Theorems for Near Stable Points on a Near Banach Space Furnished with Graph

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Abstract:

The paper introduces a novel method for defining the graph associated with a near Banach space. In mathematics, a graph typically represents relationships between objects. Here, it seems the graph is being defined in the context of a near Banach space, which is a generalization of Banach spaces allowing the norm to take infinite values. An iteration function is utilized to define the subgraph of the graph associated with the near Banach space. This subgraph likely captures specific properties or relationships within the original graph. The paper presents near-fixed point theorems by well-known math- ematicians such as Banach, Kannan, Chatterja, and Ciric [2][18][7][9]. These theorems deal with the existence of points that are approximately fixed under certain mappings or operations. The near-fixed point theorems mentioned are obtained or derived using the new approach introduced for defining the graph and its subgraph associated with the near Banach space [20]. This suggests that the new approach is effective in providing a framework for proving these theorems or extending their applicability to near Banach spaces. The paper discusses a fresh method for defining the graph of a near Banach space, employs an iteration function to define its subgraph, and then demonstrates the utility of this approach by deriving near-fixed point theorems by eminent mathemati- cians in the field [14][15[16].

Keywords: Cauchy Sequence, near fixed Point, Banach hyperspace, iterated function, graph, subgraph, Wˆ -sequence.

Subject Classification: 47H10, 54H25

I INTRODUCTION

Jachymiski's [17] generalization of the Banach contraction principle by combining fixed point theory and graph theory sounds like an intriguing extension. By incorporating graph theory into the framework, it likely allows for the consideration of more complex structures or relationships between points in the space, beyond just metric properties.

In traditional fixed-point theory, the Banach contraction principle [2] guarantees the existence and uniqueness of fixed points for contraction mappings in complete metric spaces. However, this principle may not directly apply in settings where the underlying space has a more intricate structure, such as when relationships between points are described by a graph. By leveraging graph theory concepts, Jachymiski's [17] generalization may provide a way to handle mappings that interact with the underlying graph structure in some manner. This could involve mappings that respect certain graph-theoretic properties or have dependencies on the graph edges or vertices. The significance of this extension likely lies in its applicability to problems where the traditional Banach contraction principle cannot be directly applied due to the presence of a graph structure. It opens up new avenues for studying fixed point properties [7][9][18][20] in spaces that exhibit both metric and graph-theoretic characteristics. It can be speculated that Jachymiski's [17] generalization offers a way to deal with mappings that interact with the graph structure underneath. This interaction could take many different forms, for example, mappings that depend on the graph's vertices and edges or that adhere to certain graph-theoretic features. This expansion provides opportunities to examine fixed point attributes in spaces that have both metric and graph-theoretic properties; it also indicates a more nuanced understanding of the interactions between points. The possible applicability of this extension to problems where the presence of a graph structure poses difficulties for the classic Banach contraction principle demonstrate the significance of this work [1][20].

Jachymiski's generalization not only broadens the application of the Banach contraction principle but also provides a flexible means of examining spaces with a variety of structural features by bridging the gap between fixed point theory and graph theory. To delve deeper into the specifics of Jachymiski's [17] generalization and its applications, it would be necessary to refer to the original paper [17] and explore how the combination of fixed point theory and graph theory is utilized to establish existence and uniqueness results for fixed points in this extended framework.

The graph was defined in previous studies of fixed point theorems on metric spaces S endowed with a graph by considering the vertex set to be the set S and the edge set

to be the diagonal of the cartesian product $S \times S$. In another way, it was assumed that the graph would have loops at each vertex. If a loop exists at vertex A in the graph, then that vertex is the fixed point of the mapping under consideration. The iteration function is necessary to take the above graph and turn it into a subgraph that shows various contraction concepts. The behavior of mappings inside the specified graph structure can be better seen and understood thanks to this iterative procedure. Researchers can investigate fixed point theorems in a more complex context—one in which the interactions between points are impacted by the underlying graph structure in addition to metric properties [22] — by combining these graph-based concerns. Using the iteration function, a subgraph of the aforementioned graph is generated to demonstrate different contraction concepts.

II   PRELIMINARIES

A mathematical framework known as Banach hyperspace (In short, BHS) allows one to analyse the characteristics and connections between compact sets in metric spaces by applying the methods and structures of Banach spaces. These spaces find use in many different areas of mathematics, such as topology, functional analysis, and set-valued analysis.

