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A Two-Warehouse Inventory System with Time-Dependent Demand and Preservation Technology

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Abstract:
Because of rapid environmental changes in current worldwide society, preserving technique is becoming increasingly crucial. With the aim to examine this idea, we created a model for two-warehouse inventory system that incorporates preservation technique investments. The model considers time as a linear function representing the rate of demand. Time-dependent demand is a common feature in many industries, where demand fluctuates over time due to various factors such as seasonality and market trends. The stock is kept in the restricted storage area (OW) and other warehouse (rented) is utilized for keep any excess stock that exceeds the OW's storage limit. The stock in RW is consumed first, followed by the stock in OW, as the cost of holding of the rented warehouse (RW) is more than that of the owned warehouse (OW). This paper does not allow for shortages. We use numerical simulations to illustrate the effectiveness of our proposed model under various parameters.

Keywords: Inventory system, preservation technique, rented warehouse, time-dependent demand.

1. Introduction

To keep long-term relationships with clients, every firm keeps things in storage. Stock exceeding the owned warehouse's storage space, which is only capable of holding a particular amount of inventory, is a common problem. In this case, the company pays a high storage cost to rent a warehouse due to good preservation techniques in the rented warehouse. The two-warehouse inventory model is a well-established approach for managing inventory in such environments, where one warehouse serves as a regular storage facility while the other acts as a backup or emergency storage. In real-world scenarios, demand for several products is not constant but changes over time due to aspects like seasonality, promotions and consumer preferences. Before storing the units in the rented warehouse, it is reasonable to keep the units in the owned warehouse first as the cost of holding of rented warehouse is comparatively greater than the cost of holding of owned warehouse, which results in reduction of inventory cost.

For deteriorating items, Hartley [1] & Sarma [2] initiated to introduce the idea of model for two-warehouse. Singh & Rathore [3], Kumar et al. [4], Singh & Rathore [5], Sharma & Chaudhary [6] , Agrawal & Banerjee [7] and Singh et al. [8] considered the two-warehouse issue having shortages & backlogging. Zhou & Yang [9] examined the two-warehouse issue for demand as stock dependent. Gupta et al. [10, 11] conducted an investigation on applications of warehouse and supply chain management. Saxena [12, 13] also evaluated the EOQ models for degrading items. Rastogi et al. [14] presented a model for dual-warehouse incorporating price-sensitive demand with decaying characteristics. Mandal & Phaujdar [15] also worked on dual warehouse having stock-dependent demand. Technology for product preservation is becoming increasingly crucial and necessary because of environment's continuous changes. The rate of deterioration can be reduced to a definite extent by using safety techniques and some procedural changes.

In this study, we have formed an inventory control model for two-warehouse having deteriorating items by considering the facilities of preservation. Firstly, the items are consumed from rented warehouse and then consumed from owned warehouse owing to the higher holding cost of RW than owned warehouse. The demand's rate is taken as time dependent. However, shortages aren't allowed. We have also considered a linear function of demand. The primary focus of formulation of this model aims to reduce the total inventory cost. In the final, numerical case is given to verify the model.

2. Notations

Z (I(t)): Instantaneous time dependent rate of demand;

q: Quantity of Order;

θ: constant rate of deterioration;

m(ξ): (= (1-$e^{-a\xi}$)) reduction in deterioration rate ξ>0;

σ_θ: Aggregate rate of deterioration σ = (θ- m (ξ));

O: Cost of ordering;

H_1: Cost of holding of Rented Warehouse (RW);

H_2: Cost of holding of Owned Warehouse (OW);

x: Cost of unit purchasing;

S_o: Amount of Inventory in Own Warehouse;

S_1: Amount of inventory in Rented Warehouse;

t_r: The period of time in which RW inventory level drops to zero;

t_o: The period of time in which OW inventory level drops to zero;

T: Cycle length;

$I_1(t)$: Stock level in RW during [0, t_r];

$I_2(t)$: Stock level in OW during [0, t_r];

