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On a Certain Subclass of Analytic Functions Defined by Bessel Functions

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Abstract: In this work, we introduce and investigate a new subclass of analytic functions in the open unit disc U with negative coefficients. The object of the present paper is to determine the coefficient estimates, extreme points, integral means inequalities and subordination results for this class.
Keywords: analytic function, uniformly starlike function, coefficient estimate, subordination.

1.Introduction:

Let A be the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

and T denote the class of functions in the form of

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, (a_n \geq 0) \qquad (1.2),$$

which are analytic in the open unit disk $U = \{z: z \in \mathcal{C} \text{ and } |z| < 1\}$. This subclass was given in . Let $T^*(\alpha)$ and $C(\alpha)$ be indicate starlike and convex functions of order $\alpha, (0 \leq \alpha < 1)$, respectively. The classes $UCV(\alpha, \sigma)$ consists of uniform $\sigma -$convex functions of order α and $SP(\alpha, \sigma)$ consists parabolic $\sigma -$ starlike functions of order $\alpha, -1 < \alpha \leq 1, \sigma \geq 0$, generalizes the class UCV and SP respectively, were given in such that

$$UCV(\alpha, \sigma) = \left\{f \in A: Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} > \sigma\left\{\frac{zf''(z)}{f'(z)}\right\}, z \in U\right\} \qquad (1.3)$$

and

$$SP(\alpha, \sigma) = \left\{f \in A: Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > \sigma\left\{\frac{zf'(z)}{f(z)} - 1\right\}, z \in U\right\}. \qquad (1.4)$$

It is obvious from (1.3) and (1.4) that $f \in UCV(\alpha, \sigma)$ if and only if $zf'(z) \in SP(\alpha, \sigma)$. Some interesting situations of the class of starlike and convex of order α associated with Bessel functions (as hypergeometric function), finding condition on the triple p, b and c such that the function $u_{p,b,c}$ is starlike and convex of order α and finding conditions on the parameters for which the Gaussian hypergeometric functions belong to the various classes of functions have discussed in the references [1,2,4,9,10] . Let us take into consideration second order linear homogenous differential equation (see [3]).

$$z^2\omega''^{(z)} + bz\omega'^{(z)} + [cz^2 - p^2 + (1-b)p]\omega(z) = 0, \quad (p,b,c \in \mathcal{C}). \qquad (1.5)$$

As a particular solition of (1.5) generalized Bessel function of the first kind of order p, is defined in as following:

$$\omega(z) = \omega_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{n!\,\Gamma\left(p+n+\frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2n+p}, \ z \in C, \qquad (1.6)$$

where Γ stands for the Euler gamma function and $\tau = p + \frac{b+1}{2} \notin Z_0 = \{0,-1,-2,\cdots\}$. Though the series given in (1.6) is convergent everywhere, the function $\omega_{p,b,c}$ is not univalent in U. Specially, choosing $b = c = 1$ in (1.6), we get Bessel function of the first kind of order p given in as

$$J_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(p+n+1)} \left(\frac{z}{2}\right)^{2n+p}, z \in C. \qquad (1.7)$$

Choosing $b = 1$ and $c = -1$ in (1.6), we get the modified Bessel function of the first kind order of p given in as

$$I_p(z) = \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(p+n+1)} \left(\frac{z}{2}\right)^{2n+p}, z \in C. \qquad (1.8)$$

Further choosing $b = 2$ and $c = 1$ in (1.6), the functions $\omega_{p,b,c}$ reduces to $\sqrt{2}\frac{j_p}{\sqrt{\pi}}$, where j_p is the spherical Bessel function of the first kind of order p, given in as

$$j_p(z) = \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma\left(p+n+\frac{3}{2}\right)} \left(\frac{z}{2}\right)^{2n+p}, z \in C. \qquad (1.9)$$

