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spgωα-Continuous Multifunctions

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Abstract: The purpose of this paper is to introduce and study spgωα-continuous multifunctions such as upper (lower) spgωα-continuous multifunctions, basic characterizations and several properties concerning to upper and lower spgωα-continuous multifunctions are investigated. Further, upper (lower) spgωα-irresolute multifunctions are also studied with basic properties.

Keywords: spgωα-closed set, spgωα-open set, spgωα-T_i-spaces, spgωα-continuous function.
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1. Introduction

Many authors have continued their research and studied several stronger and weaker forms of continuous functions and multifunctions in topological spaces. It is observed that continuity and multifunctions are basic topics in general topology and in set valued analysis in mathematics. Continuous functions and continuous multifunctions stand among the most fundamental and most related points in the whole of mathematical sciences.

In the literature, the upper and lower continuity for multifunctions were firstly studied and introduced by Berge[2]. After the work of him, many authors turned continued their research to investigate several weak and strong forms of continuity. In 1999, Mahmoad[12] introduced the concept of pre-irresolute multivalued functions. Neubrunn [13] introduced and studied the notion of upper(lower) α-continuous multifunctions.

The purpose of this paper is to give a new an idea about new weaker form of continuous functions called upper spgωα-continuous multifunctions and lower spgωα-continuous multifunctions. Further, basic properties and preservation theorems of these upper spgωα-continuous multifunctions (lower spgωα-continuous multifunctions) are studied.

In the next section, the notion of upper spgωα-irresolute multifunctions (lower spgωα-continuous multifunctions) is introduced and characterizations and some basic properties are investigated.

Throughout this paper (R, τ), (S, σ) and (Q, η) stands for topological spaces with no separation axioms are assumed, for any set A of a space R, closure of A and interior of A is denoted by cl(A) and int(A).

A multifunction P: R → S is a point to set correspondence and always we assume that P(p) ≠ ϕ for every p ∈ R. Let A be any subset of R and B be any subset of S. Then P(A) = ∪ {P(p) :

p $\in$ A}. For a multifunction P : R $\to$ S, following [8] we will denote the upper and lower inverse of a set B of S by $P^+(B)$ and $P^-(B)$ respectively,
that is P+(B) = $\cup$\{p $\in$ R : P(p) $\subset$ B\} and P-(B) = \{p $\in$ R : P(p) $\cap$ B $\neq$ 0\}. So P^- : S $\to$ R(p) and if s $\in$ S. Then $P^-(s)$ = \{p $\in$ S : s $\in$ P(p)\} where P(A) be the collection of the subsets of P. Thus, for a subset B in S, $P^-(B) = \cup\{P^-(s) : s \in B\}$. Then P is said to be a surjection if P(p) = S or equivalently, if for each s $\in$ S, there exists p $\in$ R such that s $\in$ P(p).

For multifunction P : R $\to$ S, the graph multifunction GP : R $\to$ R X S is defined as
GP (p) = \{p\} x P(p) for each p $\in$ R and the subset \{\{p\} x P(p) : p $\in$ R\} $\subset$ R x S is called the multifunction of P and is denoted by G(P)[8].

2. spgωα-Continuous Multifunctions

This section deals with the characterizations of upper and lower spgωα-continuous multifunctions and introduced basic properties related to them in topological spaces.

Definition 2.1. A multifunction P : R $\to$ S is called
1. Upper spgωα-continuous (briefly u.spgωα.C) at a point r $\in$ R, if for each V $\in$ O(S) with P(r) $\subset$ V , there exist U $\in$ spgωα-O(R, r) such that P(U) $\subset$ V .
2. Lower spgωα-continuous (briefly l.spgωα.C) at a point r $\in$ R, if for every V $\in$ O(S) with P(r) $\cap$ V $\neq \phi$, then there exist V $\in$ spgωα-O(R, z) such that P(z) $\cap$ V $\neq \phi$ holds for each z $\in$ R.
3. Upper(lower) spgωα-continuous, if it is upper(lower) spgωα-continuous at every point of R.

Example 2.1. Let R = S = $\{p_1, p_2, p_3\}$, $\tau = \{R, \phi, \{p_1\}\}$ and $\sigma = \{S, \phi, \{p_1\}\}, \{p_2\}\}$
Let P be m.f and P_1 be the identity m.f from R to S.
spgωα-O(R) = R, ϕ, $\{p_1\}$, $\{p_1, p_2\}$, $\{p_1, p_3\}$.
spgωα-O(S) = S, ϕ, $\{p_1\}$, $\{p_2\}$, $\{p_1, p_2\}$, $\{p_1, p_3\}$, $\{p_2, p_3\}$.
Let $p_1 \in$ R and V = $\{p_1, p_2\}$ is a open set of S where $P(\{p_1\}) = \{p_1\} \subset V$.
Then, there exists a spgωα-open set U = $\{p_1, p_2\}$ in R containing the point p_1 with P(U) = $P(\{p_1, p_2\}) = \{p_1, p_2\} \subset V$
Thus P(U) $\subset$ V and so P_1 is u.spgωα.C.