$I_3(t)$: Stock level in OW during [t_r, t_o];

TRC: Total relevant cost;

PC: Purchasing Cost;

HC: Holding Cost;

3. Assumptions

- Rate of demand is time dependent & taken as

$$Z\big(I(t)\big) = a + bt$$

- No shortages are allowed.
- Time horizon is taken as infinite.
- Preservation technique is used for reducing deterioration.
- Lead time is assumed as zero and replenishment rate is taken infinite.
- The owned warehouse (OW) has only W2 units of space, and unlimited amount of space in rented warehouse.
- Because the amount of holding (H_1) of RW is greater than cost of holding (H_2) of OW, utilization of inventory in OW begins only when the inventory in RW reaches zero.
- The price of transportation and time for both RW and OW are lesser.

4. Formulation of Mathematical Model

The inventory is S at t = 0, with S_2 units kept in OW & rest of S_1 units kept in RW. Because of demand & deterioration the inventory of rented warehouse decreases in the interval [o, t_r] and become zero at t = t_r. After time t_r, the demand for items is met by using OW inventory at [t_r; t_o].

The depletion in level of inventory in rented warehouse is defined via the equation:

$$\frac{d(I_1(t))}{dt} + \sigma_\theta I_1(t) = -(a + bt); \qquad 0 \le t \le t_r \tag{1}$$

And working of inventory in Owned warehouse is give as follows:

$$\frac{d(I_2(t))}{dt} + \sigma_\theta I_2(t) = 0; \quad 0 \le t \le t_r \tag{2}$$

$$\frac{d\big(I_3(t)\big)}{dt} + \sigma_\theta I_3(t) = -(a + bt); \quad t_r \le t \le t_o \tag{3}$$

Now solving the equation (1), (2) & (3), we get

$$I_1 = -\frac{a}{\sigma_\theta} - \frac{bt}{\sigma_\theta} + \frac{b}{\sigma_\theta^2} + c_1 e^{-\sigma_\theta t} \tag{4}$$

$$I_2 = e^{-\sigma_\theta t} c_2 \tag{5}$$

$$I_3 = -\frac{a}{\sigma_\theta} - \frac{bt}{\sigma_\theta} + \frac{b}{\sigma_\theta^2} + c_3 e^{-\sigma_\theta t} \tag{6}$$

Now using the boundary conditions,

$$I_1(t = 0) = S_1, I_1(t = t_r) = 0, I_2(t = 0) = S_2,$$

$$I_2(t = t_r) = I_3(t = t_r), I_3(t = t_o = T) = 0$$

$$I_1 = S_1 e^{-\sigma_\theta t} + \frac{a}{\sigma_\theta}(e^{-\sigma_\theta t} - 1) - \frac{b}{\sigma_\theta^2}(e^{-\sigma_\theta t} - 1) - \frac{bt}{\sigma_\theta} \quad (7)$$

$$I_2 = e^{-\sigma_\theta t} S_2 \quad (8)$$

$$I_3 = -\frac{a}{\sigma_\theta} - \frac{bt}{\sigma_\theta} + \frac{b}{\sigma_\theta^2} + S_2 e^{-\sigma_\theta t} + e^{-\sigma_\theta (t+t_r)}\left[\frac{a}{\sigma_\theta} + \frac{bt_r}{\sigma_\theta} - \frac{b}{\sigma_\theta^2}\right] \quad (9)$$

$$S_1 = \left[-\frac{a}{\sigma_\theta} + \frac{b}{\sigma_\theta^2}\right](1 - e^{\sigma_\theta t_r}) + \frac{bt_r}{\sigma_\theta} e^{\sigma_\theta t_r} \quad (10)$$

$$S_2 = e^{\sigma_\theta T}\left[\frac{a}{\sigma_\theta} + \frac{bT}{\sigma_\theta} - \frac{b}{\sigma_\theta^2}\right] - e^{\sigma_\theta t_r}\left[\frac{a}{\sigma_\theta} + \frac{bt_r}{\sigma_\theta} - \frac{b}{\sigma_\theta^2}\right] \quad (11)$$