Let us examine a Banach hyperspace, $(K(S), |\cdot|)$, where the space $K(S)$ is the collection of all closed subsets of a metric space S that are not empty. The norm of this space is $|\cdot|$, and it is typically defined with the help of the Hausdorff metric. The "closeness" between two sets can be expressed in terms of their Hausdorff distances using the Hausdorff metric.

Let Ω now be a null set. We wish to show that the null equality is satisfied by the norm $|\cdot|$. This indicates, in mathematical words, that for any set A that is a member of the null set Ω, $|A| = 0$. To elucidate, let us examine many fundamental ideas concerning Banach hyperspaces:

1. Banach Hyperspace Sequences: [14][15][16]

   Sequences of sets in the Banach hyperspace can be studied, and their convergence qualities under a selected norm can be examined. This is figuring out when a series of sets in the hyperspace converge to a limit set.

2. Graphs and Sub-graphs:

   Graphs connected to sets in the metric space can be examined thanks to the Banach hyperspace. Investigating sub-graphs and their characteristics is one way to learn about the organisation and connectivity of compact collections.

3. Near-Static Points:

In order to study near-fixed points, one must comprehend sets that, under a particular transformation or mapping, are almost fixed. The stability of sets under specific operations or mappings can be studied in the context of Banach hyperspaces.

The behaviour and interactions of compact sets in metric spaces can be better understood by mathematicians [12][14][15][16][22] by exploring these ideas within the context of the Banach hyperspace framework. An essential component is the proof of null equality, which guarantees that the norm accurately describes the characteristics of sets in the Banach hyperspace.

2.1 Sequences

In functional analysis, sequences are essential tools for examining the characteristics of operators and functions defined on Banach spaces. Understanding the behaviour of functions and the boundaries of different operations in these spaces requires an understanding of the convergence of sequences. Sequences are important in the setting of Banach spaces, especially when talking about convergence and completeness. A description of sequences in Banach spaces is as follows:

Definition 2.1.1:

A sequence $\{Y_n\}_{n=1}^{\infty}$ is said to be convergent in the Banach hyperspace $(K(S), \|.\|)$ if, given a $\varepsilon > 0$, there exists $N_1 \in I$ such that $\| Y_n - Y \| < \varepsilon, n \geq N_1$.

2.2 Graphs

A graph is made up of nodes, or vertices, and the edges that join node pairs. A variety of ideas and structures are involved in the study of graphs. Analysing and comprehending graphs requires investigating attributes such as routes, cycles, connectivity, and the structural makeup of nodes and edges. Some necessary notions related to graphs are listed below:

Definition 2.2.1:

An ordered pair (V, E), with V representing the set of points known as vertices and E representing the set of lines known as edges, constitutes a graph H.

Definition 2.2.2:

A graph $H_0 = (V_0, E_0)$ is said to be a sub-graph of $H = (V, E)$ if $V_0 \subset V$ and $E_0 \subset E$.

We do not like some special edge types as they cause complications in calculations:

DEFINITION **2.2.3:**

If an edge in a graph H has the same initial and terminal vertices, it is referred to as a loop. Multiple edges are those that connect the same pair of vertices with two or more edges. A simple graph is one that has neither loops nor many edges.

DEFINITION **2.2.4:**

The term "weighted graph" refers to a basic graph in which each edge has a numerical value assigned to it. Therefore, the vertex set, edge set, and weight of each edge make up a weighted graph.

2.3 Near Fixed Point Theorems

The classical fixed point theorems are extended to the context of Banach hyperspaces by near fixed point theorems. Hyperspaces are spaces of closed sets with an appropriate topology. A Banach space with non-empty closed subsets of a specified metric space, frequently furnished with the Hausdorff metric, is called a Banach hyperspace [14][15][16]. Classical fixed point theorems can be extended to include near fixed point theorems. They deal with cases where a mapping almost has a fixed point, rather than demanding a rigid fixed point. A proximity or approximation notion is used to quantify "nearness". The concept of "nearness" in the context of mappings is introduced by near fixed point theorems in Banach hyperspaces, which expand the traditional fixed point theory to the space of closed sets. These theorems have applications in many mathematical fields and are important for understanding the dynamics of mappings on closed sets.

