big(\textstyle\sum_{\psi=1}^3 w_\psi\Big) + \mathcal{F}\Big(-\textstyle\sum_{\psi=1}^3 w_\psi\Big)\Big\} + \textstyle\sum_{\psi=1}^3 \Big\{\mathcal{F}(w_\psi) - \tfrac{5}{2}\Big[\mathcal{F}(w_\psi) + \mathcal{F}(-w_\psi)\Big]\Big\}, \Lambda\Big) \\ &\qquad\qquad \geq \mu'(\delta, \Lambda), \\ &\nu\Big(\mathcal{F}(3w_1 + w_2 + w_3) + \mathcal{F}(w_1 + 3w_2 + w_3) + \mathcal{F}(w_1 + w_2 + 3w_3) - 6\mathcal{F}\Big(\textstyle\sum_{\psi=1}^3 w_\psi\Big) \\ &\quad - \tfrac{1}{2}\Big\{\mathcal{F}\Big(\textstyle\sum_{\psi=1}^3 w_\psi\Big) + \mathcal{F}\Big(-\textstyle\sum_{\psi=1}^3 w_\psi\Big)\Big\} + \textstyle\sum_{\psi=1}^3 \Big\{\mathcal{F}(w_\psi) - \tfrac{5}{2}\Big[\mathcal{F}(w_\psi) + \mathcal{F}(-w_\psi)\Big]\Big\}, \Lambda\Big) \\ &\qquad\qquad \leq \nu'(\delta, \Lambda), \end{aligned}\right\} \qquad (3.2)$$

$$\left.\begin{array}{r}\mu\left(\mathcal{F}(3w_1+w_2+w_3)+\mathcal{F}(w_1+3w_2+w_3)+\mathcal{F}(w_1+w_2+3w_3)-6\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)\right.\\ \left.-\frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)+\mathcal{F}\left(-\sum_{\psi=1}^{3}w_\psi\right)\right\}+\sum_{\psi=1}^{3}\left\{\mathcal{F}(w_\psi)-\frac{5}{2}\left[\mathcal{F}(w_\psi)+\mathcal{F}(-w_\psi)\right]\right\},\Lambda\right)\\ \geq\mu'\left(\delta\sum_{\psi=1}^{3}|w_\psi|^{\varphi},\Lambda\right),\\ \nu\left(\mathcal{F}(3w_1+w_2+w_3)+\mathcal{F}(w_1+3w_2+w_3)+\mathcal{F}(w_1+w_2+3w_3)-6\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)\right.\\ \left.-\frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)+\mathcal{F}\left(-\sum_{\psi=1}^{3}w_\psi\right)\right\}+\sum_{\psi=1}^{3}\left\{\mathcal{F}(w_\psi)-\frac{5}{2}\left[\mathcal{F}(w_\psi)+\mathcal{F}(-w_\psi)\right]\right\},\Lambda\right)\\ \leq\nu'\left(\delta\sum_{\psi=1}^{3}|w_\psi|^{\varphi},\Lambda\right),\end{array}\right\}\quad(3.3)$$

$$\left.\begin{array}{r}\mu\left(\mathcal{F}(3w_1+w_2+w_3)+\mathcal{F}(w_1+3w_2+w_3)+\mathcal{F}(w_1+w_2+3w_3)-6\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)\right.\\ \left.-\frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)+\mathcal{F}\left(-\sum_{\psi=1}^{3}w_\psi\right)\right\}+\sum_{\psi=1}^{3}\left\{\mathcal{F}(w_\psi)-\frac{5}{2}\left[\mathcal{F}(w_\psi)+\mathcal{F}(-w_\psi)\right]\right\},\Lambda\right)\\ \geq\mu'\left(\delta\sum_{\psi=1}^{3}|w_\psi|^{\varphi_\psi},\Lambda\right),\\ \nu\left(\mathcal{F}(3w_1+w_2+w_3)+\mathcal{F}(w_1+3w_2+w_3)+\mathcal{F}(w_1+w_2+3w_3)-6\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)\right.\\ \left.-\frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)+\mathcal{F}\left(-\sum_{\psi=1}^{3}w_\psi\right)\right\}+\sum_{\psi=1}^{3}\left\{\mathcal{F}(w_\psi)-\frac{5}{2}\left[\mathcal{F}(w_\psi)+\mathcal{F}(-w_\psi)\right]\right\},\Lambda\right)\\ \leq\nu'\left(\delta\sum_{\psi=1}^{3}|w_\psi|^{\varphi_\psi},\Lambda\right),\end{array}\right\}\quad(3.4)$$

$$\left.\begin{array}{r}\mu\left(\mathcal{F}(3w_1+w_2+w_3)+\mathcal{F}(w_1+3w_2+w_3)+\mathcal{F}(w_1+w_2+3w_3)-6\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)\right.\\ \left.-\frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)+\mathcal{F}\left(-\sum_{\psi=1}^{3}w_\psi\right)\right\}+\sum_{\psi=1}^{3}\left\{\mathcal{F}(w_\psi)-\frac{5}{2}\left[\mathcal{F}(w_\psi)+\mathcal{F}(-w_\psi)\right]\right\},\Lambda\right)\\ \geq\mu'\left(\delta\prod_{\psi=1}^{3}|w_\psi|^{\varphi},\Lambda\right),\\ \nu\left(\mathcal{F}(3w_1+w_2+w_3)+\mathcal{F}(w_1+3w_2+w_3)+\mathcal{F}(w_1+w_2+3w_3)-6\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)\right.\\ \left.-\frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)+\mathcal{F}\left(-\sum_{\psi=1}^{3}w_\psi\right)\right\}+\sum_{\psi=1}^{3}\left\{\mathcal{F}(w_\psi)-\frac{5}{2}\left[\mathcal{F}(w_\psi)+\mathcal{F}(-w_\psi)\right]\right\},\Lambda\right)\\ \leq\nu'\left(\delta\prod_{\psi=1}^{3}|w_\psi|^{\varphi},\Lambda\right),\end{array}\right\}\quad(3.5)$$

$$\left.\begin{array}{r}\mu\left(\mathcal{F}(3w_1+w_2+w_3)+\mathcal{F}(w_1+3w_2+w_3)+\mathcal{F}(w_1+w_2+3w_3)-6\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)\right.\\ \left.-\frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)+\mathcal{F}\left(-\sum_{\psi=1}^{3}w_\psi\right)\right\}+\sum_{\psi=1}^{3}\left\{\mathcal{F}(w_\psi)-\frac{5}{2}\left[\mathcal{F}(w_\psi)+\mathcal{F}(-w_\psi)\right]\right\},\Lambda\right)\\ \geq\mu'\left(\delta\prod\limits_{\psi=1}^{3}|w_\psi|^{\varphi_\psi},\Lambda\right),\\ \nu\left(\mathcal{F}(3w_1+w_2+w_3)+\mathcal{F}(w_1+3w_2+w_3)+\mathcal{F}(w_1+w_2+3w_3)-6\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)\right.\\ \left.-\frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)+\mathcal{F}\left(-\sum_{\psi=1}^{3}w_\psi\right)\right\}+\sum_{\psi=1}^{3}\left\{\mathcal{F}(w_\psi)-\frac{5}{2}\left[\mathcal{F}(w_\psi)+\mathcal{F}(-w_\psi)\right]\right\},\Lambda\right)\\ \leq\nu'\left(\delta\prod\limits_{\psi=1}^{3}|w_\psi|^{\varphi_\psi},\Lambda\right),\end{array}\right\}\quad(3.6)$$

$$\left.\begin{array}{r}\mu\left(\mathcal{F}(3w_1+w_2+w_3)+\mathcal{F}(w_1+3w_2+w_3)+\mathcal{F}(w_1+w_2+3w_3)-6\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)\right.\\ \left.-\frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)+\mathcal{F}\left(-\sum_{\psi=1}^{3}w_\psi\right)\right\}+\sum_{\psi=1}^{3}\left\{\mathcal{F}(w_\psi)-\frac{5}{2}\left[\mathcal{F}(w_\psi)+\mathcal{F}(-w_\psi)\right]\right\},\Lambda\right)\\ \geq\mu'\left(\delta\left\{\sum_{\psi=1}^{3}|w_\psi|^{3\varphi}+\prod_{\psi=1}^{3}|w_\psi|^{\varphi}\right\},\Lambda\right),\\ \nu\left(\mathcal{F}(3w_1+w_2+w_3)+\mathcal{F}(w_1+3w_2+w_3)+\mathcal{F}(w_1+w_2+3w_3)-6\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)\right.\\ \left.-\frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)+\mathcal{F}\left(-\sum_{\psi=1}^{3}w_\psi\right)\right\}+\sum_{\psi=1}^{3}\left\{\mathcal{F}(w_\psi)-\frac{5}{2}\left[\mathcal{F}(w_\psi)+\mathcal{F}(-w_\psi)\right]\right\},\Lambda\right)\\ \leq\nu'\left(\delta\left\{\sum_{\psi=1}^{3}|w_\psi|^{3\varphi}+\prod_{\psi=1}^{3}|w_\psi|^{\varphi}\right\},\Lambda\right),\end{array}\right\}\quad(3.7)$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with δ be a positive constant and φ be any real number.

3.1. **Definitions and Notations of Intuitionistic Fuzzy Banach Space.** Now, we recall the basic definitions and notations in the setting of intuitionistic fuzzy normed space given in [27].

Definition 3.1. [27] *A binary operation* $* : [0,1] \times [0,1] \longrightarrow [0,1]$ *is said to be continuous t-norm if* $*$ *satisfies the following conditions:*

($*1$) $*$ *is commutative and associative;*
($*2$) $*$ *is continuous;*
($*3$) $a * 1 = a$ *for all* $a \in [0,1]$*;*
($*4$) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [0,1]$.

Definition 3.2. [27] *A binary operation* $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ *is said to be continuous t-conorm if* $\diamond$ *satisfies the following conditions:*

($\diamond 1$) $\diamond$ *is commutative and associative;*
($\diamond 2$) $\diamond$ *is continuous;*
($\diamond 3$) $a \diamond 0 = a$ *for all* $a \in [0,1]$*;*
($\diamond 4$) $a \diamond b \leq c \diamond d$ *whenever* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [0,1]$.

Definition 3.3. [27] *The five-tuple* $(X, \mu, \nu, *, \diamond)$ *is said to be an intuitionistic fuzzy normed space (for short, IFNS) if* X *is a vector space,* $*$ *is a continuous t-norm,* $\diamond$ *is a continuous* $t-$ *conorm, and* μ, ν *are fuzzy sets on* $X \times (0, \infty)$ *satisfy the following conditions. For every* $x, y \in X$ *and* $s, t > 0$

(IFN1) $\mu(x,t) + \nu(x,t) \leq 1$;
(IFN2) $\mu(x,t) > 0$;
(IFN3) $\mu(x,t) = 1$, *if and only if* $x = 0$;
(IFN4) $\mu(dx,t) = \mu\left(x, \frac{t}{d}\right)$ *for each* $d \neq 0$;
(IFN5) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$;
(IFN6) $\mu(x,\cdot) : (0,\infty) \to [0,1]$ *is continuous;*
(IFN7) $\lim_{t\to\infty} \mu(x,t) = 1$ *and* $\lim_{t\to 0} \mu(x,t) = 0$;
(IFN8) $\nu(x,t) < 1$;
(IFN9) $\nu(x,t) = 0$, *if and only if* $x = 0$;
(IFN10) $\nu(dx,t) = \nu\left(x, \frac{t}{d}\right)$ *for each* $d \neq 0$;
(IFN11) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y, t+s)$;
(IFN12) $\nu(x,\cdot) : (0,\infty) \to [0,1]$ *is continuous;*
(IFN13) $\lim_{t\to\infty} \nu(x,t) = 0$ *and* $\lim_{t\to 0} \nu(x,t) = 1$.

In this case, (μ, ν) *is called an intuitionistic fuzzy norm.*

Example 3.4. [27] *Let* $(X, \|\cdot\|)$ *be a normed space. Let* $a * b = ab$ *and* $a \diamond d = \min\{a+b, 1\}$ *for all* $a, b \in [0,1]$. *For all* $x \in X$ *and every* $t > 0$, *consider*

$$\mu(x,t) = \begin{cases} \frac{t}{t+\|x\|} & if \quad t > 0; \\ 0 & if \quad t \leq 0; \end{cases} \qquad and \qquad \nu(x,t) = \begin{cases} \frac{\|x\|}{t+\|x\|} & if \quad t > 0; \\ 0 & if \quad t \leq 0. \end{cases}$$

Then $(X, \mu, \nu, *, \diamond)$ *is an IFN-space.*

Definition 3.5. [27] *Let* $(X, \mu, \nu, *, \diamond)$ *be an IFNS. Then, a sequence* $x = \{x_k\}$ *is said to be intuitionistic fuzzy convergent to a point* $L \in X$ *if*

$$\lim\ \mu(x_k - L, t) = 1 \quad and \quad \lim\ \nu(x_k - L, t) = 0,$$

for all $\rho > 0$. *In this case, we write*

$$x_k \xrightarrow{IF} L \qquad as \qquad k \to \infty.$$

Definition 3.6. [27] *Let* $(X, \mu, \nu, *, \diamond)$ *be an IFN-space. Then,* $x = \{x_k\}$ *is said to be intuitionistic fuzzy Cauchy sequence if*

$$\mu\left(x_{k+p} - x_k, t\right) = 1 \qquad and \qquad \nu\left(x_{k+p} - x_k, t\right) = 0,$$

for all $\rho > 0$, *and* $p = 1, 2 \cdots$.