Total Order Quantity $q = S_1 + S_2$

$$q = \left[-\frac{a}{\sigma_\theta} + \frac{b}{\sigma_\theta^2}\right](1 - e^{\sigma_\theta t_r}) + \frac{bt_r}{\sigma_\theta} e^{\sigma_\theta t_r} + e^{\sigma_\theta T}\left[\frac{a}{\sigma_\theta} + \frac{bT}{\sigma_\theta} - \frac{b}{\sigma_\theta^2}\right] - e^{\sigma_\theta t_r}\left[\frac{a}{\sigma_\theta} + \frac{bt_r}{\sigma_\theta} - \frac{b}{\sigma_\theta^2}\right] \quad (12)$$

The following cost parameters are part of the overall relevant cost:

1. Cost of ordering $OC = O_c$
2. The amount of Carrying Cost is obtained by:
   $HC = H_1\left[\int_0^{t_r} I_1(t)dt\right] + H_2\left[\int_0^{t_r} I_2(t)dt + \int_{t_r}^{T} I_3(t)\,dt\right]$

$$HC = H_1\left[(1 - e^{-\sigma_\theta t_r})\left[\frac{S_1}{\sigma_\theta} + \frac{a}{\sigma_\theta^2} - \frac{b}{\sigma_\theta^3}\right] - \frac{at_r}{\sigma_\theta} + \frac{bt_r}{\sigma_\theta^2} - \frac{bt_r^2}{2\sigma_\theta}\right] + H_2\left[\left[\frac{S_2}{\sigma_\theta}(1 - e^{-\sigma_\theta t_r})\right] + \frac{a}{\sigma_\theta}(t_r - T) + \frac{b}{2\sigma_\theta}(t_r^2 - T^2) - \frac{b}{\sigma_\theta^2}(t_r - T) + \frac{S_2}{\sigma_\theta}(e^{-\sigma_\theta t_r} - e^{-\sigma_\theta T}) + \left[\frac{a}{\sigma_\theta^2} + \frac{bT}{\sigma_\theta^2} - \frac{b}{\sigma_\theta^3}\right]\left(e^{-\sigma_\theta (t_r+T)} - e^{-2\sigma_\theta T}\right)\right] \quad (13)$$

Now, the complete relevant cost is given by:

$TRC = \frac{1}{T}[OC + HC]$

$$TRC = \frac{1}{T}\left[0 + H_1\left[(1 - e^{-\sigma_\theta t_r})\left[\frac{S_1}{\sigma_\theta} + \frac{a}{\sigma_\theta^2} - \frac{b}{\sigma_\theta^3}\right] - \frac{at_r}{\sigma_\theta} + \frac{bt_r}{\sigma_\theta^2} - \frac{bt_r^2}{2\sigma_\theta}\right] + H_2\left[\left[\frac{S_2}{\sigma_\theta}(1 - e^{-\sigma_\theta t_r})\right] + \frac{a}{\sigma_\theta}(t_r - T) + \frac{b}{2\sigma_\theta}(t_r^2 - T^2) - \frac{b}{\sigma_\theta^2}(t_r - T) + \frac{S_2}{\sigma_\theta}(e^{-\sigma_\theta t_r} - e^{-\sigma_\theta T}) + \left[\frac{a}{\sigma_\theta^2} + \frac{bT}{\sigma_\theta^2} - \frac{b}{\sigma_\theta^3}\right]\left(e^{-\sigma_\theta (t_r+T)} - e^{-2\sigma_\theta T}\right)\right]\right] \quad (14)$$

Next, we will differentiate the equation (14) with respect to $t_1, \xi \; and \; T$, to reduce the overall relevant cost.

