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Altering Point Results involving C-Class Functions in Partial Metric Spaces

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Abstract:

The purpose of this paper is to explore the existence of altering points $(x^*, y^*) \in X \times Y$ satisfying the system of equations

$$\begin{cases} x^* \in G(y^*) \\ y^* \in F(x^*) \end{cases} \qquad \text{(AP)}$$

using the notion of C-class functions. Here, X and Y are two 0-complete partial metric spaces, and F and G are set-valued mappings. The key idea is to use the properties of C-class functions to establish the existence of a sequence of points that converge to the altering point, thereby demonstrating the existence of such points in the first place. By analyzing the properties of F and G and their interaction with C-class functions, we can gain insight into the nature of the altering points and the underlying spaces. The findings of this study provide a generalization and extension of various results in the existing literature, highlighting the significance of the proposed approach.

Keywords: Altering point, partial metric space, set-valued mapping, C-class function

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1. Introduction

Since 1922, when the celebrated Banach contraction principle was introduced, fixed point theory has fascinated many researchers as one of the most dynamic areas. Till now, it has been the fastest growing branch of mathematics, with several applications to real-world problems.

There is a vast literature on fixed point theory, and it is currently a very active field of research. As a tool for proving the existence and uniqueness of solutions to different mathematical problems, fixed point theorems are essential in many theoretical and applied branches of mathematics, including optimisation problems, dynamical systems, nonlinear analysis, integral equations, partial differential equations, variational inequalities, fractals, economics, game theory, and nonlinear analysis.

In many distinct abstract spaces, diverse results about fixed points, common fixed points, coincidence points, altering points, and so on for single-valued and set-valued mappings have been investigated for various contractive conditions, and this tradition continues.

Partial metric spaces are an extension of conventional metric spaces where the value of self-distance for each point does not have to be zero. Matthews first discussed partial metric spaces in [5]

. Romaguera presented the concept of a 0-Cauchy sequence in a partial metric space, followed by the concept of a 0-complete partial metric space in [8].

Sahu first proposes the idea of altering points in [9] for the single-valued case in order to introduce a parallel S-iteration process to solve a system of operator equations. These points are more general than the fixed points, and they are extended by Petrusel, Yao in [7] for the set-valued case. For more information on altering points and applications to variational problems one can refer to [9,10]. Nedelcheva [6] presented one of the altering point results and extended the fixed point result mentioned in [3] for the objective of composing two set-valued mappings on complete metric spaces.

The following is how this work is organized: Section 2 presents some preliminary observations and definitions. Section 3 establishes the primary conclusion by extending both Theorem 11 in [6] and Theorem 3.4 in [2] for a composition of two set-valued mappings using the idea of C-class functions. Finally, we will discuss some relevant corollaries.

2. Preliminary results

Let us begin by providing an overview of some of the key definitions and characteristics of partial metric spaces.

A partial metric on a non-empty set X is defined as a function $p: X \times X \to \mathbb{R}^+$ that satisfies the following conditions for all $x, y, z \in X$:

$(\mathbf{P}_1)$ $x = y \Leftrightarrow p(x,x) = p(y,y) = p(x,y)$;

$(\mathbf{P}_2)$ $p(x,x) \leq p(x,y)$;

$(\mathbf{P}_3)$ $p(x,y) = p(y,x)$;

$(\mathbf{P}_4)$ $p(x,y) + p(z,z) \leq p(x,z) + p(z,y)$.

If p satisfies the above conditions, then the pair (X,p) is referred to as a partial metric space. This definition was introduced by Matthews [5].

The closed p-ball with radius r and x as its center is denoted by $\overline{\mathbb{B}_{p,r}}(x)$ and opened p -ball by $\mathbb{B}_{p,r}(x)$, where

$$\overline{\mathbb{B}_{p,r}}(x) = \{y \in X : p(x,y) \leq p(x,x) + r\}, \quad \text{and}$$
$$\mathbb{B}_{p,r}(x) = \{y \in X : p(x,y) < p(x,x) + r\}.$$

$\mathbb{B}_{+\infty}(x)$ represents X for convenience.

The function $p^s : X \times X \to \mathbb{R}^+$ provided by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X if p is a partial metric on X.

Consider (X,p) to be a partial metric space. Then:

- If $p(x,x)=\lim_{n\to+\infty} p(x,x_n)$, then $\{x_n\}$ is said to converge to a point $x\in X$.
- If there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n,x_m)$, a sequence $\{x_n\}$ is called a Cauchy sequence. $\{x_n\}$ is referred to as a 0-Cauchy sequence if $\lim_{n,m\to+\infty} p(x_n,x_m)=0$.
- In X, every Cauchy sequence $\{x_n\}$ that converges to a point x in such a way that $p(x,x)=\lim_{n,m\to+\infty} p(x_n,x_m)$ indicates that (X,p) is complete.
- Regarding τ_p, every 0-Cauchy sequence $\{x_n\}$ in X converges to a point $x\in X$ where $p(x,x)=0$. A 0-complete (X,p) is what this is called.
- In (X,p), any 0-Cauchy sequence is a Cauchy in (X,p^s).
- Complete partial metric spaces are all 0-complete. However, the reverse is not true.

In the partial metric space $(X,\widetilde{p})$, the $C^{\widetilde{p}}(X)$ represents the family of all closed and nonempty subsets. With $x\in X$ and $A,B\in C^{\widetilde{p}}(X)$, we establish

$$H_p(A,B)=\max\{\delta_p(A,B),\delta_p(B,A)\},$$
$$=\max\{\sup\{p(a,B)\mid a\in A\},\sup\{p(b,A)\mid b\in B\}\},$$

such that

$$p(x,A)=\inf\{p(x,a)\mid a\in A\}. \tag{2.1}$$

in conformity with the convention

$$\widetilde{p}(x,\emptyset)=+\infty,\quad e_{\widetilde{p}}(\emptyset,B)=0. \tag{2.2}$$

The following concepts will be used throughout the article.

Let $F:X\to C^{\sigma}(Y)$ denotes a set-valued mapping defined in the partial metric space (X,p) with closed values in the partial metric space (Y,σ). We define the composition of mappings $G:Z\to C^{\sigma}(Y)$ and $F:X\to C^{\widetilde{p}}(Z)$ as the mapping $\Phi:=G\circ F:X\to C^{\sigma}(Y)$ such that for all $x\in X$, $\Phi(x)=\{y\in G(z)\mid z\in F(x)\}$.

The sets J and J' in the following text represent intervals on $\mathbb{R}^+$ that contain 0, like $[0,a)$, $[0,a]$, or $[0,+\infty)$. We also introduce H, a set-valued mapping defined on X for $x,y\in(X,\widetilde{p})$. We denote $M_{\widetilde{p},H}(x,y)$ as

$$M_{\widetilde{p},H}(x,y)=\max\left\{\widetilde{p}(x,y),\widetilde{p}(x,H(x)),\widetilde{p}(y,H(y)),\frac{\widetilde{p}(x,H(y))+\widetilde{p}(y,H(x))}{2}\right\}.$$

Lemma 2.1. [2] Let (X,p) be a partial metric space and let $A\subset X$. Then, the following statements hold:

1. $p(a,A)=p(a,a)\Leftrightarrow a\in\overline{A}$.
2. $p(a,A)=0\Leftrightarrow p(a,a)=0 \text{ and } a\in\overline{A}$.

Given the partial metric p, the closure of A is represented here as $\overline{A}$. It is worth mentioning that A is closed in (X,p) if and only if $\overline{A}=A$.

Lemma 2.2. [3] Consider $x\in X$ and $A\in C^p(X)$ in a partial metric space (X,p). If $\mu>0$ and $p(x,A)<\mu$, we can find that there is an element $a\in A$ such that $p(x,a)<\mu$.