The function $\vartheta_{p,b,c}$ is defined in as

$$\vartheta_{p,b,c}(z) = 2^p \Gamma\left(p + \frac{b+1}{2}\right) z^{1-\frac{p}{2}} \omega_{p,b,c}\left(\sqrt{z}\right) \qquad (1.10)$$

in terms of generalized Bessel function $\omega_{p,b,c}$. By the help of Pochhammer symbol, Gamma function is defined as

and we get $\vartheta_{p,b,c}$ given in (1.10) as

$$\vartheta_{p,b,c}(z) = z + \sum_{n=1}^{\infty} \frac{(-c)^n}{4^n (\tau)_n n!}, \qquad (1.11)$$

where $\tau = p + \frac{b+1}{2} \notin Z_0$ and $N = \{1,2,3,\cdots\}$. We will write $\vartheta_{\tau,c}(z) = \vartheta_{p,b,c}(z)$ for convenience. Now, we consider S_τ^c operator given as

$$S_\tau^c f(z) = \vartheta_{\tau,c}(z) * f(z) = z + \sum_{n=1}^{\infty} \frac{(-c)^n a_{n+1}}{4^n(\tau)_n n!} z^{n+1}$$

$$= z + \sum_{n=2}^{\infty} \frac{(-c)^{n-1} a_n}{4^{n-1}(\tau)_{n-1}(n-1)!} z^n = z + \sum_{n=2}^{\infty} E\,(c,\tau,n) a_n z^n$$

where $$E(c,\tau,n) = \frac{(-c)^{n-1}}{4^{n-1}(\tau)_{n-1}(n-1)!},\ \tau = \left(p + \frac{b+1}{2}\right) \neq 0, -1, -2, \cdots. \qquad (1.12)$$

For $\alpha \geq 0, 0 \leq \beta < 1$, we set $S_\tau^c(\alpha,\beta)$ be the subclass of A consisting of functions of the form (1.1) and satisfy

$$Re\left(\frac{S_\tau^c f(z)}{z}\right) \geq \alpha \left| \left(S_\tau^c f(z)\right)' - \frac{S_\tau^c f(z)}{z} \right| + \beta \qquad (1.13)$$

where $S_\tau^c f(z)$ is given by (1.12).
We further let $TS_\tau^c(\alpha,\beta) = S_\tau^c(\alpha,\beta) \cap T$.

In this paper, we obtain coefficient inequalities, extreme points, integral means inequalities for the functions in the class $TS_\tau^c(\alpha,\beta)$ and also subordination results for the class of function $f \in S_\tau^c(\alpha,\beta)$.

2. Coefficient Estimates

Theorem 2.1. *The function f defined by* (1.1) *is in the class $S_\tau^c(\alpha,\beta)$ if*

$$\sum_{n=2}^{\infty}[1+\alpha(n-1)]\, E(c,\tau,n)|a_n| \leq 1 - \beta, \qquad (2.1)$$

where $\alpha \geq 0, 0 \leq \beta < 1$ and $E(c,\tau,n)$ is given by (1.12).

Proof. It suffices to show that

$$\alpha \left| \left(S_\tau^c f(z)\right)' - \frac{S_\tau^c f(z)}{z} \right| - Re\left\{\frac{S_\tau^c f(z)}{z} - 1\right\} \leq 1 - \beta.$$

We have

$$\alpha \left| \left(S_\tau^c f(z)\right)' - \frac{S_\tau^c f(z)}{z} \right| - Re\left\{\frac{S_\tau^c f(z)}{z} - 1\right\}$$

$$\leq \alpha \left| \frac{\sum_{n=2}^{\infty}(n-1)\, E(c,\tau,n) a_n z^n}{z} \right| + \left| \frac{\sum_{n=2}^{\infty} E\,(c,\tau,n) a_n z^n}{z} \right|$$

$$\leq \alpha \sum_{n=2}^{\infty}(n-1)\, E(c,\tau,n)|a_n| + \sum_{n=2}^{\infty} E\,(c,\tau,n)|a_n|$$

$$= \sum_{n=2}^{\infty}[1+\alpha(n-1)]\, E(c,\tau,n)|a_n|.$$

The last expression is bounded above by $(1-\beta)$ if

$$\sum_{n=2}^{\infty}[1+\alpha(n-1)]\, E(c,\tau,n)|a_n| \leq 1 - \beta$$

and the proof of theorem is completed.

In the following theorem, we obtain necessary and sufficient conditions for functions in $TS_\tau^c(\alpha,\beta)$.