Required conditions for ideal values are:

$$\frac{\partial TRC(t_r, \xi, T)}{\partial t_r} = 0, \qquad \frac{\partial TRC(t_r, \xi, T)}{\partial \xi} = 0,$$

$$\frac{\partial TRC(t_r,\xi,T)}{\partial T} = 0,$$

These values give the determinant value of principal minor of hessian matrix and det(H1)> 0, det(H2)> 0, det(H3) >0. H1, H2 & H3 are the principal minor of the Hessian Matrix.

The total function as a Hessian matrix is as follows:

$$\begin{bmatrix} \frac{\partial^2 TRC}{\partial t_r^2} & \frac{\partial^2 TRC}{\partial t_r \partial \xi} & \frac{\partial^2 TRC}{\partial t_r \partial T} \\ \frac{\partial^2 TRC}{\partial \xi \partial t_r} & \frac{\partial^2 TRC}{\partial \xi^2} & \frac{\partial^2 TRC}{\partial T \partial \xi} \\ \frac{\partial^2 TRC}{\partial T \partial t_r} & \frac{\partial^2 TRC}{\partial \xi \partial T} & \frac{\partial^2 TRC}{\partial T^2} \end{bmatrix}$$

From equation (14) we get

$$\frac{\partial TRC(t_r,\xi,T)}{\partial t_r} = \frac{1}{T}\left[H_1\left[S_1 e^{-\sigma_\theta t_r} + \frac{a}{\sigma_\theta}(e^{-\sigma_\theta t_r}) + \frac{b}{\sigma_\theta^2}(e^{-\sigma_\theta t_r} - 1) - \frac{bt_r}{\sigma_\theta}\right] + H_2\left[\frac{a}{\sigma_\theta} + \frac{bt_r}{\sigma_\theta} - \frac{b}{\sigma_\theta^2} - e^{-\sigma_\theta(t_r+T)}\left[\frac{a}{\sigma_\theta} + \frac{bT}{\sigma_\theta} - \frac{b}{\sigma_\theta^2}\right]\right]\right] \quad (15)$$

$$\frac{\partial TRC(t_r,\xi,T)}{\partial T} = -\frac{1}{T^2}\left[0 + H_1\left[(1 - e^{-\sigma_\theta t_r})\left[\frac{S_1}{\sigma_\theta} + \frac{a}{\sigma_\theta^2} - \frac{b}{\sigma_\theta^3}\right] - \frac{at_r}{\sigma_\theta} + \frac{bt_r}{\sigma_\theta^2} - \frac{bt_r^2}{2\sigma_\theta}\right]\right] + H_2\left[-\frac{S_2}{T^2\sigma_\theta}(1 - e^{-\sigma_\theta t_r}) - \frac{at_r}{\sigma_\theta T^2} - \frac{b}{2\sigma_\theta}\left(\frac{t_r^2}{T^2} + 1\right) + \frac{bt_r}{\sigma_\theta^2 T^2} - \frac{S_2}{T^2\sigma_\theta^2}(e^{-\sigma_\theta t_r} - e^{-\sigma_\theta T}) + \frac{S_2 e^{-\sigma_\theta T}}{T\sigma_\theta} + \frac{b}{\sigma_\theta^2}\left(e^{-\sigma_\theta(t_r+T)} - e^{-2\sigma_\theta T}\right) + \left[\frac{a}{\sigma_\theta} + \frac{bT}{\sigma_\theta} - \frac{b}{\sigma_\theta^2}\right]\left(2e^{-2\sigma_\theta T} - e^{-\sigma_\theta(t_r+T)}\right)\right] = 0 \quad (16)$$