Definition 1. [4] A Bianchini-Grandolfi gauge function on J is defined as a nondecreasing function $\phi: J\to J$ that satisfies the following condition:

$$\widetilde{s}(t):=\sum_{n=0}^{\infty}\phi^n(t)<+\infty \text{ for all } t\in J, \tag{2.3}$$

where ϕ^n denotes the n-th iteration of the function ϕ and $\phi^0(t)=t$. In other words, $\phi^n(t)$ is defined recursively as $\phi^0(t)=t$, $\phi^1(t)=\phi(t)$, $\phi^2(t)=\phi(\phi(t))$, and so on

The associated estimate function is known as the sum (2.3), and it was observed that ϕ fulfils the functional equation shown below

$$\widetilde{s}(t)=t+\widetilde{s}(\phi(t)). \tag{2.4}$$

In [6], the author has defined Ψ as the set of pairs of increasing functions φ and ψ that map from J to J, satisfying the following three conditions:

$(\Psi 1)$ $\varphi(t)\leq t, \forall t\in J$

$(\Psi 2)$ $\lim_{t\downarrow 0}\varphi(t)=\lim_{t\downarrow 0}\psi(t)=0$

$(\Psi 3)$ $\psi\circ\varphi$ f is a Bianchini-Grandolfi gauge function with the property that

$$s(t):=\sum_{n=0}^{\infty}(\psi\circ\varphi)^n(t)<\infty \quad \text{for all } t\in J.$$

Then the theorem mentioned in [6] reads as follows:

Theorem 2.1. [6] *Let (X,p) and (Y,σ) be complete partial metric spaces, and let $\bar{x}\in X$ and $\bar{y}\in Y$. Let $r>0$ be a constant, and let F: $\overline{\mathbb{B}_{p,r}}(\bar{x})\to C^\sigma(Y)$ and $G:\overline{\mathbb{B}_{\sigma,r}}(\bar{y})\to C^p(X)$ be two set-valued mappings such that $\bar{y}\in F\bar{x})$. Let φ and $\psi: J\to J$ be increasing functions such that $(\psi,\varphi)\in\Psi$, where Ψ is a collection of pairs of increasing functions from J to J, subject to the following three conditions. Assume there exists $\alpha\in J$ such that the following assumptions hold:*

(a) $\max\{p(\bar{x},G(\bar{y})),\sigma(\bar{y},F(\bar{x}))\}<\alpha,$ where $s(\alpha)\leq\min\{p(\bar{x},\bar{x}),\sigma(\bar{y},\bar{y})\}+r,$

(b) $e_\sigma\left(F\left(x_1\right)\cap\mathbb{B}_{\sigma,r}(\bar{y}),F\left(x_2\right)\right)\leq\varphi\left(p\left(x_1,x_2\right)\right), \quad$ for all $x_1,x_2\in\mathbb{B}_{p,r}(\bar{x}),$

(c) $e_p\left(G\left(y_1\right)\cap\mathbb{B}_{p,r}(\bar{x}),G\left(y_2\right)\right)\leq\psi\left(\sigma\left(y_1,y_2\right)\right), \quad$ for all $y_1,y_2\in\mathbb{B}_{\sigma,r}(\bar{y}).$

Then there exist $x^*\in\overline{\mathbb{B}_{p,r}}(\bar{x})$ and $y^*\in\overline{\mathbb{B}_{\sigma,r}}(\bar{y})$ such that $y^*\in F(x^*)$ and $x^*\in G(y^*)$ i.e., (x^*,y^*) is an altering point of F and G. If F and G are single valued mappings, and $p(\bar{x},\bar{x})+2r\in J$, then x^* is the unique fixed point of $G\circ F$ in $\overline{\mathbb{B}_{p,r}}(\bar{x})$, and y^* is the unique fixed point of $F\circ G$ in $\overline{\mathbb{B}_{\sigma,r}}(\bar{y})$.

Remark 1. The theorem mentioned above is satisfied without using the assumption $\bar{x} \in G(\bar{y})$. See the proof of [6, Theorem 11] for further details.

For $X = Y$, $p = \sigma$, $F = Id_X$ and $\varphi = Id_J$ where Id_E the identity function defined on E, we have

Corollary 2.2. *Theorem 3.2 in [3].*

A notion of C-class functions was presented by A.H. Ansari in [1].

Definition 2. (C-class functions) [1,2] Assume that there is a continuous mapping $\mathfrak{F} : J \times J' \to \mathbb{R}$. If $\mathfrak{F}$ satisfies these requirements, we will classify it as a C-class function.

$(\mathfrak{F}_1)$ $s \geq \mathfrak{F}(s,t)$, for any $(s,t) \in J \times J'$

$(\mathfrak{F}_2)$ $\mathfrak{F}(s,t) = s$ implies that $st = 0$.

$\mathfrak{F}(0,0) = 0$ and $\mathcal{C}$ denotes the set of all C-class functions on $J \times J'$.

Example 1. Here are some examples of C-class functions with $J = [0,a)$ and $J' = [0,b)$, where $a, b > 0$:

1. $\text{F(s,t)}=\begin{cases} s, & \text{if } t = 0 \\ \frac{st}{s+t}, & \text{otherwise} \end{cases}$.

   This function satisfies $(\mathfrak{F}_1)$ and $(\mathfrak{F}_2)$, and it is known as the harmonic mean. It is used, for example, to calculate the average rate of speed of two objects moving at different constant speeds.

2. $\text{F(s,t)}=\begin{cases} 0, & \text{if } s = t = 0 \\ \frac{s}{s+t}, & \text{otherwise} \end{cases}$. This function satisfies $(\mathfrak{F}_1)$ and $(\mathfrak{F}_2)$, and it is known as the normalized power mean of order 1. It is used, for example, in statistics to calculate the average of a set of non-negative numbers.

In [2], the authors presented the following collections of C-class functions:

Definition 3. [2] The set of functions of the C-class that satisfy these criteria is called $\mathcal{C}_I$:

- $\mathfrak{F}(s,t)$ is non-decreasing for both s and t when $(s,t) \in J \times J'$.
- For every $t \in J'$ that are fixed, the series

  $$\widetilde{\mathfrak{w}}(s,t) := \sum_{n=0}^{\infty} \mathfrak{F}^n(s,t)$$

  converges for all $s \in J$. The function $\mathfrak{F}$ is defined as follows, and $\mathfrak{F}^n$ represents the n-th iteration of this function:

  $\mathfrak{F}^0(s,t) = s,\ \mathfrak{F}^1(s,t) = \mathfrak{F}(s,t),$ and $\mathfrak{F}^{n+1}(s,t) = \mathfrak{F}(\mathfrak{F}^n(s,t),t)$.

Definition 4. [2] $\mathcal{C}_{II}$ comprises a set of C-class functions that adhere to the following specifications:
$\mathfrak{F}(s,t)$ exhibits non-decreasing behavior in s and non-increasing behavior in t.