Theorem 2.2. For $\alpha \geq 0, 0 \leq \beta < 1$, a function f of the form (1.2) to be in the class $TS_\tau^c(\alpha,\beta)$ if and only if

$$\sum_{n=2}^{\infty}[1+\alpha(n-1)]\,E(c,\tau,n)|a_n| \leq 1-\beta.$$

Proof. Suppose $f(z)$ of the form (1.2) is in the class $TS_\tau^c(\alpha,\beta)$. Then

$$Re\left\{\frac{S_\tau^c f(z)}{z}\right\} - \alpha\left|\left(S_\tau^c f(z)\right)' - \frac{S_\tau^c f(z)}{z}\right| \geq \beta.$$

Equivalently

$$Re\left[1-\sum_{n=2}^{\infty} E\,(c,\tau,n)|a_n|z^{n-1}\right] - \alpha\left[\sum_{n=2}^{\infty}(n-1)\,E(c,\tau,n)a_n z^{n-1}\right] \geq \beta.$$

Letting z to be real values and as $|z| \to 1$, we have

$$1-\sum_{n=2}^{\infty} E\,(c,\tau,n)|a_n| - \alpha\sum_{n=2}^{\infty}(n-1)\,E(c,\tau,n)|a_n| \geq \beta$$

which implies

$$\sum_{n=2}^{\infty}[1+\alpha(n-1)]\,E(c,\tau,n)|a_n| \leq 1-\beta,$$

where $\alpha \geq 0,\ 0 \leq \beta < 1, E(c,\tau,n)$ is given by (1.12) and the sufficiency follows from Theorem 2.1.

Corollary 2.3. If $f \in TS_\tau^c(\alpha,\beta)$ then

$$a_n \leq \frac{1-\beta}{[1+\alpha(n-1)]E(c,\tau,n)}.$$

Equality holds for the function

$$f(z) = z - \frac{1-\beta}{[1+\alpha(n-1)]E(c,\tau,n)} z^n,$$

$\alpha \geq 0,\ \ 0 \leq \beta < 1, E(c,\tau,n)$ is given by (1.12).

3. Extreme Points

Theorem 3.1. *Let* $f_1(z) = z$ and $f_n(z) = z - \frac{1-\beta}{[1+\alpha(n-1)]E(c,\tau,n)} z^n, n \geq 2$ *for* $\alpha \geq 0, 0 \leq \beta < 1, E(c,\tau,n)$ is given by (1.12) Then $f(z)$ is in the class $E(c,\tau,n)$ if and only if it can be expressed in the form

$f(z) = \sum_{n=1}^{\infty} \lambda_n\, f_n(z)$, where λ_n and $\sum_{n=1}^{\infty} \lambda_n = 1$.

Proof. If $f(z)=\sum_{n=1}^{\infty}\lambda_n f_n(z)$ with $\lambda_n \geq 0$ and $\sum_{n=1}^{\infty}\lambda_n = 1$. Then

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty}\lambda_n f_n(z) \\ &= \lambda_1 f_1(z) + \sum_{n=2}^{\infty}\lambda_n f_n(z) \\ &= \left(1-\sum_{n=2}^{\infty}\lambda_n\right)z + \sum_{n=2}^{\infty}\left[\lambda_n\left(z-\frac{1-\beta}{[1+\alpha(n-1)]E(c,\tau,n)}z^n\right)\right] \\ &= z-\sum_{n=2}^{\infty}\frac{1-\beta}{[1+\alpha(n-1)]E(c,\tau,n)}z^n. \end{aligned}$$

Now $$\sum_{n=2}^{\infty}\frac{[1+\alpha(n-1)]E(c,\tau,n)}{1-\beta}\frac{1-\beta}{[1+\alpha(n-1)]E(c,\tau,n)}\lambda^n$$

$$=\sum_{n=2}^{\infty}\lambda_n = 1-\lambda_1 \leq 1.$$

Then $f \in TS_{\tau}^{c}(\alpha,\beta)$. Conversely suppose that $f \in TS_{\tau}^{c}(\alpha,\beta)$. Then Corollary 2.3 gives

$$a_n \leq \frac{1-\beta}{[1+\alpha(n-1)]E(c,\tau,n)},\ n\geq 2$$

set $$\lambda_n = \frac{[1+\alpha(n-1)]E(c,\tau,n)}{1-\beta}a_n,\ n\geq 2$$

where $$\lambda_n = 1-\sum_{n=2}^{\infty}\lambda_n\,.$$

Then

$$\begin{aligned} f(z) &= z-\sum_{n=2}^{\infty}a_n z^n \\ &= z-\sum_{n=2}^{\infty}\lambda_n\frac{1-\beta}{[1+\alpha(n-1)]E(c,\tau,n)} \\ &= z-\left[1-\sum_{n=2}^{\infty}\lambda_n\right]+\sum_{n=2}^{\infty}\lambda_n f_n(z) \\ &= \lambda_1 f_1(z)+\sum_{n=2}^{\infty}\lambda_n f_n(z) \\ &= \sum_{n=1}^{\infty}\lambda_n f_n(z). \end{aligned}$$

The poof of theorem is completed .