And find also,

$$\frac{\partial^2 TRC}{\partial t_r^2} = \frac{1}{T}\left[H_1\left[-S_1\sigma_\theta e^{-\sigma_\theta t_r} - \frac{b}{\sigma_\theta}[e^{-\sigma_\theta t_r} + 1]\right] + H_2\left[-S_2\sigma_\theta e^{-\sigma_\theta t_r} + \frac{b}{\sigma_\theta} + e^{-2\sigma_\theta t_r}(4a + 4bt_r) - e^{-\sigma_\theta(t_r+T)}\left[a + b - \frac{b}{\sigma_\theta}\right]\right]\right] \quad (17)$$

$$\frac{\partial^2 TRC}{\partial T^2} = -\frac{1}{T^2}\left[0 + H_1\left[-S_1\frac{e^{-\sigma_\theta t_r}}{\sigma_\theta} - \frac{a}{\sigma_\theta}\left(\frac{e^{-\sigma_\theta}}{\sigma_\theta} - t_r\right) + \frac{b}{\sigma_\theta^2}\left(\frac{e^{-\sigma_\theta t_r}}{\sigma_\theta} - t_r\right) - \frac{bt_r^2}{2\sigma_\theta}\right]\right] + H_2\left[\frac{-2}{T^3\sigma_\theta}(S_2 e^{-\sigma_\theta t_r} + at_r) + \frac{bt_r}{T^3\sigma_\theta^2}(t_r - 2) - \frac{e^{-\sigma_\theta t_r}}{\sigma_\theta^2}\left[a + bt_r + \frac{b}{\sigma_\theta}\right]\left[\frac{2}{T^3}(e^{-\sigma_\theta t_r} - e^{-\sigma_\theta T}) - \frac{\sigma_\theta e^{-\sigma_\theta T}}{T}\left[\frac{2}{T} - \sigma_\theta\right]\right]\right] \quad (18)$$

$$\frac{\partial^2 TRC}{\partial t_r \partial T} = \frac{\partial^2 TRC}{\partial T \partial t_r} = -\frac{1}{T^2}\left[H_1\left[S_1 e^{-\sigma_\theta t_r} + \frac{a}{\sigma_\theta} + \frac{b}{\sigma_\theta^2}(-e^{\sigma_\theta t_r} - 1) - \frac{b t_r}{\sigma_\theta}\right] + H_2\left[-\frac{1}{T^2}\left[S_2 e^{-\sigma_\theta t_r} - \frac{a}{\sigma_\theta} + \frac{b t_r}{\sigma_\theta} - \frac{b}{\sigma_\theta^2} + e^{-2\sigma_\theta t_r}\left[-\frac{2a}{\sigma_\theta} - \frac{2b t_r}{\sigma_\theta} - \frac{b}{\sigma_\theta^2}\right] - \frac{e^{-\sigma_\theta(t_r+T)}}{T}\left[\sigma_\theta + \frac{1}{T}\right]\right]\right]\right] \quad (19)$$

5. Effect of Preservation Technology

Within this framework, we have included preservation technology in order to reduce the deterioration rate. We have taken the following values

$$a = 3;\ \xi = 0.0136073;\ \theta = 0.09$$

We get the value of $m(\xi) = 0.04$ & $\sigma_\theta = 0.05$

6. Numerical Illustration

In order to demonstrate the model, we have taken the following inventory system:

$$O = 600;\ a = 250;\ b = 25;\ H_1 = 0.75;\ H_2 = 0.4;$$
$$S_1 = 500;\ S_2 = 300;\ \sigma_\theta = 0.05$$

We observed the optimal solution as

$$t_r^* = 2.97148;\ T^* = 10;\ TRC^* = 665.0132$$

7. Sensitivity Analysis

To analyse the sensitivity of the model, we execute a sensitivity analysis by changing the values of various key parameters such as demand parameters & deterioration rate. Table 1, Table 2 & Table 3 depicts the impact of parameter changes.