Example 2.2. Let R = S = $\{p_1, p_2, p_3\}$ $\tau = \{R, \phi, \{p_1\}, \{p_1, p_2\}\}$.
Here spgωα-open sets are: R, ϕ, $\{p_1\}$, $\{p_2\}$, $\{p_1, p_2\}$
$\sigma = \{S, \phi, \{p_1\}\}$. Here spgωα-open sets are: S, ϕ, $\{p_1\}$, $\{p_1, p_2\}$, $\{p_1, p_3\}$
Let P_1 : R $\to$ S be a m.f and R = $\{p_2\} \in$ R
spgωα-open sets in S containing p_2 are :S, $\{p_1, p_2\}$
Let V = S.
V = S = $\{p_1, p_2, p_3\} \to P(\{p_2\}) \cap V = \{p_2\} \cap \{p_1, p_2, p_3\} = \{p_2\} \neq \phi$
Consider V = $\{p_1, p_2\}$
V = $\{p_1, p_2\} \to P(\{p_2\}) \cap V = \{p_2\} \cap \{p_1, p_2\} = \{p_2\} \neq \phi$.
Then there exists spgωα-open set U = $\{p_1, p_2\}$ containing $\{p_2\}$ are: S, $\{p_1, p_2\}$

1. $\{p_2\} \cap \{p_1, p_2\} = \{p_2\} \neq \phi$.
2. $\{p_2\} \cap N = \{p_2\} \cap \{p_1, p_2, p_3\} = \{p_2\} \neq \phi$.

Thus P_1 is a l.spgωα.C.

Theorem 2.1. If R and S be any TS. Then for a m.f P: R $\rightarrow$ S the following properties are equivalent:

1. R is u.spgωα.C.
2. for each $r \in R$, for each $V \in O(S)$ such that $r \in P^+(V)$, there exists $U \in$ spgωα-O(R, r) with $U \subset P^+(V)$.
3. for each $r \in R$ and each $K \in C(S)$ such that $r \in P^+(S - K)$, there exists $H \in$ spgωα-C(R) such that $r \in S - H$ and $P^-(K) \subset H$.
4. for each $V \in O(S)$, $P^+(V) \in$ spgωα-O(R).
5. for each $K \in C(S)$, $P^-(V) \in$ spgωα-C(R).
6. for each $r \in R$ and for each nbd. V of P(R), $P^+(V)$ is spgωα-nbd. of r.
7. for each $r \in R$ and each nbd. V of P(R), there exists a spgωα-nbd. U of r such that $P(U) \subset V$.

Proof: (1) $\rightarrow$ (2): Obvious from the definition.

(2) $\rightarrow$ (3): Let $r \in R$ and $K \in C(S)$ with $r \in P^+(S - K)$. From (2) there exists $U \in$spgωα-O(R, r) such that $U \in P^+(S - K)$. Put $H = S - K$, then $H \in$ spgωα-C(R) with $r = R - H$. Also $U \subset P^+(S - K) = R – P^-(K)$, that is $P^-(K) \subset R - V = H$.

(3) $\rightarrow$ (2): Let $r \in R$ and $V \in O(S)$ with $r \in P^+(V)$. Put $K = S - V$, where $K \in C(S)$ with $r \in P^+(S - K)$. From (3), there exists $H \in$ spgωα-C(R) such that $r \in R - H$ and $P^+(K) \subset H$. Let $U = R - H$ then $U \in$ spgωα-O(R, r) and also $P^-(K) \subset H$. Thus, $R- P^-(S - K) \subset H$ and so $R - H \subset P^-(S - K)$. So $U \subset P+(V)$.

(2) $\rightarrow$ (4): Let $V \in O(S)$ and $r \in P^-(V)$. Then from (2), there exists $U \in$ spgωα-O(R, r) with $U \subset P^+(V)$. So $P^+(V) = \bigcup_{r \in P^+(V)} UU$. We know that, arbitrary union of spgωα-open set is again spgωα-open and so $P^+(V) \in$ spgωα-O(R).

(4) $\rightarrow$ (2): Let $r \in R$ and $V \in O(S)$ with $r \in P^+(V)$. From (4) $P+(V) \in$ spgωα-O(R). Let $U = P^+(V)$, then $U \in$ spgωα-O(R, r) and so $U \in P^+(V)$.