- For any given $t \in J'$, the series
$$\widetilde{\mathfrak{w}}(s,t) := \sum_{n=0}^{\infty} \mathfrak{F}^n(s,t)$$
converges for every $s \in J$, where the n-th iteration of the function $\mathfrak{F}$ with the following recurrence relation is represented as $\mathfrak{F}^n$:
$$\mathfrak{F}^0(s,t) = s,\ \mathfrak{F}^1(s,t) = \mathfrak{F}(s,t), \text{ and } \mathfrak{F}^{n+1}(s,t) = \mathfrak{F}(\mathfrak{F}^n(s,t), \mathfrak{F}^n(s,t))$$
For more examples and properties on $\mathcal{C}_I$ and $\mathcal{C}_{II}$ one can refer to [2]. We recall a class of functions Ξ that were mentioned in [2]. One important need is satisfied by these functions, denoted as $\tau : X^2 \times (2^X)^2 \to J'$. Being more explicit, $\tau(x,y,A,C) = 0$ means that $x = y$ or $p(x,y) = 0$ is true for every $x, y \in X$ and $A, C \in 2^X$. Furthermore, we prove that $\tau \in \Xi$ is nondecreasing in the (X,p) space, as proved by the following inequality:
$$p(x,y) \le p(a,b) \Rightarrow \tau(x,y,A_x,C_y) \le \tau(a,b,A_a,C_b) \quad \forall A_x, A_a, C_y, C_b \in 2^X.$$
So, the Theorem [2, Theorem 3.4], which extends Theorem [3,Theorem 3.2], is as follows:
Theorem 2.3. [2] *Assuming (X,p) is a partial metric space with $\bar{x} \in X$ and r>0 such that $\overline{\mathbb{B}_{p,r}}(x)$ is a 0-complete subspace of X. Let $F : \overline{\mathbb{B}_{p,r}}(\bar{x}) \to C^p(X)$ be a set-valued mapping. Consider $\mathfrak{F} \in \mathcal{C}$, $\tau \in \Xi$ and $\alpha \in J$ satisfying one of the following conditions:*

- $\mathfrak{F} \in \mathcal{C}_I$ *and τ is nondecreasing.*
- $\mathfrak{F} \in \mathcal{C}_{II}$ *and* $\tau(x,y,F(x),F(y)) \ge \alpha$ *for* $x, y \in \overline{\mathbb{B}_{p,r}}(\bar{x})$.

We suppose the following two conditions are met:

(a) $e_p(F(x) \cap \overline{\mathbb{B}_{p,r}}(x), F(y)) \le \mathfrak{F}(M_{p,F}(x,y), \tau(x,y,F(x),F(y))) \ \forall \ x, y \in \overline{\mathbb{B}_{p,r}}(x),$

(b) $p(\bar{x}, F(\bar{x})) < \alpha$ where $\widetilde{w}(\alpha, \cdot) \le p(\bar{x}, \bar{x}) + r$

hence, in $\overline{\mathbb{B}_{p,r}}(x)$, there is a fixed point $x*$ in F. In the set $\overline{\mathbb{B}_{p,r}}(x)$, the unique fixed point of F is $x*$ if F is a single-valued mapping and $p(\bar{x}, \bar{x}) + 2r \in J$.

3. Main results

We need to review certain concepts and establish some definitions before we can present our major finding.

Definition 5. (Altering points). [7, 9] Let X and Y be two partial metric spaces and let $F : X \to 2^Y$, $G : Y \to 2^X$ be two set-valued mappings. The couple $(x^*, y^*) \in X \times Y$ is called altering point of F and G if

$$\begin{cases} x^* \in G(y^*) \\ y^* \in F(x^*) \end{cases}$$

There are some special cases that can be deduced from this definition, which can be summarized in the following table

Special cases

$X = Y$	$x^* = y^*$	$\begin{cases} F = Id_X & \text{or} \\ F = G \end{cases}$	$x^* \in G(x^*)$	Fixed point of G
		$\begin{cases} F \neq Id_X & \text{and} \\ F \neq G \end{cases}$	$\begin{cases} x^* \in G(x^*) \\ x^* \in F(x^*) \end{cases}$	Common fixed point of F and G
	$x^* \neq y^*$	$F = G$	$\begin{cases} x^* \in G(y^*) \\ y^* \in G(x^*) \end{cases}$	Couple fixed point of $\tilde{G}(x, y) = G(y)$

According to Theorem 2.3 and definitions 3, 4, and 5, we will establish new theorems with respect to set-valued mappings on 0-complete partial metric spaces that extend and generalize Theorem 2.1 for altering points of two set-valued mappings and generalized Theorem 2.3 [2,Theorem 3.4] and [3,Theorem 3.2]{benterki2016} for the fixed points of a two-set-valued mapping composition.

At beginning, we set $\mathbb{F}$ the family of pairs of an increasing function $\varphi : J \to J$ and an increasing function in first variable $\mathfrak{F} : J \times J' \to J$ satisfying the following three assumptions:

($\mathbb{F}1$) $\varphi(t) \leq t, \forall t \in J$

($\mathbb{F}2$) $\lim_{t \downarrow 0} \varphi(t) = 0$

($\mathbb{F}3$) $\mathfrak{F}_\varphi \in \mathcal{C}_I \cup \mathcal{C}_{II}$, where $\mathfrak{F}_\varphi(s,t) = \mathfrak{F}(\varphi(s), t)$ for all $(s,t) \in J \times J'$ such that $\mathfrak{F}_\varphi^0(s,t) = s$, $\mathfrak{F}_\varphi^n(s,t) = \begin{cases} \mathfrak{F}_\varphi(\mathfrak{F}_\varphi^{n-1}(s,t), t) & \text{If } \mathfrak{F}_\varphi \in \mathcal{C}_I \\ \mathfrak{F}_\varphi(\mathfrak{F}_\varphi^{n-1}(s,t), \mathfrak{F}_\varphi(s,t)) & \text{If } \mathfrak{F}_\varphi \in \mathcal{C}_{II} \end{cases}$ and

$$w(s,t) := \sum_{n=0}^{\infty} \mathfrak{F}_\varphi^n(s,t) \text{ is convergent for any } s \in J.$$

Remark 2. If $\mathfrak{F}(s,t) = \psi(s)$ such that $\lim_{s \downarrow 0} \psi(s) = 0$ then the subset $\mathbb{F}$ will be the subset Ψ.

As a consequence, we express and demonstrate our main finding as follows

Theorem 3.1. *Let (X, p) and (Y, σ) be partial metric spaces with $\overline{\mathbb{B}_{p,r}}(\bar{x})$ and $\overline{\mathbb{B}_{\sigma,r}}(\bar{y})$ as 0-complete subspaces of X and Y, respectively, where $\bar{x} \in X$, $\bar{y} \in Y$, and $r > 0$. Consider set-valued mappings $F : \overline{\mathbb{B}_{p,r}}(\bar{x}) \to C^\sigma(Y)$ and $G : \overline{\mathbb{B}_{\sigma,r}}(\bar{y}) \to C^p(X)$ such that $\bar{y} \in F(\bar{x})$. Let $(\mathfrak{F}, \varphi) \in \mathbb{F}$, $\tau \in \Xi$ on (Y, σ), and $\alpha \in J$ which satisfies at least one of the conditions:*

(i) $\mathfrak{F}_\varphi \in \mathcal{C}_I$ and τ is nondecreasing,

(ii) $\mathfrak{F}_\varphi \in \mathcal{C}_{II}$ and $\tau(y_1, y_2, F \circ G(y_1), F \circ G(y_2)) \geq \alpha$ where $y_1, y_2 \in \mathbb{B}_{\sigma,r}(\bar{y})$.

Given these conditions, we are going to suppose that the following is true:

(a) $\max\{p(\bar{x}, G(\bar{y})), \sigma(\bar{y}, F(\bar{x}))\} < \alpha$, where $w(\alpha, \cdot) \leq \min\{p(\bar{x}, \bar{x}), \sigma(\bar{y}, \bar{y})\} + r$,

(b) $e_\sigma\left(F\left(x_1\right) \cap \mathbb{B}_{\sigma,r}(\bar{y}), F\left(x_2\right)\right) \leq \varphi\left(M_{p,G\circ F}(x_1, x_2)\right), \qquad \forall x_1,\ x_2 \in \mathbb{B}_{p,r}(\bar{x})$,

(c) $e_p\left(G\left(y_1\right) \cap \mathbb{B}_{p,r}(\bar{x}), G\left(y_2\right)\right) \leq \mathfrak{F}\left(M_{\sigma,F\circ G}(y_1, y_2), \tau(y_1, y_2, F\circ G(y_1), F\circ G(y_2))\right)$
$\forall y_1, y_2 \in \mathbb{B}_{\sigma,r}(\bar{y})$.