Definition 3.7. [27] *Let $(X, \mu, \nu, *, \diamond)$ be an IFN-space. Then $(X, \mu, \nu, *, \diamond)$ is said to be complete if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \diamond)$ is intuitionistic fuzzy convergent $(X, \mu, \nu, *, \diamond)$.*

3.2. Oddness of $\mathcal{F}$: Additive Case Stability Results : Direct Method.

Theorem 3.8. *Suppose that an odd function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (3.1) where $\Psi : \mathcal{W}_1^3 \to [0, \infty)$ with the conditions*

$$\left.\begin{array}{l} \mu'\left(\Psi\left(5^{\ell m}w_1, 5^{\ell m}w_2, 5^{\ell m}w_3\right), \Lambda\right) \geq \mu'\left(I^{\ell m}\Psi\left(w_1, w_2, w_3\right), \Lambda\right) \\ \nu'\left(\Psi\left(5^{\ell m}w_1, 5^{\ell m}w_2, 5^{\ell m}w_3\right), \Lambda\right) \leq \nu'\left(I^{\ell m}\Psi\left(w_1, w_2, w_3\right), \Lambda\right) \end{array}\right\} \tag{3.8}$$

and

$$\left.\begin{array}{l} \lim\limits_{\ell\to\infty} \mu'\left(\Psi\left(5^{\ell m}w_1, 5^{\ell m}w_2, 5^{\ell m}w_3\right), 5^{\ell m}\Lambda\right) = 1 \\ \lim\limits_{\ell\to\infty} \nu'\left(\Psi\left(5^{\ell m}w_1, 5^{\ell m}w_2, 5^{\ell m}w_3\right), 5^{\ell m}\Lambda\right) = 0 \end{array}\right\} \tag{3.9}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with $m = \pm 1$ and $0 < \left(\frac{I}{5}\right)^{\mu} < 1$. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality

$$\left.\begin{array}{l} \mu\left(\mathcal{A}(w_1) - \mathcal{F}(w_1), \Lambda\right) \geq \mu'\left(\Psi_A\left(w_1\right), \frac{3\Lambda}{4}\left|5 - I\right|\right) \\ \qquad = \mu'\left(\Psi\left(w_1, w_1, w_1\right), \frac{3\Lambda}{4}\left|5 - I\right|\right) * \mu'\left(\Psi\left(w_1, w_1, -w_1\right), \frac{3\Lambda}{4}\left|5 - I\right|\right) \\ \nu\left(\mathcal{A}(w_1) - \mathcal{F}(w_1), \Lambda\right) \leq \nu'\left(\Psi_A\left(w_1\right), \frac{3\Lambda}{4}\left(5 - I\right)\right) \\ \qquad = \nu'\left(\Psi\left(w_1, w_1, w_1\right), \frac{3\Lambda}{4}\left(5 - I\right)\right) \diamond \nu'\left(\Psi\left(w_1, w_1, -w_1\right), \frac{3\Lambda}{4}\left(5 - I\right)\right) \end{array}\right\} \tag{3.10}$$

and the mapping $\mathcal{A}(w_1)$ is obtained by

$$\left.\begin{array}{l} \lim\limits_{\ell\to\infty} \mu\left(\frac{1}{5^{\ell m}}\mathcal{F}\left(5^{\ell m}w_1\right) - \mathcal{A}(w_1), \Lambda\right) = 1 \\ \lim\limits_{\ell\to\infty} \nu\left(\frac{1}{5^{\ell m}}\mathcal{F}\left(5^{\ell m}w_1\right) - \mathcal{A}(w_1), \Lambda\right) = 0 \end{array}\right\} \tag{3.11}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$.

Proof. Using oddness of $\mathcal{F}$ in (2.1), we get

$$\left.\begin{array}{l} \mu\left(\mathcal{F}(3w_1 + w_2 + w_3) + \mathcal{F}(w_1 + 3w_2 + w_3) + \mathcal{F}(w_1 + w_2 + 3w_3) - 6\mathcal{F}\left(\sum_{\psi=1}^{3} w_\psi\right) + \sum_{\psi=1}^{3}\mathcal{F}(w_\psi), \Lambda\right) \\ \qquad \geq \mu'\left(\Psi\left(w_1, w_2, w_3\right), \Lambda\right) \\ \nu\left(\mathcal{F}(3w_1 + w_2 + w_3) + \mathcal{F}(w_1 + 3w_2 + w_3) + \mathcal{F}(w_1 + w_2 + 3w_3) - 6\mathcal{F}\left(\sum_{\psi=1}^{3} w_\psi\right) + \sum_{\psi=1}^{3}\mathcal{F}(w_\psi), \Lambda\right) \\ \qquad \leq \nu'\left(\Psi\left(w_1, w_2, w_3\right), \Lambda\right) \end{array}\right\} \tag{3.12}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$. Interchanging (w_1, w_2, w_3) by (w_1, w_1, w_1) in (3.12), we obtain

$$\left.\begin{array}{l} \mu\left(3\mathcal{F}(5w_1) - 6\mathcal{F}\left(3w_1\right) + 3\mathcal{F}(w_1), \Lambda\right) \geq \mu'\left(\Psi\left(w_1, w_1, w_1\right), \Lambda\right) \\ \nu\left(3\mathcal{F}(5w_1) - 6\mathcal{F}\left(3w_1\right) + 3\mathcal{F}(w_1), \Lambda\right) \leq \nu'\left(\Psi\left(w_1, w_1, w_1\right), \Lambda\right) \end{array}\right\} \tag{3.13}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Again interchanging (w_1, w_2, w_3) by $(w_1, w_1, -w_1)$ in (3.12) and using (IFN4), (IFN10), we have

$$\left.\begin{array}{l} \mu\left(2\mathcal{F}(3w_1) - 6\mathcal{F}\left(w_1\right), \Lambda\right) \geq \mu'\left(\Psi\left(w_1, w_1, -w_1\right), \Lambda\right) \\ \nu\left(2\mathcal{F}(3w_1) - 6\mathcal{F}\left(w_1\right), \Lambda\right) \leq \nu'\left(\Psi\left(w_1, w_1, -w_1\right), \Lambda\right) \end{array}\right\}$$

$$\Rightarrow \left.\begin{array}{l} \mu\left(6\mathcal{F}(3w_1) - 18\mathcal{F}\left(w_1\right), 3\Lambda\right) \geq \mu'\left(\Psi\left(w_1, w_1, -w_1\right), \Lambda\right) \\ \nu\left(6\mathcal{F}(3w_1) - 18\mathcal{F}\left(w_1\right), 3\Lambda\right) \leq \nu'\left(\Psi\left(w_1, w_1, -w_1\right), \Lambda\right) \end{array}\right\} \tag{3.14}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Combining (3.13) and (3.14) using (IFN5), (IFN11), we arrive

$$\left.\begin{aligned} \mu\left(3\mathcal{F}(5w_1) - 15\mathcal{F}(w_1), 4\Lambda\right) &\geq \mu\left(3\mathcal{F}(5w_1) - 6\mathcal{F}\left(3w_1\right) + 3\mathcal{F}(w_1), \Lambda\right) * \mu\left(6\mathcal{F}(3w_1) - 18\mathcal{F}\left(w_1\right), 3\Lambda\right) \\ &\geq \mu'\left(\Psi\left(w_1, w_1, w_1\right), \Lambda\right) * \mu'\left(\Psi\left(w_1, w_1, -w_1\right), \Lambda\right) = \mu'\left(\Psi_A\left(w_1\right), \Lambda\right) \\ \nu\left(3\mathcal{F}(5w_1) - 15\mathcal{F}(w_1), 4\Lambda\right) &\leq \nu\left(3\mathcal{F}(5w_1) - 6\mathcal{F}\left(3w_1\right) + 3\mathcal{F}(w_1), \Lambda\right) \diamond \nu\left(6\mathcal{F}(3w_1) - 18\mathcal{F}\left(w_1\right), 3\Lambda\right) \\ &\leq \nu'\left(\Psi\left(w_1, w_1, w_1\right), \Lambda\right) \diamond \nu'\left(\Psi\left(w_1, w_1, -w_1\right), \Lambda\right) = \nu'\left(\Psi_A\left(w_1\right), \Lambda\right) \end{aligned}\right\} \tag{3.15}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Using (IFN4), (IFN10), one can see from (3.15) that

$$\left.\begin{aligned} \mu\left(\frac{1}{5}\mathcal{F}(5w_1) - \mathcal{F}(w_1), \frac{4}{3}\cdot\frac{1}{5}\Lambda\right) \geq \mu'\left(\Psi_A\left(w_1\right), \Lambda\right) \\ \nu\left(\frac{1}{5}\mathcal{F}(5w_1) - \mathcal{F}(w_1), \frac{4}{3}\cdot\frac{1}{5}\Lambda\right) \leq \nu'\left(\Psi_A\left(w_1\right), \Lambda\right) \end{aligned}\right\} \tag{3.16}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Changing w_1 by $5^\ell w_1$ in (3.16), and using (IFN4), (IFN10), (3.8), we get

$$\left.\begin{aligned} \mu\left(\frac{1}{5^{\ell+1}}\mathcal{F}(5^{\ell+1}w_1) - \frac{1}{5^\ell}\mathcal{F}(5^\ell w_1), \frac{4}{3\cdot 5}\cdot\frac{1}{5^\ell}\Lambda\right) &\geq \mu'\left(\Psi_A\left(5^\ell w_1\right), \Lambda\right) \geq \mu'\left(I^\ell \Psi_A\left(w_1\right), \Lambda\right) \\ &= \mu'\left(\Psi_A\left(w_1\right), \frac{1}{I^\ell}\Lambda\right) \\ \nu\left(\frac{1}{5^{\ell+1}}\mathcal{F}(5^{\ell+1}w_1) - \frac{1}{5^\ell}\mathcal{F}(5^\ell w_1), \frac{4}{3\cdot 5}\cdot\frac{1}{5^\ell}\Lambda\right) &\leq \nu'\left(\Psi_A\left(5^\ell w_1\right), \Lambda\right) \leq \nu'\left(I^\ell \Psi_A\left(w_1\right), \Lambda\right) \\ &= \nu'\left(\Psi_A\left(w_1\right), \frac{1}{I^\ell}\Lambda\right) \end{aligned}\right\} \tag{3.17}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$ also $\ell > 0$. Changing Λ by $I^\ell \Lambda$ in (3.17), we see

$$\left.\begin{aligned} \mu\left(\frac{1}{5^{\ell+1}}\mathcal{F}(5^{\ell+1}w_1) - \frac{1}{5^\ell}\mathcal{F}(5^\ell w_1), \frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^\ell \Lambda\right) \geq \mu'\left(\Psi_A\left(w_1\right), \Lambda\right) \\ \nu\left(\frac{1}{5^{\ell+1}}\mathcal{F}(5^{\ell+1}w_1) - \frac{1}{5^\ell}\mathcal{F}(5^\ell w_1), \frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^\ell \Lambda\right) \leq \nu'\left(\Psi_A\left(w_1\right), \Lambda\right) \end{aligned}\right\} \tag{3.18}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. It is easy to check that

$$\frac{1}{5^\ell}\mathcal{F}(5^\ell w_1) - \mathcal{F}(w_1) = \sum_{\eta=0}^{\ell-1} \frac{1}{5^{\eta+1}}\mathcal{F}(5^{\eta+1}w_1) - \frac{1}{5^\eta}\mathcal{F}(5^\eta w_1) \tag{3.19}$$

for all $w_1 \in \mathcal{W}_1$. Using (IFN5), (IFN11), it follows from (3.18) and (3.19), we obtain