(4) $\rightarrow$ (5): $U \in C(S)$ and so $S - U = O(S)$. But from (4), $P+(S - U) \in$ spgωα-O(R), since $P+(S - U) = R – P-(U)$ and so $R – P-(U) \in$ spgωα-O(R). Thus $P-(U) \in$ spgωα-O(R).

(5) $\rightarrow$ (6): $V \in O(S)$ and so $S - V \in C(S)$. But from (6), $P- (S - V) \in$ spgωα-C(R), since $P-(S - V) = R - P+(V)$ and so $R - P+(U) \in$ spgωα-C(R). Thus $P+(V) \in$ spgωα-O(R)

(4) $\rightarrow$ (6): Let $r \in R$ and V be a nbd. of P(r). Then $U \in O(N)$ such that $P(r) \subset U \subset V$, that is $r \in P+(U) \subset P+(V)$. But from (4), $P+(U) \in$ spgωα-O(R) and so $P+(V) \in$ spgωα.nbd. of r.

(6) $\rightarrow$ (7): Let $r \in R$ and V be a nbd. of P(r). From (6), P+(V) is a spgωα-nbd. of the point r. Put $U = P+(V)$, so U is a spgωα-nbd. of r with $P(U) \subset V$.

(7) $\rightarrow$ (1): Let $r \in R$ and $V \in O(S)$ with $P(r) \subset V$. Then V is a nbd. of P(r). By (7), there exists spgωα-nbd. U of r such that $P(U) \subset V$. So, there exists a $G \in$ spgωα-O(R) with $r \in G \subset U$ and so $P(r) \in P(G) \subset P(U) \subset V$. Thus R is u.spgωα.C for each point $r \in R$.

Theorem 2.2. The following properties are equivalent for a m.f P: R → S:

1. P is l.spgωα.C.
2. for each r ∈ R and V ∈ O(R) with r ∈ P-(V) there exists U ∈ spgωα-O(R) containing r such that U ⊂ P^-(V).
3. for each r ∈ R and K ∈ C(S) with r ∈ P-(S - K) there exists H ∈ spgωα-C(R) such that r ∈ R - H and P+(K) ⊂ H.
4. for each V ∈ O(S), P-(V) ∈ spgωα-O(R).
5. for each K ∈ C(S), P+(K) ∈ spgωα-C(R).

Definition 2.2: [4] A space R is said to be Submaximal if every dense subset of R is open.

Theorem 2.3. If a m.f P: R → S is u.p.C and S is submaximal, then P is u.spgωα.C.
Proof. Let A ∈ p-O(S). As S is submaximal, A ∈ O(S). Since P is u.p.C, P^+(A) ∈ p-O(R) and hence P^+(A) ∈ spgωα-O(R). Thus P is u.spgωα.C.

Theorem 2.4. A m.f P: R → S is u.spgωα.C if and only if for all B ∈ O(S), P^-(B) ∈ O(R).
Proof. Let B ∈ O(S) and r ∈ P^+(B). Then by u.spgωα.C, there exists V ∈ spgωα-O(R) with P(V) ⊂ B, where P+(B) ∈ O(R). Let P^+(B) is open and r ∈ P^-(B). Then P^+(B) = {r ∈ B : P(R) ⊂ B}. So P is u.spgωα.C.

Theorem 2.5. A m.f P: R → S is l.spgωα.C if and only if for all open set B in S, P^-(B) ∈ O(R).
Proof. Let B ∈ O(S) and r ∈ P^+(B). Then by l.spgωα.C, there exists V ∈ spgωα-O(R) with P(V) ∩ B ≠ ϕ. As v ∈ V then P-(B) ∈ O(R). Suppose P^-(B) ∈ O(R) and r ∈ P^-(B), then P^-(B) = {r ∈ R : P(R) ∩ B ≠ ϕ}. So P is l.spgωα.C.

Theorem 2.6. The following holds good for a m.f P: R → S

1. P is u.spgωα.C.
2. P(spgωα-cl(B)) ⊂ cl(P(B)) for every B ⊂ R.
3. spgωα-cl(P^+(A)) ⊂ P^+(cl(A)) for every A ⊂ S.
4. P^-(int(A)) ⊂ spgωα-int(P^-(A)) for every A ⊂ S.
5. int(P(B)) ⊂ P(spgωα-int(B)) for every B ⊂ R.