Then there exist (x^*, y^*) an altering point of F and G in $\overline{\mathbb{B}_{p,r}}(\bar{x}) \times \overline{\mathbb{B}_{\sigma,r}}(\bar{y})$. If F and G are both single-valued mappings and $p(\bar{x}, \bar{x}) + 2r \in J$, then x^* is the unique fixed point of $G\circ F$ in $\overline{\mathbb{B}_{p,r}}(\bar{x})$ and y^* is the unique fixed point of $F\circ G$ in $\overline{\mathbb{B}_{\sigma,r}}(\bar{y})$.

Proof. The proof is complete if $\bar{x} \in G(\bar{y})$. So we assume that $\bar{x} \notin G(\bar{y})$. By assumption **(a)**, we have
$$p(\bar{x}, G(\bar{y})) < \alpha.$$

Now, using Lemma 2.2, there exists $x_1 \in G(\bar{y})$ such that
$$p\left(\bar{x}, x_1\right) = \begin{cases} \mathfrak{F}_\varphi^0(p(\bar{x}, x_1), \tau(\bar{y}, \cdot, F\circ G(\bar{y}), F\circ G(\cdot))) < \alpha \quad \text{or} \\ \mathfrak{F}_\varphi^0(p(\bar{x}, x_1), p(\bar{x}, x_1)) < \alpha, \end{cases} \tag{3.1}$$
$$\leq w(\alpha, \cdot) \leq p(\bar{x}, \bar{x}) + r.$$
Then $x_1 \in G(\bar{y}) \cap \mathbb{B}_{p,r}(\bar{x})$. Now, denoting $x_0 = \bar{x}$ and $y_0 = \bar{y}$. the proof is complete if $y_0 \in F(x_1)$. So we assume that $y_0 \notin F(x_1)$ and by using **(b)** we have
$$\begin{aligned}\sigma\left(y_0, F\left(x_1\right)\right) &\leq e_\sigma\left(F\left(x_0\right) \cap \mathbb{B}_{\sigma,r}(\bar{y}), F\left(x_1\right)\right) \\ &\leq \varphi\left(M_{p,G\circ F}(x_0, x_1)\right)\end{aligned}$$
with
$$\begin{aligned}M_{p,G\circ F}(x_0, x_1) &= \max\left\{p(x_0, x_1), p(x_0, G(F(x_0))), p(x_1, G(F(x_1))), \frac{p(x_0, G(F(x_1))) + p(x_1, G(F(x_0)))}{2}\right\} \\ &\leq \max\left\{p(x_0, x_1), p(x_0, G(y_0)), p(x_1, G(F(x_1))), \frac{p(x_0, G(F(x_1))) + p(x_1, G(y_0))}{2}\right\} \\ &\leq \max\left\{p(x_0, x_1), p(x_0, x_1), p(x_1, G(F(x_1))), \frac{p(x_0, G(F(x_1))) + p(x_1, x_1)}{2}\right\} \\ &\leq \max\left\{p(x_0, x_1), p(x_1, G(F(x_1))), \frac{p(x_0, x_1) + p(x_1, G(F(x_1)))}{2}\right\} \\ &= \max\left\{p(x_0, x_1), p(x_1, G(F(x_1)))\right\}.\end{aligned}$$

However, we also have

$$\begin{aligned}p\left(x_1, G\left(F(x_1)\right)\right) &\leq e_p\left(G(F(x_0)) \cap \mathbb{B}_{p,r}(\bar{x}), G\left(F(x_1)\right)\right) \\ &\leq \mathfrak{F}\left(\varphi\left(M_{p,G\circ F}(x_0, x_1)\right), \tau(y_0, z, F\circ G(y_0), F\circ G(z))\right); \quad z \in F(x_1) \\ &\leq \mathfrak{F}_\varphi\left(M_{p,G\circ F}(x_0, x_1), \tau(y_0, z, F\circ G(y_0), F\circ G(z))\right) \\ &\leq M_{p,G\circ F}(x_0, x_1).\end{aligned}$$
If $M_{p,G\circ F}(x_0, x_1) \leq p\left(x_1, G\left(F(x_1)\right)\right)$ and $\max\{p(x_0, x_1), p(x_1, G(F(x_1)))\} = p(x_1, G(F(x_1)))$, then $\mathfrak{F}_\varphi\left(M_{p,G\circ F}(x_0, x_1), \tau(y_0, z, F\circ G(y_0), F\circ G(z))\right) = M_{p,G\circ F}(x_0, x_1)$ wich implies that $M_{p,G\circ F}(x_0, x_1) = 0$ or $y_0 = z \in F(x_1)$.

Thus, $p\left(x_1, G\left(F(x_1)\right)\right) = 0 < p(x_0, x_1)$ or $y_0 \in F(x_1)$ which is a contradiction.

So, we assume that $\max\left\{p(x_0, x_1), p(x_1, G(F(x_1)))\right\} = p(x_0, x_1) \in J$ and $p\left(x_1, G\left(F(x_1)\right)\right) < M_{p,G\circ F}(x_0, x_1) \leq p(x_0, x_1)$. Then, by using Lemma 2.2, relation (3.1) and non empty of F and G, there exists $y_1 \in F(x_1)$ and $x_2 \in G(y_1)$ such that

$$p(x_1, x_2) < M_{p,G\circ F}(x_0, x_1) \leq p(x_0, x_1) \in J,$$

and

$$\begin{aligned}\sigma\left(y_0, y_1\right) &\leq \varphi\left(M_{p,G\circ F}(x_0, x_1)\right) \\ &\leq \varphi\left(p(x_0, x_1)\right) \\ &= \varphi\left(\mathfrak{F}^0_{\varphi}(p(x_0, x_1), \cdot)\right).\end{aligned}$$

Consequently,

$$\sigma\left(y_0, y_1\right) < \varphi(\mathfrak{F}^0_{\varphi}(\alpha, \cdot)) < w(\alpha, \cdot) \leq \sigma(\bar{y}, \bar{y}) + r.$$

This means that $y_1 \in F(x_1) \cap \mathbb{B}_{\sigma,r}(\bar{y})$.

By induction we construct two sequences $\{x_k\}$ and $\{y_k\}$ satisfying:

$$\begin{aligned} &x_0 = \bar{x} \text{ and } x_{k+1} \in G(y_k) \cap \mathbb{B}_{p,r}(\bar{x}) \\ &y_0 = \bar{y} \text{ and } y_{k+1} \in F(x_{k+1}) \cap \mathbb{B}_{\sigma,r}(\bar{y}) \\ &p(x_k, x_{k+1}) \leq \psi^k(p(x_0, x_1)) \leq p(x_0, x_1) \\ &\sigma(y_k, y_{k+1}) \leq \varphi(\psi^k(p(x_0, x_1)))\end{aligned} \tag{3.2}$$

where

$$\psi^k(p(x_0, x_1)) = \begin{cases} \mathfrak{F}^k_{\varphi}(p(x_0, x_1), \tau(y_0, y_1, F\circ G(y_0), F\circ G(y_1)), & \text{if } (i) \text{ is satisfied;} \\ \mathfrak{F}^k_{\varphi}(p(x_0, x_1), p(x_0, x_1)), & \text{if } (ii) \text{ is satisfied.}\end{cases}$$

Assuming that $x_k \neq x_{k+1}$, $x_k \notin G(y_k)$ and $y_k \neq y_{k+1}$ for all $k \in \mathbb{N}$, we can conclude that if $x_k = x_{k+1}$ or $y_k = y_{k+1}$ for some $k \in \mathbb{N}$, then we are finished. Otherwise, we have $p(x_k, x_{k+1}) \geq p(x_k, G(y_k)) > 0$ and $\sigma(y_k, y_{k+1}) > 0$.