$$\left.\begin{aligned} &\mu\left(\frac{1}{5^\ell}\mathcal{F}(5^\ell w_1) - \mathcal{F}(w_1), \sum_{\eta=0}^{\ell-1}\frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^\eta \Lambda\right) \\ &\qquad = \mu\left(\sum_{\eta=0}^{\ell-1}\frac{1}{5^{\eta+1}}\mathcal{F}(5^{\eta+1}w_1) - \frac{1}{5^\eta}\mathcal{F}(5^\eta w_1), \sum_{\eta=0}^{\ell-1}\frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^\eta \Lambda\right) \\ &\qquad \geq \prod_{\eta=0}^{\ell-1}\mu\left(\frac{1}{5^{\eta+1}}\mathcal{F}(5^{\eta+1}w_1) - \frac{1}{5^\eta}\mathcal{F}(5^\eta w_1), \frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^\eta \Lambda\right) \\ &\qquad \geq \textstyle\prod_{\eta=0}^{\ell-1}\mu'\left(\Psi_A\left(w_1\right), \Lambda\right) = \mu'\left(\Psi_A\left(w_1\right), \Lambda\right) \\ &\nu\left(\frac{1}{5^\ell}\mathcal{F}(5^\ell w_1) - \mathcal{F}(w_1), \sum_{\eta=0}^{\ell-1}\frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^\eta \Lambda\right) \\ &\qquad = \nu\left(\sum_{\eta=0}^{\ell-1}\frac{1}{5^{\eta+1}}\mathcal{F}(5^{\eta+1}w_1) - \frac{1}{5^\eta}\mathcal{F}(5^\eta w_1), \sum_{\eta=0}^{\ell-1}\frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^\eta \Lambda\right) \\ &\qquad \leq \coprod_{\eta=0}^{\ell-1}\nu\left(\frac{1}{5^{\eta+1}}\mathcal{F}(5^{\eta+1}w_1) - \frac{1}{5^\eta}\mathcal{F}(5^\eta w_1), \frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^\eta \Lambda\right) \\ &\qquad \leq \textstyle\coprod_{\eta=0}^{\ell-1}\nu'\left(\Psi_A\left(w_1\right), \Lambda\right) = \nu'\left(\Psi_A\left(w_1\right), \Lambda\right) \end{aligned}\right\} \tag{3.20}$$

where

$$\prod_{\eta=0}^{\ell-1} \mu = \mu * \mu * \mu * ... \quad and \quad \coprod_{\eta=0}^{\ell-1} \nu = \nu \diamond \nu \diamond \nu \diamond ...$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Again changing w_1 by $5^{\ell_1} w_1$ in (3.20), and using (IFN4), (IFN10), (3.8) in that changing Λ by $I^{\ell_1}\Lambda$, we have

$$\left.\begin{array}{l} \mu\left(\frac{1}{5^{\ell+\ell_1}}\mathcal{F}(5^{\ell+\ell_1}w_1) - \frac{1}{5^{\ell_1}}\mathcal{F}(\ell_1 w_1), \sum_{\eta=0}^{\ell-1}\frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^{\eta+\ell_1}\Lambda\right) \geq \mu'\left(\Psi_A\left(w_1\right), \Lambda\right) \\ \nu\left(\frac{1}{5^{\ell+\ell_1}}\mathcal{F}(5^{\ell+\ell_1}w_1) - \frac{1}{5^{\ell_1}}\mathcal{F}(\ell_1 w_1), \sum_{\eta=0}^{\ell-1}\frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^{\eta+\ell_1}\Lambda\right) \leq \nu'\left(\Psi_A\left(w_1\right), \Lambda\right) \end{array}\right\} \tag{3.21}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$ also $\ell, \ell_1 > 0$. It follows from (3.21) that

$$\left.\begin{array}{l} \mu\left(\frac{1}{5^{\ell+\ell_1}}\mathcal{F}(5^{\ell+\ell_1}w_1) - \frac{1}{5^{\ell_1}}\mathcal{F}(\ell_1 w_1), \Lambda\right) \geq \mu'\left(\Psi_A\left(w_1\right), \frac{\Lambda}{\sum_{\eta=0}^{\ell-1}\frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^{\eta+\ell_1}}\right) \\ \nu\left(\frac{1}{5^{\ell+\ell_1}}\mathcal{F}(5^{\ell+\ell_1}w_1) - \frac{1}{5^{\ell_1}}\mathcal{F}(\ell_1 w_1), \Lambda\right) \leq \nu'\left(\Psi_A\left(w_1\right), \frac{\Lambda}{\sum_{\eta=0}^{\ell-1}\frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^{\eta+\ell_1}}\right) \end{array}\right\} \tag{3.22}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. By data, the Cauchy criterion for convergence in Intuitionistic Fuzzy normed space gives that the sequence $\left\{\frac{1}{5^\ell}\mathcal{F}(5^\ell w_1)\right\}$, is Cauchy in $(\mathcal{W}_2, \mu', \nu')$ and it is a complete Intuitionistic Fuzzy normed space, this sequence converges to some point $\mathcal{A}(w_1)$ in $(\mathcal{W}_2, \mu', \nu')$ for all $w_1 \in \mathcal{W}_1$. So, by notation, we write

$$\left.\begin{array}{l} \lim_{\ell\to\infty}\mu\left(\frac{1}{5^\ell}\mathcal{F}\left(5^\ell w_1\right) - \mathcal{A}(w_1), \Lambda\right) = 1; \\ \lim_{\ell\to\infty}\nu\left(\frac{1}{5^\ell}\mathcal{F}\left(5^\ell w_1\right) - \mathcal{A}(w_1), \Lambda\right) = 0; \end{array}\right\} \tag{3.23}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Letting $\ell_1 = 0$ and $\ell \to \infty$ in (3.22) and using (3.23), we arrive

$$\left.\begin{array}{l} \mu\left(\mathcal{A}(w_1) - \mathcal{F}(w_1), \Lambda\right) \geq \mu'\left(\Psi_A\left(w_1\right), \frac{3\Lambda}{4}\left(5-I\right)\right) \\ \qquad = \mu'\left(\Psi\left(w_1, w_1, w_1\right), \frac{3\Lambda}{4}\left(5-I\right)\right) * \mu'\left(\Psi\left(w_1, w_1, -w_1\right), \frac{3\Lambda}{4}\left(5-I\right)\right) \\ \nu\left(\mathcal{A}(w_1) - \mathcal{F}(w_1), \Lambda\right) \leq \nu'\left(\Psi_A\left(w_1\right), \frac{3\Lambda}{4}\left(5-I\right)\right) \\ \qquad = \nu'\left(\Psi\left(w_1, w_1, w_1\right), \frac{3\Lambda}{4}\left(5-I\right)\right) \diamond \nu'\left(\Psi\left(w_1, w_1, -w_1\right), \frac{3\Lambda}{4}\left(5-I\right)\right) \end{array}\right\} \tag{3.24}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Thus, (3.10) and (3.11) holds for $\mu = 1$. Interchanging

$$(w_1, w_2, w_3) = \left(5^\ell w_1, 5^\ell w_2, 5^\ell w_3\right),$$

in (3.1) and using (IFN4), (IFN10), we have

$$\left.\begin{array}{l} \mu\left(\frac{1}{5^\ell}\left\{\mathcal{F}(5^\ell(3w_1+w_2+w_3)) + \mathcal{F}(5^\ell(w_1+3w_2+w_3)) + \mathcal{F}(5^\ell(w_1+w_2+3w_3)) - 6\mathcal{F}\left(\sum_{\psi=1}^3 5^\ell w_\psi\right)\right.\right. \\ \quad \left.\left. -\frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^3 5^\ell w_\psi\right) + \mathcal{F}\left(-\sum_{\psi=1}^3 5^\ell w_\psi\right)\right\} + \sum_{\psi=1}^3\left\{\mathcal{F}(5^\ell w_\psi) - \frac{5}{2}\left[\mathcal{F}(5^\ell w_\psi) + \mathcal{F}(-5^\ell w_\psi)\right]\right\}\right\}, \Lambda\right) \\ \qquad \geq \mu'\left(\Psi\left(5^\ell w_1, 5^\ell w_2, 5^\ell w_3\right), 5^\ell\Lambda\right) \\ \nu\left(\frac{1}{5^\ell}\left\{\mathcal{F}(5^\ell(3w_1+w_2+w_3)) + \mathcal{F}(5^\ell(w_1+3w_2+w_3)) + \mathcal{F}(5^\ell(w_1+w_2+3w_3)) - 6\mathcal{F}\left(\sum_{\psi=1}^3 5^\ell w_\psi\right)\right.\right. \\ \quad \left.\left. -\frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^3 5^\ell w_\psi\right) + \mathcal{F}\left(-\sum_{\psi=1}^3 5^\ell w_\psi\right)\right\} + \sum_{\psi=1}^3\left\{\mathcal{F}(5^\ell w_\psi) - \frac{5}{2}\left[\mathcal{F}(5^\ell w_\psi) + \mathcal{F}(-5^\ell w_\psi)\right]\right\}\right\}, \Lambda\right) \\ \qquad \leq \nu'\left(\Psi\left(5^\ell w_1, 5^\ell w_2, 5^\ell w_3\right), 5^\ell\Lambda\right) \end{array}\right\} \tag{3.25}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$. Now,

$$\left.\begin{array}{l}
\mu\left(\mathcal{A}(3w_1+w_2+w_3)+\mathcal{A}(w_1+3w_2+w_3)+\mathcal{A}(w_1+w_2+3w_3)-6\mathcal{A}\left(\sum_{\psi=1}^{3} w_\psi\right)\right.\\
\left.-\frac{1}{2}\left\{\mathcal{A}\left(\sum_{\psi=1}^{3} w_\psi\right)-\mathcal{A}\left(-\sum_{\psi=1}^{3} w_\psi\right)\right\}+\sum_{\psi=1}^{3}\left\{\mathcal{A}(w_\psi)-\frac{5}{2}\Big[\mathcal{A}(w_\psi)+\mathcal{A}(-w_\psi)\Big]\right\},\Lambda\right)\\
\geq \mu\left(\mathcal{A}(3w_1+w_2+w_3)-\frac{1}{5^\ell}\mathcal{F}(5^\ell(3w_1+w_2+w_3)),\frac{\Lambda}{7}\right)*\\
\mu\left(\mathcal{A}(w_1+3w_2+w_3)-\frac{1}{5^\ell}\mathcal{F}(5^\ell(w_1+3w_2+w_3)),\frac{\Lambda}{7}\right)*\\
\mu\left(\mathcal{A}(w_1+w_2+3w_3)-\frac{1}{5^\ell}\mathcal{F}(5^\ell(w_1+w_2+3w_3)),\frac{\Lambda}{7}\right)*\\
\quad\mu\left(-6\mathcal{A}\left(\sum_{\psi=1}^{3} w_\psi\right)+\frac{1}{5^\ell}6\mathcal{F}\left(\sum_{\psi=1}^{3}5^\ell w_\psi,\frac{\Lambda}{7}\right)\right)*\\
\quad\mu\left(\frac{1}{2}\left\{\mathcal{A}\left(\sum_{\psi=1}^{3} w_\psi\right)+\mathcal{A}\left(-\sum_{\psi=1}^{3} w_\psi\right)\right\}\right.\\
\qquad\left.-\frac{1}{5^\ell}\frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3}5^\ell w_\psi\right)+\mathcal{F}\left(-\sum_{\psi=1}^{3}5^\ell w_\psi\right)\right\},\frac{\Lambda}{7}\right)*\\
\quad\mu\left(\sum_{\psi=1}^{3}\left\{\mathcal{A}(w_\psi)-\frac{5}{2}\Big[\mathcal{A}(w_\psi)+\mathcal{A}(-w_\psi)\Big]\right\}\right.\\
\qquad\left.-\frac{1}{5^\ell}\sum_{\psi=1}^{3}\left\{\mathcal{F}(5^\ell w_\psi)-\frac{5}{2}\Big[\mathcal{F}(5^\ell w_\psi)+\mathcal{F}(-5^\ell w_\psi)\Big]\right\},\frac{\Lambda}{7}\right)*\\
\quad\mu\left(\frac{1}{5^\ell}\left\{\mathcal{F}(5^\ell(3w_1+w_2+w_3))+\mathcal{F}(5^\ell(w_1+3w_2+w_3))+\mathcal{F}(5^\ell(w_1+w_2+3w_3))\right.\right.\\
\qquad-6\mathcal{F}\left(\sum_{\psi=1}^{3}5^\ell w_\psi\right)-\frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3}5^\ell w_\psi\right)+\mathcal{F}\left(-\sum_{\psi=1}^{3}5^\ell w_\psi\right)\right\}\\
\qquad\left.\left.+\sum_{\psi=1}^{3}\left\{\mathcal{F}(5^\ell w_\psi)-\frac{5}{2}\Big[\mathcal{F}(5^\ell w_\psi)+\mathcal{F}(-5^\ell w_\psi)\Big]\right\}\right\},\frac{\Lambda}{7}\right)\\
\nu\left(\mathcal{A}(3w_1+w_2+w_3)+\mathcal{A}(w_1+3w_2+w_3)+\mathcal{A}(w_1+w_2+3w_3)-6\mathcal{A}\left(\sum_{\psi=1}^{3} w_\psi\right)\right.\\
\left.-\frac{1}{2}\left\{\mathcal{A}\left(\sum_{\psi=1}^{3} w_\psi\right)-\mathcal{A}\left(-\sum_{\psi=1}^{3} w_\psi\right)\right\}+\sum_{\psi=1}^{3}\left\{\mathcal{A}(w_\psi)-\frac{5}{2}\Big[\mathcal{A}(w_\psi)+\mathcal{A}(-w_\psi)\Big]\right\},\Lambda\right)\\
\leq \nu\left(\mathcal{A}(3w_1+w_2+w_3)-\frac{1}{5^\ell}\mathcal{F}(5^\ell(3w_1+w_2+w_3)),\frac{\Lambda}{7}\right)\diamond\\
\nu\left(\mathcal{A}(w_1+3w_2+w_3)-\frac{1}{5^\ell}\mathcal{F}(5^\ell(w_1+3w_2+w_3)),\frac{\Lambda}{7}\right)\diamond\\
\nu\left(\mathcal{A}(w_1+w_2+3w_3)-\frac{1}{5^\ell}\mathcal{F}(5^\ell(w_1+w_2+3w_3)),\frac{\Lambda}{7}\right)\diamond\\
\quad\nu\left(-6\mathcal{A}\left(\sum_{\psi=1}^{3} w_\psi\right)+\frac{1}{5^\ell}6\mathcal{F}\left(\sum_{\psi=1}^{3}5^\ell w_\psi,\frac{\Lambda}{7}\right)\right)\diamond\\
\quad\nu\left(\frac{1}{2}\left\{\mathcal{A}\left(\sum_{\psi=1}^{3} w_\psi\right)+\mathcal{A}\left(-\sum_{\psi=1}^{3} w_\psi\right)\right\}\right.\\
\qquad\left.-\frac{1}{5^\ell}\frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3}5^\ell w_\psi\right)+\mathcal{F}\left(-\sum_{\psi=1}^{3}5^\ell w_\psi\right)\right\},\frac{\Lambda}{7}\right)\diamond\\
\quad\nu\left(\sum_{\psi=1}^{3}\left\{\mathcal{A}(w_\psi)-\frac{5}{2}\Big[\mathcal{A}(w_\psi)+\mathcal{A}(-w_\psi)\Big]\right\}\right.\\
\qquad\left.-\frac{1}{5^\ell}\sum_{\psi=1}^{3}\left\{\mathcal{F}(5^\ell w_\psi)-\frac{5}{2}\Big[\mathcal{F}(5^\ell w_\psi)+\mathcal{F}(-5^\ell w_\psi)\Big]\right\},\frac{\Lambda}{7}\right)\diamond\\
\quad\nu\left(\frac{1}{5^\ell}\left\{\mathcal{F}(5^\ell(3w_1+w_2+w_3))+\mathcal{F}(5^\ell(w_1+3w_2+w_3))+\mathcal{F}(5^\ell(w_1+w_2+3w_3))\right.\right.\\
\qquad-6\mathcal{F}\left(\sum_{\psi=1}^{3}5^\ell w_\psi\right)-\frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3}5^\ell w_\psi\right)+\mathcal{F}\left(-\sum_{\psi=1}^{3}5^\ell w_\psi\right)\right\}\\
\qquad\left.\left.+\sum_{\psi=1}^{3}\left\{\mathcal{F}(5^\ell w_\psi)-\frac{5}{2}\Big[\mathcal{F}(5^\ell w_\psi)+\mathcal{F}(-5^\ell w_\psi)\Big]\right\}\right\},\frac{\Lambda}{7}\right)
\end{array}\right\} \quad (3.26)$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$. Taking limit $\ell \to \infty$ in (3.26), using (3.23) and (3.25), we get