Assume that there is a function $\mathsf{S} : K(S) \to K(S)$ that maps $K(S)$ into itself. If and only if $\mathsf{S}(A) = A$, then $A \in K(S)$ is a fixed point of S. The idea of a fixed point in set-valued functions is entirely distinct from this notion. There are some classical fixed point theorems that depend on the normed space, which is also a vector space. Since $(K(S), \|.\|)$ is not a vector space, we are unable to investigate the related fixed point theorems based on $(K(S), \|.\|)$. But we can examine the so-called near fixed point, which has the following definitions.

DEFINITION **2.3.1:**

Consider a function defined on $K(S)$ into itself, such that $\mathsf{S} : K(S) \to K(S)$. If and only if $\mathsf{S}(A) \overset{\mathsf{Q}}{=} A$, then a point $A \in K(S)$ is referred to as a near fixed point of S.

DEFINITION **2.3.2:**

Consider a pseudo-seminormed hyperspace $(K(S), \|.\|)$. If and only if there is a real number $0 < \beta < 1$ such that $\|\mathsf{S}(A) \ominus \mathsf{S}(B)\| \leq \beta\|A \ominus B\|$ for any $A, B \in K(S)$, then a function $\mathsf{S} : (K(S), \|.\|) \to (K(S), \|.\|)$ is referred to as a contraction on $K(S)$.

Using the function S, we define the iterative sequence $\{A_n\}_{n=1}^{\infty}$ given any initial ele- ment $A_0 \in K(S)$ as follows:

$$A_1 = S(A_0), A_2 = SS(A_0), \ldots$$

We will demonstrate that the series $\{A_n\}_{n=1}^{\infty}$ can converge to a close fixed point under certain appropriate circumstances. This convergence to a fixed point can occur under certain conditions, which we will now examine. Diverse criteria and theorems guarantee that a sequence will eventually converge.

Definition 2.3.3:

Let $(K(S), \|.\|)$ be a BHS. A mapping $S : K(S) \to K(S)$ is called a Kannan Mapping if there exists $\beta \in (0, \frac{1}{2})$ such that,

$$\|S(x) \ominus S(y)\| \le \beta^{\cdot} \|x \ominus S(x)\| \oplus \|y \ominus S(y)\|^{\cdot}$$

Definition 2.3.4:

Let $(K(S), \|.\|)$ be a BHS. A mapping $S : K(S) \to K(S)$ is called a Chatterjea Mapping if there exists $\beta \in (0, \frac{1}{2})$ such that,

$$\|S(x) \ominus S(y)\| \le \beta^{\cdot} \|x \ominus S(y)\| \oplus \|y \ominus S(x)\|^{\cdot}$$

Definition 2.3.5:

Let $(K(S), \|.\|)$ be a BHS. A map $S : K(S) \to K(S)$ is λ-generalized contraction if and only if for every $u, v \in K(S)$, there exist non-negative numbers $q(u, v)$, $r(u, v)$, $s(u, v)$ and $t(u, v)$ such that,

$$sup_{u,v \in S} \{q(u, v) + r(u, v) + s(u, v) + 2t(u, v)\} = \lambda < 1$$

and
$$\|u \ominus S(v)\| \le q(u, v)\|u \ominus v\| + r(u, v)\|u \ominus S(u)\| + s(u, v)\|v \ominus S(v)\| + t(u, v) [\|u - S(v)\| + \|v - S(u)\|]$$
holds for every $u, v \in K(S)$.

Definition 2.3.6:

Suppose there is a BHS $(K(S), \|.\|)$. If there is a limit point in S for every cauchy sequence $\{S^{ni}u : i \in N\}$, $u \in K(S)$, then the mapping $S : K(S) \to K(S)$ is considered T-orbitally complete.

Definition 2.3.7:

Let $(K(S), \|.\|)$ be a BHS. A mapping $S : K(S) \to K(S)$ is said to be S-orbitally continuous if for $u \in S$ then $u = lim_{i \to \infty} S^{ni}v$ for few $v \in S$, here $Su = lim_{i \to \infty} SS^{ni}v$.