First, we demonstrate that the sequence $\{x_k\}$ and $\{y_k\}$ fulfilling (3.2) provides

$$M_{\sigma,F\circ G}(y_k, y_{k+1}) \leq \sigma(y_k, y_{k+1}) \leq \varphi(M_{p,G\circ F}(x_k, x_{k+1})) \leq \varphi(p(x_k, x_{k+1})) \in J. \tag{3.3}$$

Indeed, let $k \in \mathbb{N}$ then we have

$$\begin{aligned} M_{p,G\circ F}(x_k, x_{k+1}) &= \max\left\{p(x_k, x_{k+1}), p(x_k, G(F(x_k))), p(x_{k+1}, G(F(x_{k+1}))), \frac{p(x_k, G(F(x_{k+1}))) + p(x_{k+1}, G(F(x_k)))}{2}\right\} \\ &\leq \max\left\{p(x_k, x_{k+1}), p(x_k, G(y_k)), p(x_{k+1}, G(y_{k+1})), \frac{p(x_k, G(y_{k+1})) + p(x_{k+1}, G(y_k))}{2}\right\} \\ &\leq \max\left\{p(x_k, x_{k+1}), p(x_k, x_{k+1}), p(x_{k+1}, x_{k+2}), \frac{p(x_k, x_{k+2}) + p(x_{k+1}, x_{k+1})}{2}\right\} \\ &\leq \max\left\{p(x_k, x_{k+1}), p(x_{k+1}, x_{k+2}), \frac{p(x_k, x_{k+1}) + p(x_{k+1}, x_{k+2})}{2}\right\} \\ &= \max\left\{p(x_k, x_{k+1}), p(x_{k+1}, x_{k+2})\right\}\end{aligned}$$

Similarly,

$$\begin{aligned} M_{\sigma,F\circ G}(y_k,y_{k+1}) &= \max\left\{\sigma(y_k,y_{k+1}),\sigma(y_k,F(G(y_k))),\sigma(y_{k+1},F(G(y_{k+1}))),\frac{\sigma(y_k,F(G(y_{k+1})))+\sigma(y_{k+1},F(G(y_k)))}{2}\right\} \\ &\le \max\left\{\sigma(y_k,y_{k+1}),\sigma(y_k,F(x_{k+1})),\sigma(y_{k+1},F(x_{k+2})),\frac{\sigma(y_k,F(x_{k+2}))+\sigma(y_{k+1},F(x_{k+1}))}{2}\right\} \\ &\le \max\left\{\sigma(y_k,y_{k+1}),\sigma(y_k,y_{k+1}),\sigma(y_{k+1},y_{k+2}),\frac{\sigma(y_k,y_{k+2})+\sigma(y_{k+1},y_{k+1})}{2}\right\} \\ &\le \max\left\{\sigma(y_k,y_{k+1}),\sigma(y_{k+1},y_{k+2}),\frac{\sigma(y_k,y_{k+1})+\sigma(y_{k+1},y_{k+2})}{2}\right\} \\ &= \max\left\{\sigma(y_k,y_{k+1}),\sigma(y_{k+1},y_{k+2})\right\}. \end{aligned}$$

Firstly,

$$\begin{aligned} \sigma(y_k,y_{k+1}) &\le e_\sigma\left(F\left(x_k\right)\cap\mathbb{B}_{\sigma,r}(\bar{y}),F\left(x_{k+1}\right)\right) \\ &\le \varphi\left(M_{p,G\circ F}(x_k,x_{k+1})\right). \end{aligned} \tag{3.4}$$

For the other inequalities, we will handle the following cases:

Case 1: If $\max\{p(x_k,x_{k+1}),p(x_{k+1},x_{k+2})\}=p(x_{k+1},x_{k+2})$ and $\max\{\sigma(y_k,y_{k+1}),\sigma(y_{k+1},y_{k+2})\}=\sigma(y_k,y_{k+1})$. We have

$$\begin{aligned} M_{p,G\circ F}(x_k,x_{k+1}) &\le p(x_{k+1},x_{k+2}) \\ &\le e_p(G(y_k)\cap\mathbb{B}_{p,r}(\bar{x}),G(y_{k+1})) \\ &\le \mathfrak{F}(M_{\sigma,F\circ G}(y_k,y_{k+1}),\tau(y_k,y_{k+1},F\circ G(y_k),F\circ G(y_{k+1}))) \\ &\le \mathfrak{F}(\sigma(y_k,y_{k+1}),\tau(y_k,y_{k+1},F\circ G(y_k),F\circ G(y_{k+1}))) \\ &\le \mathfrak{F}(\varphi(M_{p,G\circ F}(x_k,x_{k+1})),\tau(y_k,y_{k+1},F\circ G(y_k),F\circ G(y_{k+1}))) \\ &\le \mathfrak{F}_\varphi(M_{p,G\circ F}(x_k,x_{k+1}),\tau(y_k,y_{k+1},F\circ G(y_k),F\circ G(y_{k+1}))) \\ &\le M_{p,G\circ F}(x_k,x_{k+1}). \end{aligned}$$

This implies that

$$\mathfrak{F}_\varphi(M_{p,G\circ F}(x_k,x_{k+1}),\tau(y_k,y_{k+1},F\circ G(y_k),F\circ G(y_{k+1})))=M_{p,G\circ F}(x_k,x_{k+1})$$

which give $M_{p,G\circ F}(x_k,x_{k+1})=0$ or $\tau(y_k,y_{k+1},F\circ G(y_k),F\circ G(y_{k+1}))=0$. So, we get a contradiction.

Case 2: If $\max\{p(x_k,x_{k+1}),p(x_{k+1},x_{k+2})\}=p(x_k,x_{k+1})$ and $\max\{\sigma(y_k,y_{k+1}),\sigma(y_{k+1},y_{k+2})\}=\sigma(y_{k+1},y_{k+2})$. We have

$$\begin{aligned} M_{p,G\circ F}(x_k,x_{k+1}) &\le p(x_k,x_{k+1}) \\ &\le e_p(G(y_{k-1})\cap\mathbb{B}_{p,r}(\bar{x}),G(y_k)) \\ &\le \mathfrak{F}(M_{\sigma,F\circ G}(y_{k-1},y_k),\tau(y_{k-1},y_k,F\circ G(y_{k-1}),F\circ G(y_k))) \\ &\le \mathfrak{F}(\sigma(y_k,y_{k+1}),\tau(y_{k-1},y_k,F\circ G(y_{k-1}),F\circ G(y_k))) \\ &\le \mathfrak{F}(\varphi(M_{p,G\circ F}(x_k,x_{k+1})),\tau(y_{k-1},y_k,F\circ G(y_{k-1}),F\circ G(y_k))) \\ &\le \mathfrak{F}_\varphi(M_{p,G\circ F}(x_k,x_{k+1}),\tau(y_{k-1},y_k,F\circ G(y_{k-1}),F\circ G(y_k))) \\ &\le M_{p,G\circ F}(x_k,x_{k+1}). \end{aligned}$$

This implies that

$$\mathfrak{F}_\varphi(M_{p,G\circ F}(x_k,x_{k+1}),\tau(y_{k-1},y_k,F\circ G(y_{k-1}),F\circ G(y_k)))=M_{p,G\circ F}(x_k,x_{k+1})$$

which give $M_{p,G\circ F}(x_k,x_{k+1})=0$ or $\tau(y_{k-1},y_k,F\circ G(y_{k-1}),F\circ G(y_k))=0$. So, we get a contradiction.