$$\left.\begin{array}{l}
\mu\left(\mathcal{A}(3w_1+w_2+w_3)+\mathcal{A}(w_1+3w_2+w_3)+\mathcal{A}(w_1+w_2+3w_3)-6\mathcal{A}\left(\sum_{\psi=1}^{3} w_\psi\right)\right.\\
\left.-\frac{1}{2}\left\{\mathcal{A}\left(\sum_{\psi=1}^{3} w_\psi\right)+\mathcal{A}\left(-\sum_{\psi=1}^{3} w_\psi\right)\right\}+\sum_{\psi=1}^{3}\left\{\mathcal{A}(w_\psi)-\frac{5}{2}\Big[\mathcal{A}(w_\psi)+\mathcal{A}(-w_\psi)\Big]\right\},\Lambda\right)=1\\
\nu\left(\mathcal{A}(3w_1+w_2+w_3)+\mathcal{A}(w_1+3w_2+w_3)+\mathcal{A}(w_1+w_2+3w_3)-6\mathcal{A}\left(\sum_{\psi=1}^{3} w_\psi\right)\right.\\
\left.-\frac{1}{2}\left\{\mathcal{A}\left(\sum_{\psi=1}^{3} w_\psi\right)+\mathcal{A}\left(-\sum_{\psi=1}^{3} w_\psi\right)\right\}+\sum_{\psi=1}^{3}\left\{\mathcal{A}(w_\psi)-\frac{5}{2}\Big[\mathcal{A}(w_\psi)+\mathcal{A}(-w_\psi)\Big]\right\},\Lambda\right)=0
\end{array}\right\} \quad (3.27)$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$. Using (IFN3), (IFN9) in (3.27), we see, $\mathcal{A}(w_1)$ satisfies (1.7). In order to confirm that $\mathcal{A}(w_1)$ is unique, suppose $\mathcal{B}(w_1)$ be another mapping (1.7), (3.23) and (3.24), we obtain

$$\left.\begin{aligned}\mu\left(\mathcal{A}(w_1)-\mathcal{B}(w_1),2\Lambda\right)&=\mu\left(\mathcal{A}\left(5^\ell w_1\right)-\mathcal{B}\left(5^\ell w_1\right),5^\ell\,2\Lambda\right)\\&\geq\mu\left(\mathcal{A}\left(5^\ell w_1\right)-\mathcal{F}\left(5^\ell w_1\right),5^\ell\Lambda\right)*\mu\left(\mathcal{F}\left(5^\ell w_1\right)-\mathcal{B}\left(5^\ell w_1\right),5^\ell\Lambda\right)\\&\geq\mu'\left(\Psi_A\left(5^\ell w_1\right),\tfrac{3\Lambda}{4}\,5^\ell(5-I)\right)*\mu'\left(\Psi_A\left(5^\ell w_1\right),\tfrac{3\Lambda}{4}\,5^\ell(5-I)\right)\\&\geq\mu'\left(\Psi_A\left(w_1\right),\tfrac{3\Lambda}{4}\tfrac{5^\ell}{I^\ell}\left(5-I\right)\right)\\\nu\left(\mathcal{A}(w_1)-\mathcal{B}(w_1),2\Lambda\right)&=\nu\left(\mathcal{A}\left(5^\ell w_1\right)-\mathcal{B}\left(5^\ell w_1\right),5^\ell\,2\Lambda\right)\\&\leq\nu\left(\mathcal{A}\left(5^\ell w_1\right)-\mathcal{F}\left(5^\ell w_1\right),5^\ell\Lambda\right)\diamond\nu\left(\mathcal{F}\left(5^\ell w_1\right)-\mathcal{B}\left(5^\ell w_1\right),5^\ell\Lambda\right)\\&\leq\nu'\left(\Psi_A\left(5^\ell w_1\right),\tfrac{3\Lambda}{4}\,5^\ell(5-I)\right)\diamond\nu'\left(\Psi_A\left(5^\ell w_1\right),\tfrac{3\Lambda}{4}\,5^\ell(5-I)\right)\\&\leq\nu'\left(\Psi_A\left(w_1\right),\tfrac{3\Lambda}{4}\tfrac{5^\ell}{I^\ell}\left(5-I\right)\right)\end{aligned}\right\}\tag{3.28}$$

for all $w_1\in\mathcal{W}_1$ and all $\Lambda>0$. Taking limit $\ell\to\infty$ in (3.28), and using (IFN7), (IFN13), we arrive

$$\left.\begin{aligned}\mu\left(\mathcal{A}(w_1)-\mathcal{B}(w_1),2\Lambda\right)=1\\\nu\left(\mathcal{A}(w_1)-\mathcal{B}(w_1),2\Lambda\right)=0\end{aligned}\right\}\tag{3.29}$$

for all $w_1\in\mathcal{W}_1$ and all $\Lambda>0$. By (IFN4) and (IFN10), we get $\mathcal{A}(w_1)$ is unique. So, the Theorem holds for $m=1$.

Changing $w_1=\dfrac{w_1}{5}$ in (3.15) and using (IFN4), (IFN10), (3.8), in that changing Λ by $\frac{\Lambda}{I}$, we have

$$\left.\begin{aligned}\mu\left(\mathcal{F}(w_1)-5\mathcal{F}\left(\frac{w_1}{5}\right),\frac{4}{3\cdot I}\Lambda\right)\geq\mu'\left(\Psi_A\left(w_1\right),\Lambda\right)\\\nu\left(\mathcal{F}(w_1)-5\mathcal{F}\left(\frac{w_1}{5}\right),\frac{4}{3\cdot I}\Lambda\right)\leq\nu'\left(\Psi_A\left(w_1\right),\Lambda\right)\end{aligned}\right\}\tag{3.30}$$

for all $w_1\in\mathcal{W}_1$ and all $\Lambda>0$. Changing w_1 by $\dfrac{w_1}{5^\ell}$ in (3.30), and using (IFN4), (IFN10), (3.8) in that changing Λ by $\dfrac{\Lambda}{I^\ell}$, we get

$$\left.\begin{aligned}\mu\left(5^\ell\mathcal{F}\left(\frac{w_1}{5^\ell}\right)-5^{\ell+1}\mathcal{F}\left(\frac{w_1}{5^{\ell+1}}\right),\frac{4}{3\cdot I}\left(\frac{5}{I}\right)^\ell\Lambda\right)\leq\mu'\left(\Psi_A\left(w_1\right),\Lambda\right)\\\nu\left(5^\ell\mathcal{F}\left(\frac{w_1}{5^\ell}\right)-5^{\ell+1}\mathcal{F}\left(\frac{w_1}{5^{\ell+1}}\right),\frac{4}{3\cdot I}\left(\frac{5}{I}\right)^\ell\Lambda\right)\leq\nu'\left(\Psi_A\left(w_1\right),\Lambda\right)\end{aligned}\right\}\tag{3.31}$$

for all $w_1\in\mathcal{W}_1$ and all $\Lambda>0$ also $\ell>0$. It is easy to check that

$$\mathcal{F}(w_1)-5^\ell\mathcal{F}\left(\frac{w_1}{5^\ell}\right)=\sum_{\eta=1}^{\ell}5^{\eta-1}\mathcal{F}\left(\frac{w_1}{5^{\eta-1}}\right)-5^\eta\mathcal{F}\left(\frac{w_1}{5^\eta}\right)\tag{3.32}$$

for all $w_1\in\mathcal{W}_1$. The rest of the proof is similar to that of above case. So, the Theorem holds for $m=-1$. Hence the proof is complete. □

Corollary 3.9. *Suppose that an odd function $\mathcal{F}:\mathcal{W}_1\to\mathcal{W}_2$ satisfy the functional inequality (3.2) for all $w_1,w_2,w_3\in\mathcal{W}_1$ and all $\Lambda>0$ with δ be a positive constant and φ be any real number. Then there exists a unique additive mapping $\mathcal{A}(w_1):\mathcal{W}_1\to\mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\left.\begin{aligned}\mu\left(\mathcal{A}(w_1)-\mathcal{F}(w_1),\Lambda\right)\geq\mu'\left(\delta,|3|\,\Lambda\right),\\\nu\left(\mathcal{A}(w_1)-\mathcal{F}(w_1),\Lambda\right)\leq\nu'\left(\delta,|3|\,\Lambda\right),\end{aligned}\right\}\tag{3.33}$$

for all $w_1\in\mathcal{W}_1$.

Corollary 3.10. *Suppose that an odd function $\mathcal{F}:\mathcal{W}_1\to\mathcal{W}_2$ satisfy the functional inequality (3.3) for all $w_1,w_2,w_3\in\mathcal{W}_1$ and all $\Lambda>0$ with δ be a positive constant and φ be any real number. Then there exists a* *unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\left.\begin{aligned} \mu\left(\mathcal{A}(w_1) - \mathcal{F}(w_1), \Lambda\right) &\geq \mu'\left(\delta|w_1|^{\varphi}, \tfrac{\Lambda}{4}\left|5 - 5^{\varphi}\right|\right), \quad \varphi \neq 1, \\ \nu\left(\mathcal{A}(w_1) - \mathcal{F}(w_1), \Lambda\right) &\leq \nu'\left(\delta|w_1|^{\varphi}, \tfrac{\Lambda}{4}\left|5 - 5^{\varphi}\right|\right), \quad \varphi \neq 1, \end{aligned}\right\} \tag{3.34}$$

for all $w_1 \in \mathcal{W}_1$.