Theorem 2.3.1 (16)**:**

Let S be a λ-generalized contraction of S-orbitally BHS $K(X)$ into itself. Then

1. There is in $K(S)$ a unique near fixed point v under S,

2. $S^n x \to v$ for every $x \in K(X)$ and

3. $\|S^n x \ominus v\| \leq \frac{\lambda^n}{1-\lambda} \|x - S(x)\|$

Theorem 2.3.2 (16)**:**

Let $(K(S), \|.\|)$ be a BHS, and let $\|.\|$ satisfy the null equality, and the null set be Ω. Let $S : (K(S), \|.\|) \to (K(S), \|.\|)$ be a contraction on $K(S)$. If $S(B) \doteq B$, then S has a near fixed point $B \in K(S)$. Moreover, the near fixed point B is obtained by the limit,

$$\|B \ominus B_n\| = \|B_n \ominus B\| \to 0 \text{ as } n \to \infty$$

in which the sequence $\{B_n\}_{n=1}^{\infty}$ is generated according to $B_1 = S(B_0), B_2 = S^2(B_0) \ldots, B_n = S^n(B_0)$. We also have the following properties:

1. Since there is only one equivalence class $[B]$, no $\bar{B} \notin [B]$ may be a near fixed point. This is how uniqueness is defined.

2. In addition, every point $\bar{B} \in [B]$ that satisfies $S(\bar{B}) \doteq \bar{B}$ and $[\bar{B}] = [B]$ is a near fixed point of S.

3. $\bar{B} \in [B], i.e. [\bar{B}] = [B]$, if $\bar{B}$ is a close fixed point of S. Similarly, $B \doteq \bar{B}$ if B and $\bar{B}$ are the near fixed points of S.

III   MAIN RESULT

One method to visualise the interactions between elements in BHS is through graphs associated with them, where each node of the graph corresponds to a compact set. Certain attributes or relationships between these sets are represented by the edges connecting the nodes. The way in which these graphs are specifically constructed can change based on the qualities of interest and the environment. The structure of Banach hyperspaces and the connections between compact sets within them can be seen and examined using these graphs. In this final part of the manuscript, we list our main results:

3.1   Graphs associate with Banach hyperspaces

Let $(K(S), \|.\|)$ be a BHS, and let $\|.\|$ satisfy the null equality, and the null set be Ω. $S : K(S) \to K(S)$. Define the Banach hyperspace-related graph as follows:

DEFINITION 3.1.1:

Assume that the BHS is $(K(S), \|.\|)$. $T : K(S) \to K(S)$, let us say. Define the following weighted graph H connected to $K(S)$:

Let $H = (V, E)$, where $E = \{(x, S(x))/x \in K(S)\}$ and $V = K(S)$. The spacing between an edge's end points determines its weight. As a result, $(K(S), \|.\|)$ is transformed into a BHS and given H.

DEFINITION 3.1.2:

The sub-graph H_0 of H is defined as,

Let y_0 be any random point of S and $H_0 = (V_0, E_0)$ where $V_0 = \{y_0, Sy_0, S^2y_0, \ldots\}$ and let $E_0 = \{(y_0, Sy_0), (Sy_0, S^2y_0), \ldots\}$.

Then $V_0 \subset V$ and $E_0 \subset E$. Hence H_0 is a sub-graph of H.

DEFINITION 3.1.3:

Let $(K(S), \|.\|)$ be a BHS endowed with H. Let H_0 be the sub-graph of H defined as in Def. 3.1.2.

Let $w_n = \|S^{n-1}(y_0) \ominus S^n(y_0)\|$. Then the sequence $\{w_n\}_{n=1}^{\infty}$ is called w-sequence of real numbers associated with the graph H_0.

EXAMPLE 3.1.1:

Let $K(S) = \{0, 1, 2, 3\}$, $\|x - y\| = |x - y|$, $x, y \in K(S)$. Let $S : K(S) \to K(S)$ as,

$$S(x) = \begin{cases} 0, & x \in \{0, 1, 2\} \\ 1, & x = 3 \end{cases}$$

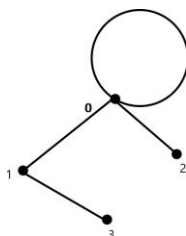$V = \{0, 1, 2, 3\}$ and $E = \{(0, 0), (1, 0), (2, 0), (3, 1)\}$

Figure 3.1: The data structure H linked to $K(S)$

Case 1 : $y_0 = 0$.