Case 3: If $\max\{p(x_k,x_{k+1}),p(x_{k+1},x_{k+2})\}=p(x_{k+1},x_{k+2})$ and $\max\{\sigma(y_k,y_{k+1}),\ \sigma(y_{k+1},y_{k+2})\}=\sigma(y_{k+1},y_{k+2})$. Then we get a contradiction with the decreasing of the sequences $u_n=p(x_n,x_{n+1})$ and $v_n=\sigma(y_n,y_{n+1})$.

In the second place, we realised that $\max\{p(x_k,x_{k+1}),p(x_{k+1},x_{k+2})\}=p(x_k,x_{k+1})$ and $\max\{\sigma(y_k,y_{k+1}),\sigma(y_{k+1},y_{k+2})\}=\sigma(y_k,y_{k+1})$ and then the inequality (3.3).

Now we prove the second step of induction (3.2). By using assumption **(c)**, inequality (3.3) and the nondecreasing of $\mathfrak{F}$ in first variable, we have

$$\begin{aligned}p(x_{k+1},G(y_{k+1}))&\le e_p(G(y_k)\cap\mathbb{B}_{p,r}(\bar{x}),G(y_{k+1}))\\&\le\mathfrak{F}(M_{\sigma,F\circ G}(y_k,y_{k+1}),\tau(y_k,y_{k+1},F\circ G(y_k),F\circ G(y_{k+1})))\\&\le\mathfrak{F}(\varphi(M_{p,G\circ F}(x_k,x_{k+1})),\tau(y_k,y_{k+1},F\circ G(y_k),F\circ G(y_{k+1})))\\&\le\mathfrak{F}_\varphi(M_{p,G\circ F}(x_k,x_{k+1}),\tau(y_k,y_{k+1},F\circ G(y_k),F\circ G(y_{k+1})))\\&\le M_{p,G\circ F}(x_k,x_{k+1}).\end{aligned}$$

If we assume that $\tau(y_k,y_{k+1},F\circ G(y_k),F\circ G(y_{k+1}))=0$ for some $k\in\mathbb{N}$ or $M_{p,G\circ F}(x_k,x_{k+1})\le p(x_{k+1},G(y_{k+1}))$, then we have

$$\mathfrak{F}_\varphi(M_{p,G\circ F}(x_k,x_{k+1}),\tau(y_k,y_{k+1},F\circ G(y_k),F\circ G(y_{k+1})))=M_{p,G\circ F}(x_k,x_{k+1})$$

which implies that $M_{p,G\circ F}(x_k,x_{k+1})=0$ or $\tau(y_k,y_{k+1},F\circ G(y_k),F\circ G(y_{k+1}))=0$ and then $x_k=x_{k+1}$ or $y_k=y_{k+1}$ which is a contradiction.

So we assume that $p(x_{k+1},G(y_{k+1}))<M_{p,G\circ F}(x_k,x_{k+1})$ for all $k\in\mathbb{N}$ and $\tau(y_k,y_{k+1},F\circ G(y_k),F\circ G(y_{k+1}))\neq 0$, then there exists $x_{k+2}\in G(y_{k+1})$ such that

$$p(x_{k+1},x_{k+2})<M_{p,G\circ F}(x_k,x_{k+1})\le p(x_k,x_{k+1}).$$

Moreover, if $\mathfrak{F}_\varphi\in\mathcal{C}_I$ and τ is nondecreasing then we have

$$\begin{aligned}p(x_{k+1},x_{k+2})&\le e_p(G(y_k)\cap\mathbb{B}_{p,r}(\bar{x}),G(y_{k+1}))\\&\le\mathfrak{F}(\sigma(y_k,y_{k+1}),\tau(y_k,y_{k+1},F\circ G(y_k),F\circ G(y_{k+1})))\\&\le\mathfrak{F}(\varphi(p(x_k,x_{k+1})),\tau(y_0,y_1,F\circ G(y_0),F\circ G(y_1)))\\&\le\mathfrak{F}_\varphi(p(x_k,x_{k+1}),\tau(y_0,y_1,F\circ G(y_0),F\circ G(y_1)))\\&\le\mathfrak{F}_\varphi(\psi^k(p(x_0,x_1)),\tau(y_0,y_1,F\circ G(y_0),F\circ G(y_1)))\\&\le\psi^{k+1}(p(x_0,x_1))\end{aligned}$$

else if $\mathfrak{F}_\varphi\in\mathcal{C}_{II}$ and $\tau(y_k,y_{k+1},F\circ G(y_k),F\circ G(y_{k+1}))\ge\alpha$

$$\begin{aligned} p(x_{k+1},x_{k+2}) &\leq e_p(G(y_k)\cap \mathbb{B}_{p,r}(\bar{x}),G(y_{k+1})) \\ &\leq \mathfrak{F}(\sigma(y_k,y_{k+1}),\tau(y_k,y_{k+1},F\circ G(y_k),F\circ G(y_{k+1}))) \\ &\leq \mathfrak{F}(\varphi(p(x_k,x_{k+1})),\tau(y_k,y_{k+1},F\circ G(y_k),F\circ G(y_{k+1}))) \\ &\leq \mathfrak{F}_\varphi(\psi^k(p(x_0,x_1)),\alpha) \\ &\leq \mathfrak{F}_\varphi(\psi^k(p(x_0,x_1)),\psi^k(p(x_0,x_1))) \\ &\leq \psi^{k+1}(p(x_0,x_1)). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sigma(y_{k+1},F(x_{k+2})) &\leq e_\sigma(F(x_{k+1})\cap \mathbb{B}_{\sigma,r}(\bar{y}),F(x_{k+2})) \\ &\leq \varphi(M_{\sigma,G\circ F}(x_{k+1},x_{k+2})). \end{aligned}$$

If we assume that $M_{p,G\circ F}(x_{k+1},x_{k+2}) \leq \sigma(y_{k+1},F(x_{k+2}))$ for some $k\in\mathbb{N}$ then we get a contradiction with the definition of φ and $M_{p,G\circ F}(x_{k+1},x_{k+2})\in J$. Then we suppose that $\sigma(y_{k+1},F(x_{k+2})) < M_{p,G\circ F}(x_{k+1},x_{k+2})$ and then there exists $y_{k+2}\in F(x_{k+2})$ such that

$$\sigma(y_{k+1},y_{k+2}) < M_{p,G\circ F}(x_{k+1},x_{k+2}) \leq p(x_{k+1},x_{k+2}).$$

And with the inequality (3.3), we have

$$\sigma(y_{k+1},y_{k+2}) \leq \varphi(p(x_{k+1},x_{k+2})) \leq \varphi(\psi^{k+1}(p(x_0,x_1))).$$

On the other hand, x_{k+2} is an element of the opened p-ball $\mathbb{B}_{p,r}(\bar{x})$ and y_{k+2} be an element of the opened σ-ball $\mathbb{B}_{\sigma,r}(\bar{y})$. Indeed,

$$\begin{aligned} p(x_{k+2},x_0) &\leq \sum_{j=0}^{k+1} p(x_{j+1},x_j) - \sum_{j=1}^{k+1} p(x_j,x_j) \\ &< \sum_{j=0}^{+\infty} \psi^j(p(x_1,x_0)) \\ &= w(\alpha,\cdot) \leq p(x_0,x_0)+r. \end{aligned}$$

and

$$\begin{aligned} \sigma(y_{k+2},y_0) &\leq \sum_{j=0}^{k+1} \sigma(y_{j+1},y_j) - \sum_{j=1}^{k+1} \sigma(y_j,y_j) \\ &< \sum_{j=0}^{+\infty} \varphi(\psi^j(p(x_1,x_0))) \\ &< w(\alpha,\cdot) \leq \sigma(y_0,y_0)+r. \end{aligned}$$