Corollary 3.11. *Suppose that an odd function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (3.4) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with δ be a positive constant and φ be any real number. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\left.\begin{aligned} \mu\left(\mathcal{A}(w_1) - \mathcal{F}(w_1), \Lambda\right) &\geq \mu'\left(\delta \textstyle\sum_{\psi=1}^{3} |w_1|^{\varphi_\psi}, \tfrac{3\Lambda}{4} \textstyle\sum_{\psi=1}^{3} \left|5 - 5^{\varphi_\psi}\right|\right), \quad \varphi_1, \varphi_2, \varphi_3 \neq 1, \\ \nu\left(\mathcal{A}(w_1) - \mathcal{F}(w_1), \Lambda\right) &\leq \nu'\left(\delta \textstyle\sum_{\psi=1}^{3} |w_1|^{\varphi_\psi}, \tfrac{3\Lambda}{4} \textstyle\sum_{\psi=1}^{3} \left|5 - 5^{\varphi_\psi}\right|\right), \quad \varphi_1, \varphi_2, \varphi_3 \neq 1, \end{aligned}\right\} \tag{3.35}$$

for all $w_1 \in \mathcal{W}_1$.

Corollary 3.12. *Suppose that an odd function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (3.5) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with δ be a positive constant and φ be any real number. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\left.\begin{aligned} \mu\left(\mathcal{A}(w_1) - \mathcal{F}(w_1), \Lambda\right) &\geq \mu'\left(\delta|w_1|^{3\varphi}, \tfrac{3\Lambda}{4}\left|5 - 5^{3\varphi}\right|\right), \quad 3\varphi \neq 1, \\ \nu\left(\mathcal{A}(w_1) - \mathcal{F}(w_1), \Lambda\right) &\leq \nu'\left(\delta|w_1|^{3\varphi}, \tfrac{3\Lambda}{4}\left|5 - 5^{3\varphi}\right|\right), \quad 3\varphi \neq 1, \end{aligned}\right\} \tag{3.36}$$

for all $w_1 \in \mathcal{W}_1$.

Corollary 3.13. *Suppose that an odd function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (3.6) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with δ be a positive constant and φ be any real number. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\left.\begin{aligned} \mu\left(\mathcal{A}(w_1) - \mathcal{F}(w_1), \Lambda\right) &\geq \mu'\left(\delta|w_\psi|^{\sum_{\psi=1}^{3}\varphi_\psi}, \tfrac{3\Lambda}{4}\left|5 - 5^{\sum_{\psi=1}^{3}\varphi_\psi}\right|\right), \quad \textstyle\sum_{\psi=1}^{3}\varphi_\psi \neq 1, \\ \nu\left(\mathcal{A}(w_1) - \mathcal{F}(w_1), \Lambda\right) &\leq \nu'\left(\delta|w_\psi|^{\sum\text{-}\psi=1^3\varphi_\psi}, \tfrac{3\Lambda}{4}\left|5 - 5^{\sum_{\psi=1}^{3}\varphi_\psi}\right|\right), \quad \textstyle\sum_{\psi=1}^{3}\varphi_\psi \neq 1, \end{aligned}\right\} \tag{3.37}$$

for all $w_1 \in \mathcal{W}_1$.

Corollary 3.14. *Suppose that an odd function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (3.7) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with δ be a positive constant and φ be any real number. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\left.\begin{aligned} \mu\left(\mathcal{A}(w_1) - \mathcal{F}(w_1), \Lambda\right) &\geq \mu'\left(2\delta|w_1|^{3\varphi}, \tfrac{3\Lambda}{4}\left|5 - 5^{3\varphi}\right|\right), \quad 3\varphi \neq 1, \\ \nu\left(\mathcal{A}(w_1) - \mathcal{F}(w_1), \Lambda\right) &\leq \nu'\left(2\delta|w_1|^{3\varphi}, \tfrac{3\Lambda}{4}\left|5 - 5^{3\varphi}\right|\right), \quad 3\varphi \neq 1, \end{aligned}\right\} \tag{3.38}$$

for all $w_1 \in \mathcal{W}_1$.

3.3. **Evenness of $\mathcal{F}$: Quadratic Case Stability Results : Direct Method.**

Theorem 3.15. *Suppose that an even function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (3.1) where $\Psi : \mathcal{W}_1^3 \to [0, \infty)$ with the conditions (3.8) and*

$$\left.\begin{aligned} \lim_{\ell\to\infty} \mu'\left(\Psi\left(5^{\ell m}w_1, 5^{\ell m}w_2, 5^{\ell m}w_3\right), 25^{\ell m}\Lambda\right) &= 1 \\ \lim_{\ell\to\infty} \nu'\left(\Psi\left(5^{\ell m}w_1, 5^{\ell m}w_2, 5^{\ell m}w_3\right), 25^{\ell m}\Lambda\right) &= 0 \end{aligned}\right\} \tag{3.39}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with $m = \pm 1$ and $0 < \left(\frac{I}{25}\right)^{\mu} < 1$. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality

$$\left.\begin{aligned} \mu\left(\mathcal{Q}(w_1)-\mathcal{F}(w_1),\Lambda\right) &\geq \mu'\left(\Psi_Q\left(w_1\right), \frac{7\Lambda}{3}\left|25-I\right|\right) \\ &= \mu'\left(\Psi\left(w_1,w_1,w_1\right), \frac{7\Lambda}{3}\left|25-I\right|\right) * \mu'\left(\Psi\left(w_1,w_1,-w_1\right), \frac{7\Lambda}{3}\left|25-I\right|\right) \\ \nu\left(\mathcal{Q}(w_1)-\mathcal{F}(w_1),\Lambda\right) &\leq \nu'\left(\Psi_Q\left(w_1\right), \frac{7\Lambda}{3}\left(25-I\right)\right) \\ &= \nu'\left(\Psi\left(w_1,w_1,w_1\right), \frac{7\Lambda}{3}\left(25-I\right)\right) \diamond \nu'\left(\Psi\left(w_1,w_1,-w_1\right), \frac{7\Lambda}{3}\left(25-I\right)\right) \end{aligned}\right\} \tag{3.40}$$

and the mapping $\mathcal{Q}(w_1)$ is obtained by

$$\left.\begin{aligned} \lim_{\ell\to\infty}\mu\left(\frac{1}{25^{\ell m}}\mathcal{F}\left(5^{\ell m}w_1\right)-\mathcal{Q}(w_1),\Lambda\right) &= 1 \\ \lim_{\ell\to\infty}\nu\left(\frac{1}{25^{\ell m}}\mathcal{F}\left(5^{\ell m}w_1\right)-\mathcal{Q}(w_1),\Lambda\right) &= 0 \end{aligned}\right\} \tag{3.41}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$.

Proof. Using evenness of $\mathcal{F}$ in (2.1), we get

$$\left.\begin{aligned} &\mu\left(\mathcal{F}(3w_1+w_2+w_3)+\mathcal{F}(w_1+3w_2+w_3)+\mathcal{F}(w_1+w_2+3w_3)-7\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)-4\sum_{\psi=1}^{3}\mathcal{F}(w_\psi),\Lambda\right) \\ &\qquad\qquad\geq \mu'\left(\Psi\left(w_1,w_2,w_3\right),\Lambda\right) \\ &\nu\left(\mathcal{F}(3w_1+w_2+w_3)+\mathcal{F}(w_1+3w_2+w_3)+\mathcal{F}(w_1+w_2+3w_3)-7\mathcal{F}\left(\sum_{\psi=1}^{3}w_\psi\right)-4\sum_{\psi=1}^{3}\mathcal{F}(w_\psi),\Lambda\right) \\ &\qquad\qquad\leq \nu'\left(\Psi\left(w_1,w_2,w_3\right),\Lambda\right) \end{aligned}\right\} \tag{3.42}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$. Interchanging (w_1, w_2, w_3) by (w_1, w_1, w_1) in (3.42), we obtain

$$\left.\begin{aligned} \mu\left(3\mathcal{F}(5w_1)-7\mathcal{F}\left(3w_1\right)-12\mathcal{F}(w_1),\Lambda\right) &\geq \mu'\left(\Psi\left(w_1,w_1,w_1\right),\Lambda\right) \\ \nu\left(3\mathcal{F}(5w_1)-7\mathcal{F}\left(3w_1\right)-12\mathcal{F}(w_1),\Lambda\right) &\leq \nu'\left(\Psi\left(w_1,w_1,w_1\right),\Lambda\right) \end{aligned}\right\} \tag{3.43}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Again interchanging (w_1, w_2, w_3) by $(w_1, w_1, -w_1)$ in (3.42) and using (IFN4), (IFN10), we have

$$\left.\begin{aligned} \mu\left(2\mathcal{F}(3w_1)-18\mathcal{F}\left(w_1\right),\Lambda\right) &\geq \mu'\left(\Psi\left(w_1,w_1,-w_1\right),\Lambda\right) \\ \nu\left(2\mathcal{F}(3w_1)-18\mathcal{F}\left(w_1\right),\Lambda\right) &\leq \nu'\left(\Psi\left(w_1,w_1,-w_1\right),\Lambda\right) \end{aligned}\right\}$$

$$\Rightarrow \left.\begin{aligned} \mu\left(7\mathcal{F}(3w_1)-63\mathcal{F}\left(w_1\right),\tfrac{2}{7}\Lambda\right) &\geq \mu'\left(\Psi\left(w_1,w_1,-w_1\right),\Lambda\right) \\ \nu\left(7\mathcal{F}(3w_1)-63\mathcal{F}\left(w_1\right),\tfrac{2}{7}\Lambda\right) &\leq \nu'\left(\Psi\left(w_1,w_1,-w_1\right),\Lambda\right) \end{aligned}\right\} \tag{3.44}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Combining (3.43) and (3.44) using (IFN5), (IFN11), we arrive

$$\left.\begin{aligned} \mu\left(3\mathcal{F}(5w_1)-75\mathcal{F}(w_1),\tfrac{9}{7}\Lambda\right) &\geq \mu\left(3\mathcal{F}(5w_1)-7\mathcal{F}\left(3w_1\right)-12\mathcal{F}(w_1),\Lambda\right) * \mu\left(7\mathcal{F}(3w_1)-63\mathcal{F}\left(w_1\right),\tfrac{2}{7}\Lambda\right) \\ &\geq \mu'\left(\Psi\left(w_1,w_1,w_1\right),\Lambda\right) * \mu'\left(\Psi\left(w_1,w_1,-w_1\right),\Lambda\right) = \mu'\left(\Psi_Q\left(w_1\right),\Lambda\right) \\ \nu\left(3\mathcal{F}(5w_1)-15\mathcal{F}(w_1),\tfrac{9}{7}\Lambda\right) &\leq \nu\left(3\mathcal{F}(5w_1)-7\mathcal{F}\left(3w_1\right)-12\mathcal{F}(w_1),\Lambda\right) \diamond \nu\left(7\mathcal{F}(3w_1)-63\mathcal{F}\left(w_1\right),\tfrac{2}{7}\Lambda\right) \\ &\leq \nu'\left(\Psi\left(w_1,w_1,w_1\right),\Lambda\right) \diamond \nu'\left(\Psi\left(w_1,w_1,-w_1\right),\Lambda\right) = \nu'\left(\Psi_Q\left(w_1\right),\Lambda\right) \end{aligned}\right\} \tag{3.45}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Using (IFN4), (IFN10), one can see from (3.45) that

$$\left.\begin{aligned} \mu\left(\frac{1}{25}\mathcal{F}(5w_1)-\mathcal{F}(w_1),\frac{9}{7\cdot 3}\cdot\frac{1}{25}\Lambda\right) &\geq \mu'\left(\Psi_Q\left(w_1\right),\Lambda\right) \\ \nu\left(\frac{1}{25}\mathcal{F}(5w_1)-\mathcal{F}(w_1),\frac{9}{7\cdot 3}\cdot\frac{1}{25}\Lambda\right) &\leq \nu'\left(\Psi_Q\left(w_1\right),\Lambda\right) \end{aligned}\right\} \tag{3.46}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. The rest of the proof is similar to that of Theorem 3.8. Hence the proof is complete. □

Corollary 3.16. *Suppose that an even function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (3.2) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with δ be a positive constant and φ be any real number. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\left.\begin{array}{l} \mu\left(\mathcal{Q}(w_1) - \mathcal{F}(w_1), \Lambda\right) \geq \mu'\left(\delta, |8|\, 7\Lambda\right), \\ \nu\left(\mathcal{Q}(w_1) - \mathcal{F}(w_1), \Lambda\right) \leq \nu'\left(\delta, |8|\, 7\Lambda\right), \end{array}\right\} \tag{3.47}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$.

Corollary 3.17. *Suppose that an even function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (3.3) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with δ be a positive constant and φ be any real number. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\left.\begin{array}{ll} \mu\left(\mathcal{Q}(w_1) - \mathcal{F}(w_1), \Lambda\right) \geq \mu'\left(\delta|w_1|^{\varphi}, \frac{7\Lambda}{9}\left|25 - 5^{\varphi}\right|\right), & \varphi \neq 2, \\ \nu\left(\mathcal{Q}(w_1) - \mathcal{F}(w_1), \Lambda\right) \leq \nu'\left(\delta|w_1|^{\varphi}, \frac{7\Lambda}{9}\left|25 - 5^{\varphi}\right|\right), & \varphi \neq 2, \end{array}\right\} \tag{3.48}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$.