Proof. (1) → (2): Let B ⊂ R. Then P(B) ⊂ cl(P(B)), where cl(P(B)) ∈ C(S). As P is u.spgωα.C, B ⊂ P^+(cl(P(B))). From Theorem 2.1, P^+(cl(P(B))) ∈ spgωα-C(R). Thus spgωα-cl(B) ⊂ P^+cl(P(B))) and so P(spgωα-cl(B)) ⊂ cl(P(B)).
(2) → (3): Let A ⊂ S and so P^+(A) ⊂ R. From (2), P(spgωα-cl(P^+(A))) ⊂ cl(P(P^+(A))) = cl(A). Thus spgωα-cl(P^+(A)) ⊂ P^+cl(A).
(3) → (4): Let A ⊂ S. Apply (3) to S - A, then spgωα-cl(P^+(S - A)) ⊂ P^+cl(S - A)) , spgωα-cl(R–P^-(A)) ⊂ P^+(S - int(A)) , R - spgωα-int(P^-(A)) ⊂ R – P^-(int(A)), P^-(int(A)) ⊂ spgωα-int(P^-(A)).

(4) → (5): Let B ⊂ R, so P(B) ⊂ S. From (4), P^-(int(P(A)) ⊂ spgωα-int(P^-(P(A))) = spgωα-int(A). Thus int(P(A)) ⊂ P(spgωα-int(A)).
(5) → (1): Let r ∈ R and A ∈ O(S, P(r)). So, r ∈ P^+(A) where P^+(A) ⊂ R. From (5), we have int(P(P^+(A))) ⊂ P(spgωα-int(P^+(A))). Then int(A) ⊂ P(spgωα-int(P^+(A))). As A ∈ O(R), then A ∈ P(spgωα-int(P^+(A))), that is P^+(A) ⊂ spgωα-int(P^+(A)). Thus P^+(A) ∈ spgωα-O(R, r) and P(P^+(A)) ⊂ A. Hence P is u.spgωα.C.

Theorem 2.7. Let P: R → S be u.spgωα.C with Q ⊂ S. If Q is closed in S, then P: R → Q is u.spgωα.C.
Proof. Let Q ⊂ C(S), so Q - S ∈ C(S). Then Q - S ∈ spgωα-C(S). By u.spgωα.C, P^+(Q - S) ∈ spgωα-C(R) with P(r) ∈ S. Thus P^+(Q) = P^+(Q - S) ∈ spgωα-C(R). From theorem 2.1 P: R → Q is u.spgωα.C.

Theorem 2.8. [6] Intersection of any two spgωα-closed set is again spgωα-closed.

Theorem 2.9. Let P: R → S be u.spgωα.C and A ∈ spgωα-C(R). Then P - A: A → S is u.spgωα.C.
Proof. Let B ∈ C(S) as P is u.spgωα.C. Then P^+(B) ∈ spgωα-C(R). As intersection of two spgωα-closed set is closed, then P^+(B) - A = A_1, where A_1 ∈ spgωα-C(R). Then $(P - A)^+$(B) = A_1 ∈ spgωα-C(R). Hence P - A is u.spgωα.C.

Theorem 2.10. If P: R → S is u.spgωα.C injective with S is T_1, then R is spgωα-T_1.
Proof. Let S be T_1 space, so for each distinct points p_1, p_2 ∈ R, there exist A, B ∈ O(S) such that P(p_1) ∈ A, P(p_2) ∉ A and P(p_1) ∉ B, P(p_2) ∈ A. Since P is u.spgωα.C, there exist U, V ∈ spgωα-O(R) with p_1 ∈ U, p_1 ∉ V and p_2 ∉ U, p_2 ∈ V, that is p_1 ∈ U, p_2 ∈ V and P_1(U) ⊂ A, P_1(V) ⊂ B. Hence R is u.spgωα-T_1.

Theorem 2.11. Let P: R → S is u.spgωα.C injective and S is T_2 -space. Then R is spgωα-T_2.
Proof. Let p_1, p_2 ∈ R be any two distinct points. Then, G, H ∈ O(S) such that P(p_1) ∈ G, P(p_2) ∈ H. As P is u.spgωα.C, there exist A, B ∈ spgωα-O(R) such that P(G) ⊂ A, P(H) ⊂ B with A ∩ B = ϕ and so G ∩ H = ϕ. Hence R is spgωα-T_2.

Theorem 2.12. For a m.f P: R → S is u.spgωα.C, image of spgωα-connected space is spgωα-connected.
Proof. Let P: R → S is u.spgωα.C. Suppose S is not connected and S = A ∪ B with a partition of S, where A ∈ O(S) and B ∈ C(S). Since P is u.spgωα.C, P^+(A), P^+(B) ∈ spgωα-O(R) where P^+(A) ∩ P^+(B) = ϕ and R = P^+(A) ∪ P^+(B) is a partition of R, which contradicts that, R is spgωα-connected.

Definition 2.3 (7). For a m.f P: R → S, the graph m.f G_p: R → R x S is defined as G_P (p) = {p} x P(p) for every p ∈ R.

Lemma 2.1 (7). For a m.f P: R → S,
1. G^+_P (A x B) = A x P^+(B)
2. G^-_P (A x B) = A x P^-(B) for A ⊂ R and B ⊂ S.