4. Integral Means Inequalities

Definition 4.1. *(***Subordination principle)** for analytic function g and h with $g(0) = h(0)$, g is said to be subordinate to h, denoted by $g \prec h$ if there exists an analytic function ω such that $\omega(0) = 0, |\omega(z)| < 1$ and $g(z) = h(\omega(z))$, for all $z \in U$.

Lemma 4.2. [6] If the function $f(z)$ and $g(z)$ are analytic *in* U with $g(z) \prec h(z)$ then

$\int_0^{2\pi} |g(re^{i\theta})|^p \, d\theta \leq \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \ \ (0 \leq r < 1, p > 0)$.

Theorem 4.3. Suppose $f \in TS_\tau^c(\alpha, \beta), p > 0, \alpha \geq 0, 0 \leq \beta < 1$ *and* $f(z)$ is defined by

$$f_2(z) = z - \frac{1-\beta}{(1+\alpha)E(c,\tau,n)}.$$

Then for $z = re^{i\theta},\ 0 \leq r < 1,$

$$\int_0^{2\pi} |f(z)|^p \, d\theta \leq \int_0^{2\pi} |f_2(z)|^p \, d\theta \tag{4.1}$$

Proof. For $f(z) = z - \sum_{n=2}^{\infty} |a_n| \, z^n$, (4.1) is equivalent to proving that

$$\int_0^{2\pi} \left| z - \sum_{n=2}^{\infty} |a_n| \, z^n \right|^p d\theta \leq \int_0^{2\pi} \left| z - \frac{1-\beta}{1+\alpha)E(c,\tau,n)} \right|^p d\theta,\ (p > 0).$$

By applying Little wood's subordination theorem (Lemma 4.2),
it would be sufficient to show that

$$1 - \sum_{n=2}^{\infty} |a_n| \, z^{n-1} \prec 1 - \frac{1-\beta}{1+\alpha)E(c,\tau,n)} z. \tag{4.2}$$

Setting

$$1 - \sum_{n=2}^{\infty} |a_n| \, z^{n-1} \prec 1 - \frac{1-\beta}{1+\alpha)E(c,\tau,n)} \omega(z).$$

We have $\omega(z) = \frac{1+\alpha)E(c,\tau,n)}{1-\beta} \sum_{n=2}^{\infty} a_n \, z^{n-1}$ and and $\omega(z)$ is analytic in U with $\omega(0) = 0$. Moreover it suffices to prove that $\omega(z)$ satisfies $|\omega(z)| < 1, z \in U$. Now

$$\begin{aligned} |\omega(z)| \quad &= \left| \sum_{n=2}^{\infty} \frac{(1+\alpha)E(c,\tau,n)}{1-\beta} a_n z^{n-1} \right| \\ &\leq |z| \sum_{n=2}^{\infty} \frac{(1+\alpha)E(c,\tau,n)}{1-\beta} |a_n| \qquad (4.3) \\ &\leq |z| < 1. \end{aligned}$$

Thus is view of the inequality (4.3) the subordination (4.2) follows, which proves the Theorem.

5. Subordination Results

Definition 5.1. **(Subordination factor sequence)** A sequence $\{ b_n \}_{n=2}^{\infty}$ of complex numbers is said to be a subordinating sequence if, whenever $f(z)=\sum_{n=2}^{\infty} a_n z^n, a_1=1$ is regular, univalent and convex in U, we

have $\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z),\ z\in U.$

Theorem 5.2. [11] The sequence $\{ b_n \}_{n=2}^{\infty}$ is a subordinating factor sequence if and only if

$$Re\{1+2\sum_{n=1}^{\infty} b_n z^n\}>0, z\in U.$$

Theorem 5.3. Let $f\in S_\tau^c(\alpha,\beta)$ *and* $g(z)$ any function in the usual class of convex function C. Then

$$\frac{(1+\alpha)E(c,\tau,n)}{2(1-\beta)+(1+\alpha)E(c,\tau,n)}(f*g)(z)\prec g(z) \tag{5.1}$$

where $\alpha\geq 0, 0\leq\beta<1$ with $E(c,\tau,n)$ is given by (1.12)

$$Re\{f(z)\}>-\frac{(1-\beta)+(1+\alpha)E(c,\tau,n)}{(1+\alpha)E(c,\tau,n)},\ z\in E. \tag{5.2}$$

The constant $\frac{(1+\alpha)E(c,\tau,n)}{2(1-\beta)+(1+\alpha)E(c,\tau,n)}$ is the best estimate.

