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Edge Irregularity Strength of Binomial Trees

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Abstract:
For a simple graph G, a vertex labeling φ: V (G) → {1, 2, · · ·, k} is called k- labeling. The weight of an edge *uv* in G, denoted by $w_φ(uv)$, is the sum of the labels of end vertices *u* and *v*. A vertex k-labeling is defined to be an edge irregular k- labeling of the graph G if for every two different edges e and f, $w_φ(e) \neq w_φ(f)$. The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength of G, denoted by es(G). In this paper, we prove that the edge irregularity strength of corona product of a tree T with K_1 is $es(T \circ K_1) = 2es(T)$. Further, we prove that the edge irregularity strength of binomial trees B_k is 2^{k-1}, for $k \geq 1$.

Keywords: edge irregularity strength; corona product of graphs; binomial trees;

1. Introduction

All the graphs considered in this paper are finite simple graphs. Terms that are not defined here can be referred from the book [10]. For a simple graph G, a vertex labeling φ: V (G) → {1, 2, · · ·, k} is called k-labeling. The weight of an edge *uv* in G, denoted by $w_φ(uv)$, is the sum of the labels of end vertices *u* and *v*. A vertex k-labeling is defined to be an edge irregular k-labeling of the graph G if for every two different edges e and f, $w_φ(e) \neq w_φ(f)$. The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength of G, denoted by es(G). In 1988, Chartrand et al. [4] introduced edge k-labeling δ of a graph G such that $w_δ(x) \neq w_δ(y)$ for all vertices x, y ∈ V (G) with x≠y. Such labelings were called irregular assignments and the irregularity strength s(G) of a graph G is known as the minimum k for which

G has an irregular assignment using labels at most k. This parameter has attracted many researchers and several articles [2, 3, 6, 9] were published based on irregularity strength of graphs.

In 2014, Ali Ahmad et al. [1] introduced a new parameter called edge irregularity strength of graphs. A vertex k-labeling φ: V (G) → {1, 2, · · ·, k} is called an edge irregular k- labeling of the graph if for every two different edges e and f there is $w_φ(e) \neq w_φ(f)$, where the weight of an edge e = xy ∈ E(G) is $w_φ(xy) = φ(x) + φ(y)$. The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength of G, denoted by es(G). For an exhaustive survey on edge irregularity strength of graphs, we refer to the dynamic survey on graph labeling by Gallian [7].

In [1], Ali Ahmad et al. proved that the lower bound for edge irregularity strength of a graph G is given by $es(G) \geq \max\left\{\left\lceil\frac{|E(G)|+1}{2}\right\rceil, \Delta(G)\right\}$. In this paper, we prove that the edge irregularity strength of corona product of a tree T with K_1 is es $(T \circ K_1) = 2es\,(T)$. Further, we prove that the edge irregularity strength of binomial trees B_k is 2^{k-1}, for $k \geq 1$. For all of our results proved in this paper, the edge irregularity strength of trees we proved attains its lower bound.

2 Corona Product of Graphs

Let G and H be two graphs and let n be the order of G. The corona product, or simply the corona, of graphs G and H is the graph $G \odot H$ obtained by taking one copy of G and n copies of H and then joining by an edge the *i*th vertex of G to every vertex in the *i*th copy of H. Given a vertex $g \in G$, the copy of H connected to g is denoted by H_g [8]. Complete graphs, stars and wheels are basic examples of corona product families.

Observation 1: When G is a tree T with m edges and $H \cong K_1$, the corona $T \circ K_1$ is also a tree with 2m + 1 edges. Thus, the number of newly added vertices in the corona product of T and K_1 will be m + 1 and all of those vertices are of degree 1.

Notation: For the sake of convenience, let $V(K_1) = \{w\}$ and if u be any vertex in the tree T, then the corresponding vertex added in the corona product $T \circ K_1$ will be denoted as w_u. In view of this notation, let us call the vertex u as the original vertex and the newly added vertex w_u as the duplicate vertex for u.

3 Main Result

In this section, we prove our main result.

Theorem 1. Let T be a tree with edge irregularity strength be es(T). Then es $(T \circ K_1) = 2es\,(T)$.