Theorem 2.13. Let P: R → S be a m.f. If the graph m.f G_P is u.spgωα.C, then P is u.spgωα.C, where G_P : R → R x S is defined as G_P (p) = {p} x P(p).

Proof. Let p ∈ R and V ∈ O(S, P(p)). Then R x V ∈ O(R x S) and G_P (p) ⊂ R x V . As G_P is u.spgωα.C, there exists G_P (V) ⊂ R x V . Thus U ⊂ G^+_P (R x V). By lemma 2.1, G^+_P (R x V) = P^+(V) and so U ⊂ P^+(V). Thus P is u.spgωα.C.

Theorem 2.14. Let P: R → S be a m.f. If the graph m.f G_P is l.spgωα.C. Then P is l.spgωα.C.

Proof. Let p ∈ R and V ∈ O(S) with p ∈ P^-(V). Then R x V is open in R x S. Also, we have G_P (p) ∩ (R x V) = ({p}) x P(p)) ∩ (R x V) = ({p}) x P(p) ∩ V ≠ ϕ. As G_P l.spgωα.C, there exists U ∈ spgωα-O(R, r) such that U ∈ G^-_P (R x V) and from lemma 2.1 G^-_P (R x V) = P^-(V) and so U x P^-(V). Thus P is l.spgωα.C.

Theorem 2.15. Let (R, τ), (S, σ), (Q, γ) be TS and P_1: R → S and P_2 : S → Q be m.f. Let P_1 x P_2 : R → S x Q be a m.f defined by (P_1 x P_2)(p) = P_1(p) x P_2(p) for each p ∈ R if (P_1 x P_2) is u.spgωα.C. Then P_1 and P_2 are u.spgωα.C.

Proof. Let p ∈ R and V x S and W x Q be open sets with p ∈ P^+_1 (V) and p ∈ P^+_2 (W) and so P_1(p) ⊂ V and P_2(p) ⊂ W. Thus (P_1 x P_2)(p) = P_1(p) x P_2(p) ⊂ V x W and so p ∈ (P_1x P_2)$^+$(V x W) as P_1 x P_2 is u.spgωα.C, there exists spgωα-open set U containing p such that U ⊂ (P_1 x P_2)$^+$(V x W) that is U ⊂ P^+_1 (V) and U ⊂ P^+_2 (W) Hence P_1 and P_2 are u.spgωα.C

Similarlly we can prove the results for l.spgωα.C.

Theorem 2.16. Let P: R → S be compact m.f. Then G_P is u.spgωα.C if and only if P is u.spgωα.C.

Proof. Suppose G_P : R → S be u.spgωα.C. Let p ∈ R and V ∈ O(S, P(p)). Since R x V is open in R x S and G_P (p) ⊂ R x S and so there exists U ∈ spgωα-O(R, p) such that G_P (U) ⊂ R x S. From lemma 2.1, U ⊂ G^+_P (R x V) = P^+(V) and P(U) x V and hence P is u.spgωα.C.

Conversely, Let P be u.spgωα.C. Let p ∈ R and W be any open set in R x S containing G_P(p). Then for each n ∈ P(p), there exists U(n) ∈ R and V (n) ∈ S such that (p, n) ∈ U(n) x V (n) ⊂ W. The family {V (n) : n ∈ P(p)} is an open cover of P(p). As P(p) is compact, there exists finite number of points say n_1, n_2, …. n_k in P(p) such that P(p) ⊂ {V(n_i): i = 1; 2; ….k}. Put U = ∩ {U(n_i): i = i=1,2….k} and V = ∪ {V (n_i): i = 1,2,…k}. Then U and V are open sets in R and S respectively and {p} x P(p) ⊂ U X V ⊂ W. As P is u.spgωα.C there exists U ∈ spgωα-

O(R, p) such that $P(U_1) \subset V$. B lemma 2.1, $U \cap U_1 \subset U \cap P^+(V) = G^+_P (U \times V) \subset G^+_P (W)$. Thus $U \cap U_1 \in$ spgωα-O(R, p) and $G_P (U \cap V) \subset W$. Hence G_P is u.spgωα.C.

Theorem 2.17. A m.f P: R → S is if and only if G_P is l.spgωα.C.

Proof. Let P be l.spgωα.C. Let $p \in R$ and $W \in O(R \times S)$ with $p \in G^-_P (W)$. As $W \cap (\{p\} \times P(p)) \neq \phi$, there exists $n \in P(p)$ with $(p, n) \in W$ and so $(p, n) \in U \times V \subset W$, where $U \in O(R)$ and $V \in O(S)$ respectively. Since $P(p) \cap V \neq \phi$, there exists $G \in$ spgωα-O(R, p) with $G \subset P^-(V)$. By Lemma 2.1, $U \cap G \subset U \cap P^-(V) \subset G^-_P (U \times V) \subset G^-_P (W)$. Thus $p \in U \times G \in$ spgωα-O(R, p) and hence G_P is l.spgωα.C.