Proof. Let $f\in S_\tau^c(\alpha,\beta)$ and $g(z)=z+\sum_{n=2}^{\infty} c_n z^n\in C$.
Then

$$\frac{(1+\alpha)E(c,\tau,n)}{2(1-\beta)+(1+\alpha)E(c,\tau,n)}(f*g)(z)$$
$$=\frac{(1+\alpha)E(c,\tau,n)}{2(1-\beta)+(1+\alpha)E(c,\tau,n)}\left(z+\sum_{n=2}^{\infty} c_n a_n z^n\right).$$

Then by Definition 5.1, the subordination result holds true if $\left\{\frac{(1+\alpha)E(c,\tau,n)}{2(1-\beta)+(1+\alpha)E(c,\tau,n)}\right\}_{n=1}^{\infty}$ is a subordinating factor sequence with $a_1=1$.

In view of Theorem 5.2, this is equivalent to the following inequality.

$$Re\left\{1+\sum_{n=1}^{\infty}\frac{(1+\alpha)E(c,\tau,n)}{(1-\beta)+(1+\alpha)E(c,\tau,n)}a_n z^n\right\}>0,\ z\in U. \tag{5.3}$$

Now for $|z|=r<1$, we have

$$\begin{aligned}
&Re\left\{1+\sum_{n=1}^{\infty}\frac{(1+\alpha)E(c,\tau,n)}{2(1-\beta)+(1+\alpha)E(c,\tau,n)}a_n z^n\right\}\\
=\quad & Re\left\{1+\frac{(1+\alpha)E(c,\tau,n)}{(1-\beta)+(1+\alpha)E(c,\tau,n)}z+\frac{\sum_{n=2}^{\infty}(1+\alpha)E(c,\tau,n)a_n z^n}{(1-\beta)+(1+\alpha)E(c,\tau,n)}\right\}\\
\geq\quad & 1-\frac{(1+\alpha)E(c,\tau,n)}{(1-\beta)+(1+\alpha)E(c,\tau,n)}r-\frac{\sum_{n=2}^{\infty}(1+\alpha)E(c,\tau,n)a_n r^n}{(1-\beta)+(1+\alpha)E(c,\tau,n)}\\
\geq\quad & 1-\frac{(1+\alpha)E(c,\tau,n)}{(1-\beta)+(1+\alpha)E(c,\tau,n)}r-\frac{1-\beta}{(1-\beta)+(1+\alpha)E(c,\tau,n)}r\\
>\quad & 0.
\end{aligned}$$

Using (2.1) and the fact that $1+\alpha(n-1)E(c,\tau,n)$ is increasing function for $n \geq 2$.

This proves the inequality (5.3) and hence also the subordination result (5.1) asserted by Theorem 5.3.

The inequality (5.2) follows from (5.1) by taking

$$g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \in C.$$

Now we consider the function $f(z) = z - \frac{1-\beta}{(1+\alpha)E(c,\tau,n)} z^2$, where $\alpha \geq 0, 0 \leq \beta < 1$.
Clearly $F \in S_\tau^c(\alpha,\beta)$. For the function (5.1) becomes

$$\frac{(1+\alpha)E(c,\tau,n)}{2(1-\beta)+(1+\alpha)E(c,\tau,n)} F(z) \prec \frac{z}{1-z}.$$

It is easily verified that

$$\min Re\left\{\frac{(1+\alpha)E(c,\tau,n)}{2(1-\beta)+(1+\alpha)E(c,\tau,n)} F(z)\right\} = \frac{-1}{2}, z \in U.$$

This shows that the constant $\frac{(1+\alpha)E(c,\tau,n)}{2(1-\beta)+(1+\alpha)E(c,\tau,n)} F(z) \prec \frac{z}{1-z}$ is best possible.

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