Case 2 : $y_0 = 1$.

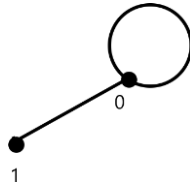Figure 3.2: Within the moment in time where by $y_0 = 0$, graph H_0

Figure 3.3: Within the moment in time where by $y_0 = 1$, graph H_0
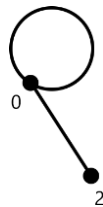
Case 3 : $y_0 = 2$.

Figure 3.4: Within the moment in time where by $y_0 = 2$, graph H_0
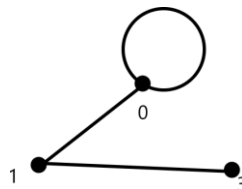
Case 4 : $y_0 = 3$.

Figure 3.5: Within the moment in time where by $y_0 = 3$, graph H_0

On the BHS $(K(S), \|.\|)$ equipped with the graph H, the near fixed point theorems are capable of being proved with the assistance of the subsequent two lemmas.

Lemma 3.1.1 (16):

Given a BHS $(K(S), \|.\|)$, let $S : K(S) \to K(S)$. Assume that H is a graph connected to $K(S)$. Let y_0 represent any random point in $K(S)$. Assume that the sub-graph H_0 of H is defined according to Definition 3.1.2. Then, if and only if the w-sequence connected to the graph H_0 is non-increasing, the sequence $\{S(A_0), S^2(A_0), \ldots\}$ is cauchy.

Lemma 3.1.2 (16):

Let $S : K(S) \to K(S)$ be a BHS, and let $(K(S), \|.\|)$ be its boundary. Assume that H is a graph connected to $K(S)$. If the graph H includes a loop at y^*, then the point y^* of $K(S)$ is a near fixed point of S.

Theorem 3.1.1:

Let $(K(S), \|.\|)$ be a BHS, and let $\|.\|$ satisfy the null equality, and the null set be Ω. Assume we have a contraction function $S : (K(S), \|.\|) \to (K(S), \|.\|)$ on $K(S)$. Assume that G is the graph connected to $K(S)$. Then, $y* \in K(S)$ is the unique near fixed point of S.

Proof :

Considering any starting element $y_0 \in K(S)$. The definitions of the graph H and sub-graph H_0 are,

"Let the Banach hyperspace $(K(S), \|.\|)$ be described. Assume that $S : K(S) \to K(S)$. As shown below, define a weighted graph G connected to $K(S)$.

With $V = K(S)$ and $E = \{x, S(x)/x \in K(S)\}$, let $H = (V, E)$. The spacing between an edge's endpoints determines its weight. At this point, $(K(S), \|.\|)$ is transformed into a BHS and given the graph H. This is the definition of H_0, the sub-graph of H.

Let $H_0 = (V_0, E_0)$, where $V_0 = \{Sy_0, S^2y_0, S^3y_0, \dots S^ny_0\}$ and

Let $E_0 = \{(Sy_0, S^2y_0), (S^2y_0, S^3y_0), \dots\}$, $E_0 \subset E$ and $V_0 \subset V$ follow. Therefore A sub-graph of H is H_0.

Consider,

The iterated sequence given any initial element $y_0 \in K(S)$, $\{Sy_0, S^2y_0, S^3y_0, \dots\} \in K(S)$.

$$y_1 = Sy_0, \quad y_2 = S(y_1) = S(Sy_0) = S^2y_0, \dots \quad y_n = S^n(y_0)$$

From lemma 3.1 of [20] to say that this sequence is cauchy. It is sufficient to show that w-sequence associates with graph G is non-increasing. Since S is a contraction on $K(X)$, we have

$$
\begin{aligned}
w_{n+1} &= \|S^ny_0 \ominus S^{n+1}y_0\| \\
&\leq \beta\|S^{n-1}y_0 \ominus S^ny_0\| \\
&= \beta w_n
\end{aligned}
$$

$$(i.e) \; w_{n+1} < w_n \quad \text{Since } 0 \leq \beta < 1, n \in I$$

Hence the w-sequence associated with H_0 is non-increasing.