Conversely, let G_P be l.spgωα.C, $p \in R$ and $V \in O(S)$ with $p \in P^-(V)$. So $R \times V \in O(R \times S)$ with $G_P (p) \cap (R \times V) = (\{p\} \times P(p)) \cap (R \times V) = (\{p\} \times (P(p) \cap V) \neq \phi$. Since G_P is l.spgωα.C, there exists $U \in$ spgωα - O(R, p) with $U \subset (R \times V)$. From Lemma 2.1, $U \subset P^-(V)$ and so P is l.spgωα.C.

3. Upper (Lower) spgωα-Irresolute Multifunctions

Definition 3.1. A m.f P: R → S is called

1. **Upper spgωα-irresolute** (briefly U.spgωα.I) if for each $p \in R$ and each $V \in$ spgωα - O(S, P(p)), there exists $U \in$ spgωα - O(R, p) such that $P(U) \subset V$.
2. **Lower spgωα-irresolute** (briefly l.spgωα.I) if for each $p \in R$ and each spgωα-open set V with $P(p) \cap V \neq \phi$, there exists $U \in$ spgωα - O(R, p) such that $U \subset P^-(V)$.

Theorem 3.1. For a m.f P: R → S the following statements are equivalent:

1. P is u.spgωα.I.
2. for each $p \in R$, for each spgωα-nbd. V of P(p), we have $P^+(V)$ is spgωα- nbd. of p.
3. for each $p \in R$, for each spgωα-nbd. V of P(p), there exists spgωα-nbd. U of p with $P(U) \subset V$.
4. $P^+(V) \in$ spgωα- O(R) for each $V \in$ spgωα - O(S).
5. $P^-(V) \in$ spgωα - C(R) for each $V \in$ sgωα - C(S).
6. spgωα - cl($P^-(B)$) $\subset P^-$(spgωα - cl(B)) for each $B \subset S$.

Proof. (1) → (2): Let $p \in R$ and W be spgωα-nbd. of P(p). Then there exists $V \in$ spgωα - O(S) with $P(p) \subset V \subset W$. As P is u.spgωα.I, there exists $U \in$ spgωα - O(R, p) such that $P(U) \subset V$. Thus $p \in U \subset P^+(V) \subset P^+(W)$ and so $P^+(W)$ is a spgωα-nbd. of p.

(2) → (3): Let $p \in R$ and V be a spgωα-nbd. of P(p). Let $U = P^+(V)$. Then by (2), U is spgωα-nbd of p with $P(U) \subset V$.

(3) → (4): Let $V \in$ spgωα -O(S) and $p \in P^+(V)$. Then there exists a spgωα - nbd G of p with $P(G) \subset V$. Thus for some $U \in$ spgωα - O(R, p) with $U \subset G$ and $P(U) \subset V$. So $p \in U \subset P^+(V)$ and hence $P^+(V) \in$ spgωα - O(S).

(4) → (5): Let $A \in$ spgωα - C(S) and so $R – P^-(A) = P^+(S - K) \in$ spgωα - O(R). Thus $P^-(A) \in$ spgωα-C(S).

(5) → (6): Let B ⊂ S. Since spgωα - cl(B) is spgωα-closed in S, so P^-(spgωα - cl(B)) ∈ spgωα -C(R) with P^-(B)) ⊂ P^-(spgωα - cl(B). Thus spgωα- cl(P^-(B)) ⊂ P^-(spgωα- cl(B)).
(6) → (1): Let p ∈ R and V ∈ spgωα- O(R) with P(p) ⊂ V . So P(p) ∩ (S - V) = ϕ. Hence p ∉ P^-(S - V). From (6), p ∈ spgωα - cl(P^-(S - V)) and so there exists U ∈ spgωα-O(R, p) such that U ∩ P^-(S - V) = ϕ. Thus P(U) ⊂ V and so P is u.spgωα.I.

Theorem 3.2. The following statements are equivalent for a m.f P: R → S:

1. P is l.spgωα.I.
2. for each V ∈ spgωα-O(S) and each p ∈ P^-(V), there exists U ∈ spgωα - O(R, p) such that U ⊂ P^-(V).
3. P^-(V) ∈ spgωα- O(R) for each V ∈ spgωα-O(S).
4. P^+(K) ∈ spgωα - C(R) for each K ∈ spgωα-C(S).
5. for each A ⊂ R, P(spgωα-cl(A)) ⊂ spgωα-cl(P(A)).
6. spgωα-cl(P^+(B)) ⊂ P^+(spgωα-cl(B)) for each B ⊂ S.