Corollary 3.18. *Suppose that an even function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (3.4) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with δ be a positive constant and φ be any real number. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\left.\begin{array}{ll} \mu\left(\mathcal{Q}(w_1) - \mathcal{F}(w_1), \Lambda\right) \geq \mu'\left(\delta \sum_{\psi=1}^{3} |w_1|^{\varphi_\psi}, \frac{7\Lambda}{3} \sum_{\psi=1}^{3}\left|25 - 5^{\varphi_\psi}\right|\right), & \varphi_1, \varphi_2, \varphi_3 \neq 2, \\ \nu\left(\mathcal{Q}(w_1) - \mathcal{F}(w_1), \Lambda\right) \leq \nu'\left(\delta \sum_{\psi=1}^{3} |w_1|^{\varphi_\psi}, \frac{7\Lambda}{3} \sum_{\psi=1}^{3}\left|25 - 5^{\varphi_\psi}\right|\right), & \varphi_1, \varphi_2, \varphi_3 \neq 2, \end{array}\right\} \tag{3.49}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$.

Corollary 3.19. *Suppose that an even function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (3.5) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with δ be a positive constant and φ be any real number. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\left.\begin{array}{ll} \mu\left(\mathcal{Q}(w_1) - \mathcal{F}(w_1), \Lambda\right) \geq \mu'\left(\delta|w_1|^{3\varphi}, \frac{7\Lambda}{3}\left|25 - 5^{3\varphi}\right|\right), & 3\varphi \neq 2, \\ \nu\left(\mathcal{Q}(w_1) - \mathcal{F}(w_1), \Lambda\right) \leq \nu'\left(\delta|w_1|^{3\varphi}, \frac{7\Lambda}{3}\left|25 - 5^{3\varphi}\right|\right), & 3\varphi \neq 2, \end{array}\right\} \tag{3.50}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$.

Corollary 3.20. *Suppose that an even function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (3.6) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with δ be a positive constant and φ be any real number. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\left.\begin{array}{ll} \mu\left(\mathcal{Q}(w_1) - \mathcal{F}(w_1), \Lambda\right) \geq \mu'\left(\delta|w_\psi|^{\sum_{\psi=1}^{3}\varphi_\psi}, \frac{7\Lambda}{3}\left|25 - 5^{\sum_{\psi=1}^{3}\varphi_\psi}\right|\right), & \sum_{\psi=1}^{3}\varphi_\psi \neq 2, \\ \nu\left(\mathcal{Q}(w_1) - \mathcal{F}(w_1), \Lambda\right) \leq \nu'\left(\delta|w_\psi|^{\sum_{\psi=1}^{3}\varphi_\psi}, \frac{7\Lambda}{3}\left|25 - 5^{\sum_{\psi=1}^{3}\varphi_\psi}\right|\right), & \sum_{\psi=1}^{3}\varphi_\psi \neq 2, \end{array}\right\} \tag{3.51}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$.

Corollary 3.21. *Suppose that an even function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (3.7) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with δ be a positive constant and φ be any real number. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\left.\begin{array}{ll} \mu\left(\mathcal{Q}(w_1) - \mathcal{F}(w_1), \Lambda\right) \geq \mu'\left(2\delta|w_1|^{3\varphi}, \frac{7\Lambda}{3}\left|25 - 5^{3\varphi}\right|\right), & 3\varphi \neq 2, \\ \nu\left(\mathcal{Q}(w_1) - \mathcal{F}(w_1), \Lambda\right) \leq \nu'\left(2\delta|w_1|^{3\varphi}, \frac{7\Lambda}{3}\left|25 - 5^{3\varphi}\right|\right), & 3\varphi \neq 2, \end{array}\right\} \tag{3.52}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$.

3.4. **Oddness and Evenness of $\mathcal{F}$: Additive Quadratic Case Stability Results : Direct Method.**

Theorem 3.22. *Suppose that a function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (3.1) where $\Psi : \mathcal{W}_1^3 \to [0,\infty)$ with the conditions (3.8), (3.9), and (3.39) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with $m = \pm 1$ and $0 < \left(\frac{I}{5}\right)^{\mu} < 1, 0 < \left(\frac{I}{25}\right)^{\mu} < 1$. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\left.\begin{aligned}
&\mu\left(\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1), 4\Lambda\right)\\
&\quad \geq \mu'\left(\Psi_A(w_1), \tfrac{3\Lambda}{4}|5-I|\right) * \mu'\left(\Psi_A(-w_1), \tfrac{3\Lambda}{4}|5-I|\right) *\\
&\qquad\qquad \mu'\left(\Psi_Q(w_1), \tfrac{7\Lambda}{3}|25-I|\right) * \mu'\left(\Psi_Q(-w_1), \tfrac{7\Lambda}{3}|25-I|\right)\\
&\quad = \mu'\left(\Psi(w_1,w_1,w_1), \tfrac{3\Lambda}{4}|5-I|\right) * \mu'\left(\Psi(w_1,w_1,-w_1), \tfrac{3\Lambda}{4}|5-I|\right) *\\
&\qquad \mu'\left(\Psi(-w_1,-w_1,-w_1), \tfrac{3\Lambda}{4}|5-I|\right) * \mu'\left(\Psi(-w_1,-w_1,w_1), \tfrac{3\Lambda}{4}|5-I|\right) *\\
&\qquad \mu'\left(\Psi(w_1,w_1,w_1), \tfrac{7\Lambda}{3}|5-I|\right) * \mu'\left(\Psi(w_1,w_1,-w_1), \tfrac{7\Lambda}{3}|5-I|\right) *\\
&\qquad \mu'\left(\Psi(-w_1,-w_1,-w_1), \tfrac{7\Lambda}{3}|5-I|\right) * \mu'\left(\Psi(-w_1,-w_1,w_1), \tfrac{7\Lambda}{3}|5-I|\right)\\
&\nu\left(\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1), 4\Lambda\right)\\
&\quad \leq \nu'\left(\Psi_A(w_1), \tfrac{3\Lambda}{4}|5-I|\right) \diamond \nu'\left(\Psi_A(-w_1), \tfrac{3\Lambda}{4}|5-I|\right) \diamond\\
&\qquad\qquad \nu'\left(\Psi_Q(w_1), \tfrac{7\Lambda}{3}|25-I|\right) \diamond \nu'\left(\Psi_Q(-w_1), \tfrac{7\Lambda}{3}|25-I|\right)\\
&\quad = \nu'\left(\Psi(w_1,w_1,w_1), \tfrac{3\Lambda}{4}|5-I|\right) \diamond \nu'\left(\Psi(w_1,w_1,-w_1), \tfrac{3\Lambda}{4}|5-I|\right) \diamond\\
&\qquad \nu'\left(\Psi(-w_1,-w_1,-w_1), \tfrac{3\Lambda}{4}|5-I|\right) \diamond \nu'\left(\Psi(-w_1,-w_1,w_1), \tfrac{3\Lambda}{4}|5-I|\right) \diamond\\
&\qquad \nu'\left(\Psi(w_1,w_1,w_1), \tfrac{7\Lambda}{3}|5-I|\right) \diamond \nu'\left(\Psi(w_1,w_1,-w_1), \tfrac{7\Lambda}{3}|5-I|\right) \diamond\\
&\qquad \nu'\left(\Psi(-w_1,-w_1,-w_1), \tfrac{7\Lambda}{3}|5-I|\right) \diamond \nu'\left(\Psi(-w_1,-w_1,w_1), \tfrac{7\Lambda}{3}|5-I|\right)
\end{aligned}\right\} \tag{3.53}$$

and the mapping $\mathcal{A}(w_1)$ and $\mathcal{Q}(w_1)$ are given in (3.11) and (3.41) for all $w_1 \in \mathcal{W}_1$.

Proof. By Theorem 3.8, it follows from (2.33), (3.1) and (3.10), we arrive

$$\left.\begin{aligned}
&\mu\left(\mathcal{A}(w_1) - \mathcal{F}_{odd}(w_1), 2\Lambda\right) \geq \mu'\left(\Psi_A(w_1), \frac{3\Lambda}{4}|5-I|\right) * \mu'\left(\Psi_A(-w_1), \frac{3\Lambda}{4}|5-I|\right)\\
&\nu\left(\mathcal{A}(w_1) - \mathcal{F}_{odd}(w_1), 2\Lambda\right) \leq \nu'\left(\Psi_A(w_1), \frac{3\Lambda}{4}|5-I|\right) \diamond \nu'\left(\Psi_A(-w_1), \frac{3\Lambda}{4}|5-I|\right)
\end{aligned}\right\} \tag{3.54}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. By Theorem 3.15, it follows from (2.37), (3.1), and (3.40), we see

$$\left.\begin{aligned}
&\mu\left(\mathcal{Q}(w_1) - \mathcal{F}_{even}(w_1), 2\Lambda\right) \geq \mu'\left(\Psi_Q(w_1), \frac{7\Lambda}{3}|25-I|\right) * \mu'\left(\Psi_Q(-w_1), \frac{7\Lambda}{3}|25-I|\right)\\
&\nu\left(\mathcal{Q}(w_1) - \mathcal{F}_{even}(w_1), 2\Lambda\right) \leq \nu'\left(\Psi_Q(w_1), \frac{7\Lambda}{3}|25-I|\right) \diamond \nu'\left(\Psi_Q(-w_1), \frac{7\Lambda}{3}|25-I|\right)
\end{aligned}\right\} \tag{3.55}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Now, it follows from (3.54), (3.55) and (2.40), we have

$$\left.\begin{aligned}
&\mu\left(\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1), 4\Lambda\right)\\
&\geq \mu\left(\mathcal{A}(w_1) - \mathcal{F}_{odd}(w_1), 2\Lambda\right) * \mu\left(\mathcal{Q}(w_1) - \mathcal{F}_{even}(w_1), 2\Lambda\right)\\
&\geq \mu'\left(\Psi_A(w_1), \frac{3\Lambda}{4}|5-I|\right) * \mu'\left(\Psi_A(-w_1), \frac{3\Lambda}{4}|5-I|\right) *\\
&\qquad\qquad \mu'\left(\Psi_Q(w_1), \frac{7\Lambda}{3}|25-I|\right) * \mu'\left(\Psi_Q(-w_1), \frac{7\Lambda}{3}|25-I|\right)\\
&\nu\left(\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1), 4\Lambda\right)\\
&\leq \nu\left(\mathcal{A}(w_1) - \mathcal{F}_{odd}(w_1), 2\Lambda\right) \diamond \nu\left(\mathcal{Q}(w_1) - \mathcal{F}_{even}(w_1), 2\Lambda\right)\\
&\leq \nu'\left(\Psi_A(w_1), \frac{3\Lambda}{4}|5-I|\right) \diamond \nu'\left(\Psi_A(-w_1), \frac{3\Lambda}{4}|5-I|\right) \diamond\\
&\qquad\qquad \nu'\left(\Psi_Q(w_1), \frac{7\Lambda}{3}|25-I|\right) \diamond \nu'\left(\Psi_Q(-w_1), \frac{7\Lambda}{3}|25-I|\right)
\end{aligned}\right\} \tag{3.56}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. □

Corollary 3.23. *Suppose that a function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (3.2) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with δ be a positive constant and φ be any real number. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\left.\begin{array}{l} \mu\left(\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1), 4\Lambda\right) \geq \mu'\left(2\delta, \left(|3| + 7\,|8|\right)\Lambda\right) \\ \nu\left(\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1), 4\Lambda\right) \leq \nu'\left(2\delta, \left(|3| + 7\,|8|\right)\Lambda\right) \end{array}\right\} \tag{3.57}$$

for all $w_1 \in \mathcal{W}_1$.

Corollary 3.24. *Suppose that a function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (3.3) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with δ be a positive constant and φ be any real number. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\left.\begin{array}{l} \mu\left(\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1), 4\Lambda\right) \geq \mu'\left(2\delta|w_1|^{\varphi}, \Lambda\left\{\frac{1}{4}\left|5 - 5^{\varphi}\right| + \frac{7}{9}\left|25 - 5^{\varphi}\right|\right\}\right), \quad \varphi \neq 1, 2, \\ \nu\left(\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1), 4\Lambda\right) \leq \nu'\left(2\delta|w_1|^{\varphi}, \Lambda\left\{\frac{1}{4}\left|5 - 5^{\varphi}\right| + \frac{7}{9}\left|25 - 5^{\varphi}\right|\right\}\right), \quad \varphi \neq 1, 2, \end{array}\right\} \tag{3.58}$$

for all $w_1 \in \mathcal{W}_1$.