Table 1: Sensitivity analysis of optimal solution with respect to variation in parameter '**a**'

	t	T	TRC
Variation in Parameter 'a'			
220	2.03169	11.2	925.9562
235	2.50437	10.6	793.9031
265	3.43325	9.4	531.157011

Table 2: Sensitivity analysis of optimal solution with respect to variation in parameter '**b**'

	t	T	TRC
Variation in Parameter 'b'			
20	4.97261	7.5	144.115
30	1.57538	11.6	1207.225
35	0.543249	12.8	1781.7228

Table 3: Sensitivity analysis of optimal solution with respect to variation in parameter '$\boldsymbol{\sigma_\theta}$' because of modification in the value of **m(*ξ*)**

			t	**T**	**TRC**
Fluctuation in the value of ξ	**Fluctuation in the value of 'm(*ξ*)'**	**Variation in value of '$\boldsymbol{\sigma_\theta}$'**			
0.0190654	0.05	0.04	1.50836	15	1501.65025
0.010735	0.03	0.06	3.90582	6.66678	238.0357

- From table 1 we can see that on increasing the value of 'a', the value of 't' and 'TRC' decreases and the value of 'T' increases.
- From table 2 we can see that on increasing the value of 'b', the value of 'T' and 'TRC' increases and the value of 't' decreases.
- From table 3 we can see that as the value of 'ξ' decreases then value of 'σ_θ' increases which results in increase in the value of 't' and decrease in the value of 'T' and 'TRC'.

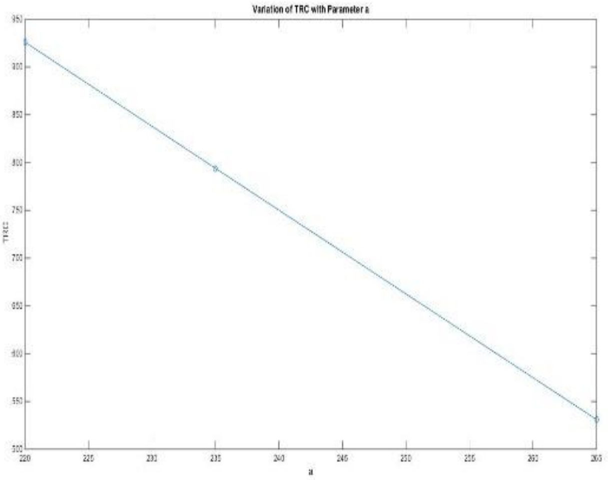

Fig (A)

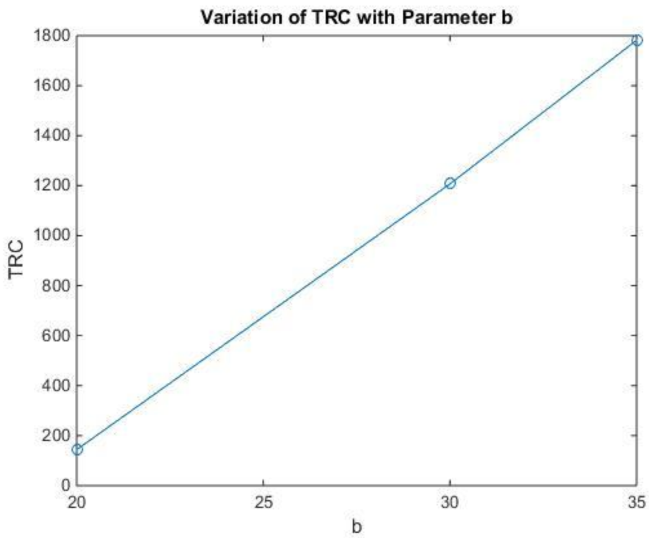

Fig (B)

8. Conclusion

For the dual warehouse model for inventory having demand as time-dependent and preservation technology, we found that the optimal strategy involves dynamically allocating inventory among the warehouses based on demand patterns and the effectiveness of preservation technology. We can see through the numerical illustration and sensitivity analysis that how the value of total relevant cost fluctuates for the different values of parameters. We can also see that by using the preservation technology we can reduce the rate of deterioration and preserves the products for the longer time. We have also drawn the graphs showing the variation in total cost with respect to the demand parameters. This approach maximizes the service levels while reducing the costs, ensuring that products will be available when needed with minimizing the risk of obsolescence. This model can be applicable for the items having linear time-dependent demand such as food items, raw materials and many more.

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