Proof. (1) → (2): Follows by the definition.
(2) → (3): Let V ∈ spgωα-O(S) with p ∈ P^-(V). From (2) there exists U ∈ spgωα-O(R, p) such that U ⊂ P^-(V). Thus, p ∈ U ⊂ cl(int(U)) ∪ int(cl(U)) ⊂ cl(int(P^-(U))) ∪ int(cl(P^-(U))). So P^-(V) ∈ spgωα-O(R).
(3) → (4): Let K ∈ spgωα-C(S). Then R - P^+(K) = P^-(S - K) ∈ spgωα-O(R) and so P^+(K) ∈ spgωα-C(R).
(4) → (5) and (5) → (6) are Straight forward.
(6) → (1): Let p ∈ R and V ∈ spgωα-O(S) with P(p) ∩ V ≠ ϕ. So P(p) ∩ (S - V) = ϕ. Then P(p) ⊄ S - V and p ∉ P^+(S - V). Since S - V ∈ spgωα - C(S) and by (6), p ∉ spgωα - cl(P^+(S - V)) and so there exists U ∈ spgωα - O(R, p) with U ∩ P^-(S - V) = U ∩ (R – P^-(V)) = ϕ. Thus U ⊂ R - (R – P^-(V)) = P^-(V), that is U ⊂ P^-(V). So P is l.spgωα.I.

Lemma 3.1. Let P be a m.f. Then (spgωα - cl(P))$^-$(V) = P^-(V) for each V ∈ spgωα - O(R).
Proof. Let V ∈ spgωα - O(S) with p ∈ (spgωα-cl(P))$^-$C(V) so V ∩ (spgωα - cl(P))(p) ≠ ϕ. As V ∈ spgωα-O(S) and so V ∩ P(p) ≠ ϕ. Thus p ∈ P^-(V).
Conversely, Let p ∈ P^-(V). Then V ∩ P(p) ⊂ (spgωα- cl(P))(p) ∩ V ≠ ϕ and so p ∈ (spgωα - cl(P))$^-$(V). Thus (spgωα-cl(P))$^-$(V) = P^-(V).

Lemma 3.2. [6] Let A, B ⊂ R. Then

1. If A ∈ spgωα-O(R) and B ∈ R then A ∩ B ∈ spgωα-O(B).
2. If A ∈ spgωα-O(B) and B ∈ spgωα-O(R) then A ∈ spgωα-O(R).

Theorem 3.3. Let P be a m.f. and U ∈ O(R). If P is u.spgωα.I (resp l.spgωα.I) then $P_{|U}$:U → S is u.spgωα.I (resp l.spgωα.I).

Proof. Let V ∈ spgωα-O(S) and p ∈ U, p ∈ $P^{-}_{\,|U}$(V). As P is l.spgωα.I there exists G ∈ spgωα-O(R, r) with G ⊂ P^{-}(V) and so p ∈ G ∩ U ∈ spgωα-O(U) with G ∩ U ⊂ $P_{|U}$ (V). Thus $P_{|U}$ is l.spgωα.I.
Similarlly, we can prove for u.spgωα.I.

Definition 3.2. A subset B of a space R is said to be
1. spgωα-compact relative to R (resp. spgωα-Lindelof relative to R) if every cover of B by spgωα-open sets of R has a finite (resp. countable) subcover.
2. spgωα-compact (spgωα-Lindelof) if R is spgωα-compact (resp spgωα-Lindelof) relative to R.

Theorem 3.4. If P be u.spgωα.I m.f and P(p) is spgωα-compact relative to S for each p ∈ R. If B is spgωα-compact relative to R, then P(B) is spgωα-compact relative to S.
Proof. Let {V_i : i ∈ Λ} be a cover of P(B) by spgωα-open sets in S. Then for each p ∈ B, there exists a finite subset Λ(p) ∈ Λ with P(p) ⊂ ∪{V_i : i ∈ Λ(p)} and so P(p) ⊂ V(p) ∈ spgωα-O(S). Thus there exists a finite number of points of B, p_1, p_2, p_3, … p_k with B ⊂ ∪{V (pi): i = 1, 2, 3…. k}.Thus P(B) ⊂ P($\bigcup_{i=1}^{k}\{V_i(p_i)\}$) ⊂ $\bigcup_{i=1}^{k} P(V_i(P_i)) \subset \bigcup_{i=1}^{k} V(P_i) \subset \bigcup_{i=1}^{k}\bigcup_{i\in\Lambda(p_i)} V_i$. Thus P(B) is spgωα-compact relative to S.

Corollary 3.1. Let P be an u.spgωα.I surjective m.f and P(B) is spgωα-compact relative to S for each p ∈ R. If R is spgωα-compact, then S is spgωα-compact.