We have for all integers n and m where $n>m$

$$\begin{aligned} p(x_n, x_m) &\le \sum_{k=m}^{n-1} p(x_k, x_{k+1}) - \sum_{k=m+1}^{n-1} p(x_k, x_k) \\ &\le \sum_{k=m}^{n-1} \psi^k(p(x_0, x_1)) \\ &\le \sum_{k=0}^{+\infty} \psi^k(p(x_0, x_1)) \\ &\le w(\alpha, \cdot). \end{aligned}$$

We know that $\{x_n\}$ is a 0-Cauchy sequence in $\overline{\mathbb{B}_{p,r}}(\bar{x})$ since $w(s, \cdot)$ converges for each $s \in J$. As $\overline{\mathbb{B}_{p,r}}(\bar{x})$ is a 0-complete subspace, $\{x_n\}$ converges to $x^* \in \overline{\mathbb{B}_{p,r}}(\bar{x})$ with respect to τ_p. This implies that

$$p(x^*, x^*) = \lim_{k \to +\infty} p(x_k, x^*) = \lim_{n,m \to +\infty} p(x_n, x_m) = 0.$$

Analogously, for a sequence $\{y_k\}$ we have

$$\begin{aligned} \sigma(y_n, y_m) &\le \varphi(p(x_n, x_m)) \\ &\le \varphi(w(\alpha, \cdot)). \end{aligned}$$

Given that $w(s, \cdot)$ is convergent for each $s \in J$ and $\overline{\mathbb{B}_{\sigma,r}}(\bar{y})$ is a 0-complete subspace, we can conclude that $\{y_n\}$ is a 0-Cauchy sequence in $\overline{\mathbb{B}_{\sigma,r}}(\bar{y})$, and thus converges with respect to τ_σ to a point $y^* \in \overline{\mathbb{B}_{\sigma,r}}(\bar{y})$.

$$p(y^*, y^*) = \lim_{k \to +\infty} \sigma(y_k, y^*) = \lim_{n,m \to +\infty} \sigma(y_n, y_m) = 0.$$

We assert now that $x^* \in G(y^*)$. The modified triangle inequality of p, $\mathfrak{F}_\varphi$ is a C-class function and assumption **(c)** give

$$\begin{aligned} p(x^*, G(y^*)) &\le p(x^*, x_{k+1}) + p(x_{k+1}, G(y^*)) - p(x_{k+1}, x_{k+1}) \\ &\le p(x^*, x_{k+1}) + e_p(G(y_k) \cap \mathbb{B}_{p,r}(\bar{x}), G(y^*)) \\ &\le p(x^*, x_{k+1}) + \mathfrak{F}\left(\sigma(y_k, y^*), \tau(y_k, y^*, F \circ G(y_k), F \circ G(y^*))\right) \\ &\le p(x^*, x_{k+1}) + \mathfrak{F}_\varphi\left(p(x_k, x^*), \tau(y_k, y^*, F \circ G(y_k), F \circ G(y^*))\right) \\ &\le p(x^*, x_{k+1}) + p(x_k, x^*) \end{aligned}$$

By taking the limit as k approaches infinity, we get $p(x^*, G(y^*)) = 0 = p(x^*, x^*)$. Applying Lemma 2.1, we conclude that

$$x^* \in \overline{G(y^*)} = G(y^*).$$

Analogously, we now claim that $y^* \in F(x^*)$. This follows from the modified triangle inequality and assumption **(b)**, which imply that

$$\begin{aligned} \sigma(y^*, F(x^*)) &\le \sigma(y^*, y_k) + \sigma(y_k, F(x^*)) - \sigma(y_k, y_k) \\ &\le \sigma(y^*, y_k) + e_\sigma(F(x_k) \cap \mathbb{B}_{\sigma,r}(\bar{y}), F(x^*)) \\ &\le \sigma(y^*, y_k) + \varphi(p(x_k, x^*)) \end{aligned}$$

Since $\lim_{t \downarrow 0} \varphi(t) = 0$, taking limit as $k \to +\infty$, we obtain

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$$\sigma(y^*, F(x^*)) = 0 = \sigma(y^*, y^*).$$

This, according to Lemma 2.1, implies $y^* \in \overline{F(x^*)} = F(x^*)$.

Assuming single-valued mappings F and G and $p(\bar{x}, \bar{x}) + 2r \in J$, let (x, y) and (x^*, y^*) be two different altering points in $\overline{\mathbb{B}_{p,r}}(\bar{x}) \times \overline{\mathbb{B}_{\sigma,r}}(\bar{y})$ for F and G, where $y = F(x)$, $y^* = F\left(x^*\right)$, $x = G(y)$, and $x^* = G\left(y^*\right)$. Then, we have:

$$M_{p,G\circ F}(x, x^*) = p(x, x^*) \leq p(x, \bar{x}) + p\left(\bar{x}, x^*\right) - p(\bar{x}, \bar{x}) \leq p(\bar{x}, \bar{x}) + 2r \in J,$$

and

$$\begin{aligned} M_{\sigma,F\circ G}(y, y^*) = \sigma\left(y, y^*\right) &= \sigma\left(y, F\left(x^*\right)\right) \\ &\leq e_\sigma\left(F\left(x\right) \cap \mathbb{B}_{\sigma,r}(\bar{y}), F\left(x^*\right)\right) \\ &\leq \varphi\left(M_{p,G\circ F}(x, x^*)\right) \\ &\leq \varphi(p\left(x, x^*\right)) \leq p\left(x, x^*\right). \end{aligned}$$

So, by using $(\mathbb{F}3)$, we get

$$\begin{aligned} p\left(x, x^*\right) &= p\left(x, G\left(y^*\right)\right) \\ &\leq e_p\left(G\left(y\right) \cap \mathbb{B}_{p,r}(\bar{x}), G\left(y^*\right)\right) \\ &\leq \mathfrak{F}\left(M_{\sigma,F\circ G}(y, y^*), \tau(y, y^*, y, y^*)\right) \\ &\leq \mathfrak{F}\left(\varphi(p\left(x, x^*\right)), \tau(y, y^*, y, y^*)\right) \\ &\leq \mathfrak{F}_\varphi\left(p\left(x, x^*\right), \tau(y, y^*, y, y^*)\right) \leq p\left(x, x^*\right) \end{aligned}$$

then we have

$$\mathfrak{F}_\varphi\left(p\left(x, x^*\right), \tau(y, y^*, y, y^*)\right) = p\left(x, x^*\right)$$

which implies that $p\left(x, x^*\right) = 0$ or $\tau(y, y^*, y, y^*) = 0$, thus $x = x^*$ or $y = y^*$ which is a contradiction and then there exist a unique altering points $(x^*, y^*) \in \overline{\mathbb{B}_{p,r}}(\bar{x}) \times \overline{\mathbb{B}_{\sigma,r}}(\bar{y})$ such that $y^* = F\left(x^*\right)$ and $x^* = G\left(y^*\right)$. By consequence, we get,

$$x^* = G\left(y^*\right) = G \circ F(x^*) \Rightarrow x^* \text{ is the only fixed point of } G \circ F \text{ in } \overline{\mathbb{B}_{p,r}}(\bar{x}),$$
$$y^* = F\left(x^*\right) = F \circ G(y^*) \Rightarrow y^* \text{ is the only fixed point of } F \circ G \text{ in } \overline{\mathbb{B}_{\sigma,r}}(\bar{y}).$$

4. Corollaries Related

As corollaries, we have an extended version of [6,Theorem 11] (i.e. Theorem 2.1), [7, Theorem 6.3] and [9, Theorem 3.1] within the context of partial metric spaces that are 0-complete.