Proof. Let k = es (T) and $\varphi: V(T) \to \{1, 2, \cdots, k\}$ be the edge irregular k-labeling of T

Case 1: k is even

Let V_1 be the set of vertices whose vertex labels are from the set $\{1, 2, \cdots, \frac{k}{2}\}$ and V_2 be the set of vertices whose vertex labels are from the set $\{\frac{k}{2}+1, \cdots, k\}$.

Let us define the function $\psi: V(T \circ K_1) \to \{1, 2, \cdots, 2k\}$ as follows: For any vertex u in $V_1 \subset V(T)$, retain the vertex label for the corresponding vertex in $T \circ K_1$. That is, $\psi(u) = \varphi(u)$, for any vertex $u \in V_1$. For any vertex v in $V_2 \subset V(T)$, $\psi(v) = \varphi(v) + k$. At this stage, the original vertices of $T \circ K_1$ are all labeled by ψ. Let X be the set of all weights of original edges of $T \circ K_1$ as induced by ψ. Let $W = \{2, 3, \cdots, 2k\}$ be the set of all weights of the edges. Let $R = W - X$ be the required set of weights of edges. Let S be the sequence of arrangement of vertex labels in the increasing order as defined by ψ. Since $T \circ K_1$ is also a tree, we have the cardinality of the required set of weights of edges is equal to the length of the sequence S. Until $R \neq \phi$, choose a minimum weight (say r) in R and correspondingly choose the first term (say q) in the sequence S. Now choose the original vertex with label q as defined by ψ in $T \circ K_1$ and label its duplicate vertex as r − q so that the induced weight of the newly added edge between the original vertex and its duplicate vertex will be r. Then, delete r from R and construct a new sequence S by deleting its first term q. This procedure can be done until R becomes an empty set. When R = ϕ, all the vertices have been labeled by the function ψ in $T \circ K_1$. Therefore, by the construction of $T \circ K_1$ and the definition of ψ, es $(T \circ K_1) = 2k$.

Case 2: k is odd:

Let V_1 be the set of vertices whose vertex labels are from the set $\{1, 2, \cdots, \lceil \frac{k}{2} \rceil + 1\}$

and V_2 be the set of vertices whose vertex labels are from the set $\{\lceil \frac{k}{2} \rceil + 2, \cdots, k\}$.

As defined for the even case, similarly we define $\psi: V(T \circ K_1) \to \{1, 2, \cdots, 2k\}$. Therefore, either es (T) is odd or even, es $(T \circ K_1) = 2$es (T).

4 Binomial trees

The binomial tree B_0 consists of a single vertex. The binomial tree B_k is an ordered tree defined recursively. The binomial tree B_k consists of two binomial trees B_{k-1} that are linked together: the root of one is the leftmost child of the root of the other. Note that there are 2^k vertices in the binomial tree B_k. For more details about binomial trees refer [5].

4.1 Edge Irregularity Strength of Binomial Trees

In this section, we will calculate the edge irregularity strength of binomial trees.

Observation 2: One can easily observe that corona product of binomial tree B_{k-1} with K_1 leads to the binomial tree B_k, for $k \geq 1$. Thus, we have $B_{k-1} \circ K_1 = B_k$.

Theorem 2. Let B_k be the binomial trees for $k \geq 1$. Then es $(B_k) = 2^{k-1}$.

Proof. We prove this by the method of induction on $k \geq 1$. It is clear that es$(B_1) = 1$ and es$(B_2) = 2$. Let us assume that es $(B_{k-1}) = 2^{k-2}$. Since binomial tree $B_k = B_{k-1} \circ K_1$ and using Theorem 1, we have es $(B_k) = 2$es $(B_{k-1}) = 2.2^{k-2} = 2^{k-1}$.

5 Conclusion

In this paper, we proved that the edge irregularity strength of corona product of a tree T with K_1 is es $(T \circ K_1) = 2$es (T). Also, we proved that the edge irregularity strength of binomial trees B_k is 2^{k-1}, for $k \geq 1$. The edge irregularity strength of corona product of a tree and K_1 attains its lower bound provided es (T) attains its lower bound. Further, the edge irregularity strength of binomial trees B_k attains its lower bound 2^{k-1}.

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