Theorem 3.5. If P is u.spgωα.I and P(p) is spgωα-Lindelof relative to S for each p ∈ R and if B is spgωα-Lindelof relative to R, then P(B) is spgωα-Lindelof relative to S.
Proof. Similar to proof of theorem 3.4.

Definition 3.3. A space R is said to be spgωα-normal (briefly spgωα.N) if for any pair of distinct spgωα-closed sets A and B in R there exists disjoint open sets U and V in R such that A ⊂ U, B ⊂ V.

Theorem 3.7. The set of a point p of R at which a m.f P is not u.spgωα.I (resp l.spgωα.I) is identical with the union of the spgωα-frontiers of the upper (lower) inverse image of spgωα-open sets containing (respectively meeting) P(p).
Proof. Let p ∈ R at which P is not u.spgωα.I. Then there exists V ∈ spgωα - O(S) containing P(p) with U ∩ (R - P^{+}(V)) ≠ ϕ for each U ∈ spgωα- O(R, p). Then p ∈ spgωα- cl(R - P^{+}(V)) as p ∈ P^{+}(V). So p ∈ spgωα - cl(P^{+}(S)) and p ∈ spgωα-F_r(P^{+}(B)). On the other hand V ∈ spgωα-O(S) containing P(p) and p ∈ spgωα-F_r(P^{+}(B)). Let P is u.spgωα.I, there exists U ∈ spgωα-O(R, p) with P(U) ⊂ V. Thus p ∈ U ⊂ spgωα-int(P^{+}(V)) which contradicts to the fact that p ∈ spgωα-F_r(P^{+}(V)). Hence P is not u.spgωα.I.
Similarlly we can prove the theorem related to l.spgωα.I.

Theorem 3.8. Let P be an u.spgωα.I injective m.f and point closed from a TS R to spgωα-N, then R is spgωα-T_2-space.

Proof. Let m, n ∈ R with m ≠ n. Then P(m) ∩ P(n) = ϕ as P is injective. By spgωα-normality of S, there exists disjoint open sets U and V containing P(m) and P(n). Thus, there exists disjoint spgωα-open sets $P^+(U)$ and $P^+(V)$ containing m and n respectively such that G ⊂ $P^+(U)$ and H ⊂ $P^+(V)$ and so G ∩ H = ϕ. Hence R is spgωα-T_2-space.

Definition 3.4. A m.f P: R → S is said to have spgωα-closed graph if for each (m, n) ∉ G(P) there exists U ∈ spgωα-O(R, m) and V ∈ spgωα-O(S, n) with (U x V) ∩ G(P) = ϕ.

Theorem 3.9. Let P be a m.f from a space R into spgωα-compact space S. If G(P) is spgωα-closed then P is u.spgωα.C.

Proof. Suppose P is not u.spgωα.C. Then there exists a non empty closed subset B with $P^-(B)$ is not spgωα-closed in R. Assume that $P^-(B)$ ≠ ϕ, then there exists a point p_0 ∈ spgωα-cl($P^-(B)$ – $P^-(B)$). So for each point n ∈ B, (p_0, n_0) ∉ G(P), as P is spgωα-closed graph. Thus there exists a spgωα-open sets U(n) and V (n) containing p_0 and n respectively with (U(n) x V (n)) ∩ G(P) = ϕ. Then {S – B} ∪ {V(n) : n ∈ B} is a spgωα-open cover of SN and so it has a subcover {S – B} ∪ {V(n_i): n_i ∈ B: 1 < i < k}. Put U = ∪U(n_i) and V = ∪V(n_i), then B ⊂ V and (U x V) ∩ G(P) = ϕ as U is spgωα-nbd. of p_0 U – $P^-(B)$ = ϕ and so ϕ ≠ (U x B) ∩ G(P) ⊂ (U x V) ∩ G(P) which is contradiction. Thus P is u.spgωα.C.

Discussion and Conclusion

Topology is a relatively new branch of Mathematics, most of the research in topology has been done since 1900. The topological structures are modelled suitably in the field of Computer graphics, Pattern recognition, Artificial intelligence, Data mining, Rough set theory, Information systems, Quantum physics etc.

The investigation on generalization of open set as well as closed set has led to significant contribution to the theory of generalization of continuity, separation axioms, covering properties and compactness with the help of open sets. Several generalized form of continuous functions has been introduced in the last decades which helps us to understand various properties of topological spaces.

In this way, this paper introduce some concepts of multifunctions in topological spaces. We first introduced the concept of upper spgωα-continuous (resp. lower spgωα-continuous) multifunctions and some properties and point out the relationship among them. Further, we introduce upper spgωα-irresolute (resp. lower spgωα-irresolute) multifunctions and several results of these spaces in topological spaces.

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