Corollary 4.1. *Let (X, p) and (Y, σ) be partial metric spaces where $\overline{\mathbb{B}_{p,r}}(\bar{x})$ and $\overline{\mathbb{B}_{\sigma,r}}(\bar{y})$ are 0-complete subspaces of X and Y respectively, for $\bar{x} \in X$, $\bar{y} \in Y$, and $r > 0$. Consider set-valued mappings $F : \overline{\mathbb{B}_{p,r}}(\bar{x}) \to C^\sigma(Y)$ and $G : \overline{\mathbb{B}_{\sigma,r}}(\bar{y}) \to C^p(X)$ such that $\bar{y} \in F(\bar{x})$. Let $(\psi, \varphi) \in \Psi$ be increasing functions $\varphi, \psi : J \to J$ and $\alpha \in J$. Assume the following:*

(a) $\max\{p(\bar{x}, G(\bar{y})), \sigma(\bar{y}, F(\bar{x}))\} < \alpha$, where $s(\alpha) \leq \min\{p(\bar{x}, \bar{x}), \sigma(\bar{y}, \bar{y})\} + r$,

(b) $e_\sigma\left(F\left(x_1\right) \cap \mathbb{B}_{\sigma, r}(\bar{y}), F\left(x2\right)\right) \leq \varphi\left(M_{p, G \circ F}(x_1, x_2)\right)$, for all $x_1, x_2 \in \mathbb{B}_{p, r}(\bar{x})$,

(c) $e_p\left(G\left(y_1\right) \cap \mathbb{B}_{p, r}(\bar{x}), G\left(y2\right)\right) \leq \psi\left(M_{\sigma, F \circ G}(y_1, y_2)\right)$, for all $y_1, y_2 \in \mathbb{B}_{\sigma, r}(\bar{y})$.

Then there exist (x^*, y^*) an altering point of F and G in $\overline{\mathbb{B}_{p,r}}(\bar{x}) \times \overline{\mathbb{B}_{\sigma,r}}(\bar{y})$. If F and G are both single-valued mappings and $p(\bar{x}, \bar{x}) + 2r \in J$, then x^* is the unique fixed point of $G \circ F$ in $\overline{\mathbb{B}_{p,r}}(\bar{x})$ and y^* is the unique fixed point of $F \circ G$ in $\overline{\mathbb{B}_{\sigma,r}}(\bar{y})$.

Proof. Take $\mathfrak{F}(s,t) = \psi(s)$ such that $\lim_{s \downarrow 0} \psi(s) = 0$ and then the subset $\mathbb{F}$ will be the subset Ψ. Since the function $\mathfrak{F}$ does not depend on the second variable t, we can choose $\tau \in \Xi$ to be non-decreasing or greater than α, and then apply the Theorem 3.1.

$\boxdot$

Using Theorem 3.1 with $X = Y$, $p = \sigma$, $F = Id_X$, and $\varphi = Id_J$, we can establish the following fixed point theorem:

Corollary 4.2. *Theorem 3.4 in [2].*

For $\varphi(t) = k_1 t$ and $\psi(s) = k_2 s$ where $\max\{k_1, k_2\} < 1$ and applying corollary 4.1 then we get

Corollary 4.3. *Let (X, p) and (Y, σ) be partial metric spaces such that $\overline{\mathbb{B}_{p,r}}(\bar{x})$ and $\overline{\mathbb{B}_{\sigma,r}}(\bar{y})$ are 0-complete subspaces of X and Y respectively, for $\bar{x} \in X, \bar{y} \in Y$, $r > 0$ and $0 \leq \max\{k_1, k_2\} < 1$. Let $F : \overline{\mathbb{B}_{p,r}}(\bar{x}) \to C^\sigma(Y)$ and $G : \overline{\mathbb{B}_{\sigma,r}}(\bar{y}) \to C^p(X)$ are set-valued mappings such that $\bar{y} \in F(\bar{x})$. Suppose that the following assumptions hold:*

(a) $\max\{p(\bar{x}, G(\bar{y})), \sigma(\bar{y}, F(\bar{x}))\} < (\min\{p(\bar{x}, \bar{x}), \sigma(\bar{y}, \bar{y})\} + r)(1 - k_1 k_2)$,

(b) $e_\sigma\left(F\left(x_1\right) \cap \mathbb{B}_{\sigma, r}(\bar{y}), F\left(x_2\right)\right) \leq k_1 M_{p, G \circ F}(x_1, x_2)$, for all $x_1, x_2 \in \mathbb{B}_{p, r}(\bar{x})$,

(c) $e_p\left(G\left(y_1\right) \cap \mathbb{B}_{p, r}(\bar{x}), G\left(y_2\right)\right) \leq k_2 M_{\sigma, F \circ G}(y_1, y_2)$, for all $y_1, y_2 \in \mathbb{B}_{\sigma, r}(\bar{y})$.

Then there exist (x^*, y^*) an altering point of F and G in $\overline{\mathbb{B}_{p,r}}(\bar{x}) \times \overline{\mathbb{B}_{\sigma,r}}(\bar{y})$. If F and G are both single-valued mappings and $p(\bar{x}, \bar{x}) + 2r \in J$, then x^* is the unique fixed point of $G \circ F$ in $\overline{\mathbb{B}_{p,r}}(\bar{x})$ and y^* is the unique fixed point of $F \circ G$ in $\overline{\mathbb{B}_{\sigma,r}}(\bar{y})$.

Proof. For $J = [0, +\infty)$ and the corresponding estimate function $s(t) = \dfrac{t}{1 - k_1 k_2}$, we select the Bianchini-Grandolfi gauge function $\psi \circ \varphi(t) = k_1 k_2 t$ in accordance with Corollary 4.1. Then we set $\alpha = (\min\{p(\bar{x}, \bar{x}), \sigma(\bar{y}, \bar{y})\} + r)(1 - k_1 k_2) \in J$ and apply Corollary 4.1.

If $r \longrightarrow +\infty$, then $\mathbb{B}_{+\infty}(\bar{x}) = X$, $\mathbb{B}_{+\infty}(\bar{y}) = Y$ and we get the extended version of [7, Theorem 6.3] as follows

Corollary 4.4. *Let (X, p) and (Y, σ) be 0-complete partial metric spaces. Let $0 \leq \max\{k_1, k_2\} < 1$. Let $F : X \to C^\sigma(Y)$ and $G : Y \to C^p(X)$ be set-valued mappings. Suppose the following assumptions hold:*

(a) $e_\sigma(F(x_1), F(x_2)) \leq k_1 M_{p,G\circ F}(x_1, x_2)$, for all $x_1, x_2 \in X$,

(b) $e_p(G(y_1), G(y_2)) \leq k_2 M_{\sigma,F\circ G}(y_1, y_2)$, for all $y_1, y_2 \in Y$.

Then there exist (x^*, y^*) an altering point of F and G in $X \times Y$. If F and G are both single-valued mappings, then x^* is the unique fixed point of $G \circ F$ in X and y^* is the unique fixed point of $F \circ G$ in Y.

Proof. Given $\bar{x} \in X$ and $\bar{y} \in Y$ such that $\bar{y} \in F(\bar{x})$, let $r > 0$ be chosen such that $\max\{p(\bar{x}, G(\bar{y})), \sigma(\bar{y}, F(\bar{x}))\} < (\min\{p(\bar{x}, \bar{x}), \sigma(\bar{y}, \bar{y})\} + r)(1 - k_1 k_2)$. Then, we can apply Corollary 4.3.

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