Corollary 3.25. *Suppose that a function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (3.4) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with δ be a positive constant and φ be any real number. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\left.\begin{array}{l} \mu\left(\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1), 4\Lambda\right) \\ \geq \mu'\left(2\delta\sum_{\psi=1}^{3}|w_\psi|^{\varphi_\psi}, \left\{\frac{3}{4}\sum_{\psi=1}^{3}\left|5 - 5^{\varphi_\psi}\right| + \frac{7}{3}\sum_{\psi=1}^{3}\left|25 - 5^{\varphi_\psi}\right|\right\}\right), \quad \varphi_1, \varphi_2, \varphi_3 \neq 1, 2, \\ \nu\left(\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1), 4\Lambda\right) \\ \leq \nu'\left(2\delta\sum_{\psi=1}^{3}|w_\psi|^{\varphi_\psi}, \left\{\frac{3}{4}\sum_{\psi=1}^{3}\left|5 - 5^{\varphi_\psi}\right| + \frac{7}{3}\sum_{\psi=1}^{3}\left|25 - 5^{\varphi_\psi}\right|\right\}\right), \quad \varphi_1, \varphi_2, \varphi_3 \neq 1, 2, \end{array}\right\} \tag{3.59}$$

for all $w_1 \in \mathcal{W}_1$.

Corollary 3.26. *Suppose that a function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (3.5) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with δ be a positive constant and φ be any real number. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\left.\begin{array}{l} \mu\left(\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1), 4\Lambda\right) \geq \mu'\left(2\delta|w_1|^{3\varphi}, \Lambda\left\{\frac{3}{4}|5 - 5^{3\varphi}| + \frac{7}{3}|25 - 5^{3\varphi}|\right\}\right), \quad 3\varphi \neq 1, 2, \\ \nu\left(\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1), 4\Lambda\right) \leq \nu'\left(2\delta|w_1|^{3\varphi}, \Lambda\left\{\frac{3}{4}|5 - 5^{3\varphi}| + \frac{7}{3}|25 - 5^{3\varphi}|\right\}\right), \quad 3\varphi \neq 1, 2, \end{array}\right\} \tag{3.60}$$

for all $w_1 \in \mathcal{W}_1$.

Corollary 3.27. *Suppose that a function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (3.6) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with δ be a positive constant and φ be any real number. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\left.\begin{array}{l} \mu\left(\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1), 4\Lambda\right) \\ \geq \mu'\left(4\delta|w_\psi|^{\sum_{\psi=1}^{3}\varphi_\psi}, 2\Lambda\left\{\frac{3}{4}\left|5 - 5^{\sum_{\psi=1}^{3}\varphi_\psi}\right| + \frac{7}{3}\left|25 - 5^{\sum_{\psi=1}^{3}\varphi_\psi}\right|\right\}\right), \quad \sum_{\psi=1}^{3}\varphi_\psi \neq 1, 2, \\ \nu\left(\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1), 4\Lambda\right) \\ \leq \nu'\left(4\delta|w_\psi|^{\sum_{\psi=1}^{3}\varphi_\psi}, 2\Lambda\left\{\frac{3}{4}\left|5 - 5^{\sum_{\psi=1}^{3}\varphi_\psi}\right| + \frac{7}{3}\left|25 - 5^{\sum_{\psi=1}^{3}\varphi_\psi}\right|\right\}\right), \quad \sum_{\psi=1}^{3}\varphi_\psi \neq 1, 2, \end{array}\right\} \tag{3.61}$$

for all $w_1 \in \mathcal{W}_1$.

Corollary 3.28. *Suppose that a function $\mathcal{F}:\mathcal{W}_1\to\mathcal{W}_2$ satisfy the functional inequality (3.7) for all $w_1,w_2,w_3\in\mathcal{W}_1$ and all $\Lambda>0$ with δ be a positive constant and φ be any real number. Then there exists a unique additive mapping $\mathcal{A}(w_1):\mathcal{W}_1\to\mathcal{W}_2$ and a unique quadratic mapping $\mathcal{Q}(w_1):\mathcal{W}_1\to\mathcal{W}_2$ which satisfies (1.7) and the functional inequality*

$$\left.\begin{array}{l}\mu\left(\mathcal{F}(w_1)-\mathcal{A}(w_1)-\mathcal{Q}(w_1),4\Lambda\right)\geq\mu'\left(4\delta|w_1|^{3\varphi},\Lambda\left\{\frac{3}{4}|5-5^{3\varphi}|+\frac{7}{3}|25-5^{3\varphi}|\right\}\right),\quad 3\varphi\neq1,2,\\ \nu\left(\mathcal{F}(w_1)-\mathcal{A}(w_1)-\mathcal{Q}(w_1),4\Lambda\right)\leq\nu'\left(4\delta|w_1|^{3\varphi},\Lambda\left\{\frac{3}{4}|5-5^{3\varphi}|+\frac{7}{3}|25-5^{3\varphi}|\right\}\right),\quad 3\varphi\neq1,2,\end{array}\right\}\tag{3.62}$$

for all $w_1\in\mathcal{W}_1$.

3.5. Oddness of $\mathcal{F}$: Additive Case Stability Results : Fixed Point Method.

Theorem 3.29. *Suppose that an odd function $\mathcal{F}:\mathcal{W}_1\to\mathcal{W}_2$ satisfy the functional inequality (3.1) where $\Psi:\mathcal{W}_1^3\to[0,\infty)$ with the condition*

$$\left.\begin{array}{l}\lim\limits_{\ell\to\infty}\mu'\left(\Psi\left(\tau_v^\ell w_1,\tau_v^\ell w_2,\tau_v^\ell w_3\right),\tau_v^\ell\Lambda\right)=1\\ \lim\limits_{\ell\to\infty}\nu'\left(\Psi\left(\tau_v^\ell w_1,\tau_v^\ell w_2,\tau_v^\ell w_3\right),\tau_v^\ell\Lambda\right)=0\end{array}\right\};\quad \tau_v=\left\{\begin{array}{l}5;v=0;\\ \frac{1}{5};v=1;\end{array}\right.\tag{3.63}$$

for all $w_1,w_2,w_3\in\mathcal{W}_1$ and all $\Lambda>0$ If there exists $L=L(v)$ be a function have the property

$$\left.\begin{array}{l}\mu\left(\Psi_A(w_1),\Lambda\right)=\mu\left(\Psi_A\left(\frac{w_1}{5}\right),\Lambda\right)\\ \nu\left(\Psi_A(w_1),\Lambda\right)=\nu\left(\Psi_A\left(\frac{w_1}{5}\right),\Lambda\right)\end{array}\right\}\quad and\quad \left.\begin{array}{l}\mu\left(\frac{1}{\tau_v}\Psi_A\left(\tau_v w_1\right),\Lambda\right)=\mu\left(L\,\Psi_A(w_1),\Lambda\right)\\ \nu\left(\frac{1}{\tau_v}\Psi_A\left(\tau_v w_1\right),\Lambda\right)=\nu\left(L\,\Psi_A(w_1),\Lambda\right)\end{array}\right\},\tag{3.64}$$

for all $w_1\in\mathcal{W}_1$ and all $\Lambda>0$. Then there exists a unique additive mapping $\mathcal{A}(w_1):\mathcal{W}_1\to\mathcal{W}_2$ which satisfies (1.7) and the functional inequality

$$\left.\begin{array}{l}\mu\left(\mathcal{A}(w_1)-\mathcal{F}(w_1),\Lambda\right)\geq\mu'\left(\dfrac{L^{1-v}}{1-L}\,\Psi_A\left(w_1\right),\dfrac{3\Lambda}{4}\right)\\ \qquad=\mu'\left(\dfrac{L^{1-v}}{1-L}\,\Psi\left(w_1,w_1,w_1\right),\dfrac{3\Lambda}{4}\right)*\mu'\left(\dfrac{L^{1-v}}{1-L}\,\Psi\left(w_1,w_1,-w_1\right),\dfrac{3\Lambda}{4}\right)\\ \nu\left(\mathcal{A}(w_1)-\mathcal{F}(w_1),\Lambda\right)\leq\nu'\left(\dfrac{L^{1-v}}{1-L}\,\Psi_A\left(w_1\right),\dfrac{3\Lambda}{4}\right)\\ \qquad=\nu'\left(\dfrac{L^{1-v}}{1-L}\,\Psi\left(w_1,w_1,w_1\right),\dfrac{3\Lambda}{4}\right)\diamond\nu'\left(\dfrac{L^{1-v}}{1-L}\,\Psi\left(w_1,w_1,-w_1\right),\dfrac{3\Lambda}{4}\right)\end{array}\right\}\tag{3.65}$$

and the mapping $\mathcal{A}(w_1)$ is obtained by

$$\left.\begin{array}{l}\lim\limits_{\ell\to\infty}\mu\left(\dfrac{1}{\tau_v^\ell}\mathcal{F}\left(\tau_v^\ell w_1\right)-\mathcal{A}(w_1),\Lambda\right)=1\\ \lim\limits_{\ell\to\infty}\nu\left(\dfrac{1}{\tau_v^\ell}\mathcal{F}\left(\tau_v^\ell w_1\right)-\mathcal{A}(w_1),\Lambda\right)=0\end{array}\right\}\tag{3.66}$$

for all $w_1\in\mathcal{W}_1$ and all $\Lambda>0$.

Proof. Assume a set $\mathcal{G}$ as in Theorem 2.7 of (2.48) and introduce the generalized metric on the above set $\mathcal{G}$ as

$$d(\mathcal{F},\mathcal{F}_1)=\inf\left\{K\in(0,\infty):\left\{\begin{array}{l}\mu\left(\mathcal{F}(w_1)-\mathcal{F}_1(w_1),\Lambda\right)\geq\mu\left(K\,\Psi(w_1,w_1,w_1),\Lambda\right)\\ \nu\left(\mathcal{F}(w_1)-\mathcal{F}_1(w_1),\Lambda\right)\leq\nu\left(K\,\Psi(w_1,w_1,w_1),\Lambda\right)\end{array}\right\}\right\}.\tag{3.67}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. It is easy to see that $(\mathcal{G}, d)$ is complete. Define a function $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ as by Theorem 2.7 of (2.50) and for $\mathcal{F}, \mathcal{F}_1 \in \mathcal{G}$ and $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$, we see

$$\begin{aligned} d(\mathcal{F},\mathcal{F}_1) \leq K \Rightarrow & \left\{ \begin{array}{l} \mu\left(\mathcal{F}(w_1) - \mathcal{F}_1(w_1), \Lambda\right) \geq \mu\left(K\,\Psi(w_1,w_1,w_1),\Lambda\right) \\ \nu\left(\mathcal{F}(w_1) - \mathcal{F}_1(w_1), \Lambda\right) \leq \nu\left(K\,\Psi(w_1,w_1,w_1),\Lambda\right) \end{array} \right\} \\ \Rightarrow & \left\{ \begin{array}{l} \mu\left(\left\|\frac{1}{\tau_v}\mathcal{F}(\tau_v w_1) - \frac{1}{\tau_v}\mathcal{F}_1(\tau_v w_1)\right\|, \Lambda\right) \geq \mu\left(\tau_v K\,\Psi(\frac{1}{\tau_v}w_1, \tau_v w_1, \tau_v w_1), \Lambda\right) \\ \nu\left(\left\|\frac{1}{\tau_v}\mathcal{F}(\tau_v w_1) - \frac{1}{\tau_v}\mathcal{F}_1(\tau_v w_1)\right\|, \Lambda\right) \leq \nu\left(\tau_v K\,\Psi(\frac{1}{\tau_v}w_1, \tau_v w_1, \tau_v w_1), \Lambda\right) \end{array} \right\} \\ \Rightarrow & \left\{ \begin{array}{l} \mu\left(\mathcal{H}\mathcal{F}(w_1) - \mathcal{H}\mathcal{F}_1(w_1), \Lambda\right) \geq \mu\left(L\,K\,\Psi(w_1,w_1,w_1),\Lambda\right) \\ \nu\left(\mathcal{H}\mathcal{F}(w_1) - \mathcal{H}\mathcal{F}_1(w_1), \Lambda\right) \leq \nu\left(L\,K\,\Psi(w_1,w_1,w_1),\Lambda\right) \end{array} \right\} \\ \Rightarrow & d(\mathcal{H}\mathcal{F}, \mathcal{H}\mathcal{F}_1) \leq L\,K, \end{aligned}$$

i.e., $\mathcal{H}$ is a strictly contractive mapping on $\mathcal{G}$ with Lipschitz constant L (see [18]).
For the case $\nu = 0$, it follows from (3.16) and with the help of (3.64), (2.50), (3.67), we get

$$\left. \begin{array}{l} \mu\left(\dfrac{1}{5}\mathcal{F}(5w_1) - \mathcal{F}(w_1), \dfrac{4}{3}\,\Lambda\right) \geq \mu'\left(\dfrac{1}{5}\Psi_A\left(w_1\right), \Lambda\right) \\ \nu\left(\dfrac{1}{5}\mathcal{F}(5w_1) - \mathcal{F}(w_1), \dfrac{4}{3}\,\Lambda\right) \leq \nu'\left(\dfrac{1}{5}\Psi_A\left(w_1\right), \Lambda\right) \end{array} \right\} \Rightarrow d(\mathcal{H}\mathcal{F}, \mathcal{F}) \leq L = L^{1-v}, \tag{3.68}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$.
For the case $\nu = 1$, it follows from (3.22) and with the help of (3.64), (2.50), (3.67), we obtain

$$\left. \begin{array}{l} \mu\left(\mathcal{F}(w_1) - 5\mathcal{F}\left(\dfrac{w_1}{5}\right), \dfrac{4}{3 \cdot I}\Lambda\right) \geq \mu'\left(\Psi_A\left(\dfrac{w_1}{5}\right), \Lambda\right) \\ \nu\left(\mathcal{F}(w_1) - 5\mathcal{F}\left(\dfrac{w_1}{5}\right), \dfrac{4}{3 \cdot I}\Lambda\right) \leq \nu'\left(\Psi_A\left(\dfrac{w_1}{5}\right), \Lambda\right) \end{array} \right\} \Rightarrow d(\mathcal{F}, \mathcal{H}\mathcal{F}) \leq 1 = L^{1-v}, \tag{3.69}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Combining (3.68) and (3.69), we have

$$d(\mathcal{F}, \mathcal{H}\mathcal{F}) \leq 1 = L^{1-v}. \tag{3.70}$$

Therefore $(FPC1)$ of Theorem 1.5 holds. The rest of the proof follows by Theorem 1.5. Hence the proof is complete. □

Corollary 3.30. *Suppose that an odd function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequalities (3.2), (3.3), (3.4), (3.5), (3.6), (3.7) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ with δ be a positive constant and φ be any real number. Then there exists a a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequalities (3.33), (3.34), (3.35), (3.36), (3.37), (3.38), respectively for all $w_1 \in \mathcal{W}_1$.*

3.6. Evenness of $\mathcal{F}$: Quadratic Case Stability Results : Fixed Point Method.

Theorem 3.31. *Suppose that an even function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (3.1) where $\Psi : \mathcal{W}_1^3 \to [0, \infty)$ with the condition*

$$\left. \begin{array}{l} \lim\limits_{\ell \to \infty} \mu'\left(\Psi\left(\tau_v^{\ell} w_1, \tau_v^{\ell} w_2, \tau_v^{\ell} w_3\right), \tau_v^{2\ell}\Lambda\right) = 1 \\ \lim\limits_{\ell \to \infty} \nu'\left(\Psi\left(\tau_v^{\ell} w_1, \tau_v^{\ell} w_2, \tau_v^{\ell} w_3\right), \tau_v^{2\ell}\Lambda\right) = 0 \end{array} \right\}; \quad \tau_v = \left\{ \begin{array}{l} 5; v = 0; \\ \frac{1}{5}; v = 1; \end{array} \right. \tag{3.71}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$. If there exists $L = L(\nu)$ be function have the property

$$\left. \begin{array}{l} \mu\left(\Psi_Q(w_1), \Lambda\right) = \mu\left(\Psi_Q\left(\frac{w_1}{5}\right), \Lambda\right) \\ \nu\left(\Psi_Q(w_1), \Lambda\right) = \nu\left(\Psi_Q\left(\frac{w_1}{5}\right), \Lambda\right) \end{array} \right\} \quad \text{and} \quad \left. \begin{array}{l} \mu\left(\frac{1}{\tau_v^2}\Psi_Q\left(\tau_v w_1\right), \Lambda\right) = \mu\left(L\,\Psi_Q(w_1), \Lambda\right) \\ \nu\left(\frac{1}{\tau_v^2}\Psi_Q\left(\tau_v w_1\right), \Lambda\right) = \nu\left(L\,\Psi_Q(w_1), \Lambda\right) \end{array} \right\}, \tag{3.72}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequality

$$\left.\begin{aligned} \mu\left(\mathcal{Q}(w_1) - \mathcal{F}(w_1), \Lambda\right) &\geq \mu'\left(\frac{L^{1-v}}{1-L}\,\Psi_Q\left(w_1\right), \frac{7\Lambda}{3}\right) \\ &= \mu'\left(\frac{L^{1-v}}{1-L}\,\Psi\left(w_1, w_1, w_1\right), \frac{7\Lambda}{3}\right) * \mu'\left(\frac{L^{1-v}}{1-L}\,\Psi\left(w_1, w_1, -w_1\right), \frac{7\Lambda}{3}\right) \\ \nu\left(\mathcal{Q}(w_1) - \mathcal{F}(w_1), \Lambda\right) &\leq \nu'\left(\frac{L^{1-v}}{1-L}\,\Psi_Q\left(w_1\right), \frac{7\Lambda}{3}\right) \\ &= \nu'\left(\frac{L^{1-v}}{1-L}\,\Psi\left(w_1, w_1, w_1\right), \frac{7\Lambda}{3}\right) \diamond \nu'\left(\frac{L^{1-v}}{1-L}\,\Psi\left(w_1, w_1, -w_1\right), \frac{7\Lambda}{3}\right) \end{aligned}\right\} \tag{3.73}$$

and the mapping $\mathcal{Q}(w_1)$ is obtained by

$$\left.\begin{aligned} \lim_{\ell\to\infty} \mu\left(\frac{1}{\tau_v^{2\ell}}\mathcal{F}\left(\tau_v^{\ell} w_1\right) - \mathcal{Q}(w_1), \Lambda\right) = 1 \\ \lim_{\ell\to\infty} \nu\left(\frac{1}{\tau_v^{2\ell}}\mathcal{F}\left(\tau_v^{\ell} w_1\right) - \mathcal{Q}(w_1), \Lambda\right) = 0 \end{aligned}\right\} \tag{3.74}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$.

Proof. Define a function $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ as by Theorem 2.9 of (2.62) and for $\mathcal{F}, \mathcal{F}_1 \in \mathcal{G}$ and $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$, we see

$$\begin{aligned} d(\mathcal{F}, \mathcal{F}_1) \leq K \Rightarrow & \left\{\begin{array}{l} \mu\left(\mathcal{F}(w_1) - \mathcal{F}_1(w_1), \Lambda\right) \geq \mu\left(K\,\Psi(w_1, w_1, w_1), \Lambda\right) \\ \nu\left(\mathcal{F}(w_1) - \mathcal{F}_1(w_1), \Lambda\right) \leq \nu\left(K\,\Psi(w_1, w_1, w_1), \Lambda\right) \end{array}\right\} \\ \Rightarrow & \left\{\begin{array}{l} \mu\left(\left\|\frac{1}{\tau_v^2}\mathcal{F}(\tau_v w_1) - \frac{1}{\tau_v^2}\mathcal{F}_1(\tau_v w_1)\right\|, \Lambda\right) \geq \mu\left(\tau_v K\,\Psi(\tau_v w_1, \tau_v w_1, \tau_v w_1), \Lambda\right) \\ \nu\left(\left\|\frac{1}{\tau_v^2}\mathcal{F}(\tau_v w_1) - \frac{1}{\tau_v^2}\mathcal{F}_1(\tau_v w_1)\right\|, \Lambda\right) \leq \nu\left(\tau_v^2 K\,\Psi(\tau_v w_1, \tau_v w_1, \tau_v w_1), \Lambda\right) \end{array}\right\} \\ \Rightarrow & \left\{\begin{array}{l} \mu\left(\mathcal{H}\mathcal{F}(w_1) - \mathcal{H}\mathcal{F}_1(w_1), \Lambda\right) \geq \mu\left(L\,K\,\Psi(w_1, w_1, w_1), \Lambda\right) \\ \nu\left(\mathcal{H}\mathcal{F}(w_1) - \mathcal{H}\mathcal{F}_1(w_1), \Lambda\right) \leq \nu\left(L\,K\,\Psi(w_1, w_1, w_1), \Lambda\right) \end{array}\right\} \\ \Rightarrow & d(\mathcal{H}\mathcal{F}, \mathcal{H}\mathcal{F}_1) \leq L\,K, \end{aligned}$$

i.e., $\mathcal{H}$ is a strictly contractive mapping on $\mathcal{G}$ with Lipschitz constant L (see [18]). The rest of the proof is similar to that of Theorem 3.29. Hence the proof is complete. □

Corollary 3.32. *Suppose that an even function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional iinequalities (3.2), (3.3), (3.4), (3.5), (3.6), (3.7) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ with δ be a positive constant and φ be any real number. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequalities (3.47), (3.48), (3.49), (3.50), (3.51), (3.52), for all $w_1 \in \mathcal{W}_1$.*

3.7. **Oddness and Evenness of $\mathcal{F}$: Additive Quadratic Case Stability Results : Fixed Point Method.**

Theorem 3.33. *Suppose that a function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequality (3.1) where $\Psi : \mathcal{W}_1^3 \to [0, \infty)$ with the conditions (3.63) and (3.71) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$. If there exists $L = L(v)$ be function have the properties (3.64) and (3.72) for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the* *functional inequality*

$$
\left.\begin{aligned}
&\mu\left(\mathcal{F}(w_1)-\mathcal{A}(w_1)-\mathcal{Q}(w_1),4\Lambda\right)\\
&\geq \mu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi_A\left(w_1\right),\tfrac{3\Lambda}{4}\right)*\mu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi_A\left(-w_1\right),\tfrac{3\Lambda}{4}\right)*\\
&\qquad\qquad \mu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi_Q\left(w_1\right),\tfrac{7\Lambda}{3}\right)*\mu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi_Q\left(-w_1\right),\tfrac{7\Lambda}{3}\right)\\
&=\mu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi\left(w_1,w_1,w_1\right),\tfrac{3\Lambda}{4}\right)*\mu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi\left(w_1,w_1,-w_1\right),\tfrac{3\Lambda}{4}\right)*\\
&\qquad \mu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi\left(-w_1,-w_1,-w_1\right),\tfrac{3\Lambda}{4}\right)*\mu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi\left(-w_1,-w_1,w_1\right),\tfrac{3\Lambda}{4}\right)*\\
&\qquad\quad \mu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi\left(w_1,w_1,w_1\right),\tfrac{7\Lambda}{3}\right)*\mu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi\left(w_1,w_1,-w_1\right),\tfrac{7\Lambda}{3}\right)*\\
&\qquad\qquad \mu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi\left(-w_1,-w_1,-w_1\right),\tfrac{7\Lambda}{3}\right)*\mu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi\left(-w_1,-w_1,w_1\right),\tfrac{7\Lambda}{3}\right)\\
&\nu\left(\mathcal{F}(w_1)-\mathcal{A}(w_1)-\mathcal{Q}(w_1),4\Lambda\right)\\
&\leq \nu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi_A\left(w_1\right),\tfrac{3\Lambda}{4}\right)\diamond\nu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi_A\left(-w_1\right),\tfrac{3\Lambda}{4}\right)\diamond\\
&\qquad\qquad \nu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi_Q\left(w_1\right),\tfrac{7\Lambda}{3}\right)\diamond\nu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi_Q\left(-w_1\right),\tfrac{7\Lambda}{3}\right)\\
&=\nu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi\left(w_1,w_1,w_1\right),\tfrac{3\Lambda}{4}\right)\diamond\nu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi\left(w_1,w_1,-w_1\right),\tfrac{3\Lambda}{4}\right)\diamond\\
&\qquad \nu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi\left(-w_1,-w_1,-w_1\right),\tfrac{3\Lambda}{4}\right)\diamond\nu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi\left(-w_1,-w_1,w_1\right),\tfrac{3\Lambda}{4}\right)\diamond\\
&\qquad\quad \nu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi\left(w_1,w_1,w_1\right),\tfrac{7\Lambda}{3}\right)\diamond\nu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi\left(w_1,w_1,-w_1\right),\tfrac{7\Lambda}{3}\right)\diamond\\
&\qquad\qquad \nu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi\left(-w_1,-w_1,-w_1\right),\tfrac{7\Lambda}{3}\right)\diamond\nu'\left(\tfrac{L^{1-v}}{1-L}\,\Psi\left(-w_1,-w_1,w_1\right),\tfrac{7\Lambda}{3}\right)
\end{aligned}\right\}\tag{3.75}
$$

and the mapping $\mathcal{A}(w_1)$ and $\mathcal{Q}(w_1)$ are given in (3.65) and (3.74) for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$.

Proof. The proof is similar ideas to that of Theorem 3.22. □

Corollary 3.34. *Suppose that a function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfy the functional inequalities (3.2), (3.3), (3.4), (3.5), (3.6), (3.7) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with δ be a positive constant and φ be any real number. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfies (1.7) and the functional inequalities (3.57), (3.58), (3.59), (3.60), (3.61), (3.62) for all $w_1 \in \mathcal{W}_1$.*

Conclusion

In this paper, we analyze the generalized Ulam-Hyers stability of a affine type AQ Functional Equation in Banach Space and Intuitionistic Fuzzy Banach Space with the help of classical Hyers direct and Radus fixed methods. The results are new, since we are getting better possible upper bound than previous stability analysis (see [6]).

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Conflict of Interest

All authors declare that they have no conflicts of interest.

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