By Lemma 3.1 of [20],

$\{Sy_0, S^2y_0, \dots\}$ is an iterated sequence that is cauchy. However, $K(S)$ is complete. We can see that, the series converges to, say, $v* \in K(S)$ as a result.

$\therefore$ The mapping τ, which is a contraction, is continuous, which is clearly visible from the above result. Thus, $\tau v*$ is the convergence point of the series $[SS^{n-1}y_0]_{n=1}^{\infty}$.

But the sequence $[SS^ny_0]_{n=1}^{\infty}$ is a sub-sequence of the sequence $[S^{n-1}y_0]_{n=1}^{\infty}$.

Hence the sub-sequence must have the same limit as the parent sequence. But the limit

of a sequence is unique.

Hence we must have,

$$S v^* = v^*$$

$$\Rightarrow (v^*, v^*) \in G$$

(i.e), H has a loop at v^*.

By lemma 3.2 of [20], v^* is a fixed point of S.

To prove uniqueness:

Let if possible W^* be any other near fixed point of S. Then $S w^* = w^*$. Since S is a contraction on $K(S)$ we have,

$$\|S v^* \ominus S w^*\| \leq \|v^* \ominus w^*\|, \quad 0 < \alpha < 1,$$

$$\|v^* \ominus w^*\| < \|v^* \ominus w^*$$

It is a contradiction.

Hence the near fixed point of S is unique.

THEOREM 3.1.2:

Given a BHS $(K(S), \|.\|)$, its null set is Ω. Let H be connected to $K(S)$, and allow $S : K(S) \to K(S)$. If S satisfy,

$$\|S(x) \ominus S(y)\| \leq \beta(\|x \ominus S(x)\| \oplus \|y \ominus S(y)\|) \tag{1}$$

for all $x, y \in K(S)$, where $\beta \in (0, \frac{1}{2})$, then S has unique near fixed point.

Proof :

Given that any initial element $y_0 \in K(S)$. The H and H_0 are defined as in Definition 3.1.1 and 3.1.2 and refer [20]

Given any initial element $y_0 \in K(S)$, we define the iterative sequence,

$$y_1 = S(y_0), \quad y_2 = S(y_1) = S(S y_0) = S^2 y_0 \ldots \quad y_n = S^n y_0$$

(i.e $\{S(y_0), S^2(y_0), S^3(y_0), \ldots\} \in K(X)$

Lemma 3.1 of [20] states that demonstrating the non-increasing nature of the w-sequence connected to the graph H_0 is sufficient to establish the cauchy nature of this sequence. From eqn 1, we have,

$$w_{n+1} = \|S^n y_0 \ominus S^{n+1} y_0\|$$

$$\leq \beta \left[\|S^{n-1} y_0 \ominus S^n y_0\| \oplus \|S^n y_0 \ominus S^{n+1} y_0\| \right]$$

$$w_{n+1} \leq \beta[w_n + w_{n+1}]$$

$$w_{n+1} \leq \frac{\beta}{1 - \beta} w_n < w_n \qquad \text{Since } 0 \leq \beta < \frac{1}{2}$$

Hence the w-sequence associated with H_0 is non-increasing.

From Lemma 3.1 of [20],

The iterated sequence $\{S(y_0), S^2(y_0), \ldots\}$ is a cauchy sequence in $K(S)$.

But $K(S)$ is complete.

Therefore the sequence converges in $K(S)$.

Let $y^* = \lim_{n \to \infty} S^n y_0$

$$\begin{aligned}
\|y^* \ominus S(y^*)\| &\leq \|y^* \ominus S^n(y_0)\| + \|S^n(y_0) \ominus S(y^*)\| \\
&\leq \|y^* \ominus S^n(y_0)\| + \beta \left[\|S^{n-1}(y_0) \ominus S^n(y_0)\| + \|y^* \ominus S(y^*)\|\right]
\end{aligned}$$

$$\therefore (1 - \beta)\|y^* \ominus S(y^*)\| \leq \|y^* \ominus S^n(y_0)\| + \alpha\|S^{n-1}(y_0) \ominus S^n(y_0)\|$$

Allow, $n \to \infty$ on both sides, Then we have,

$$(1 - \beta)\|y^* \ominus S(y^*)\| \leq \|y^* \ominus y^*\| + \beta\|y^* \ominus y^*\|$$

Hence $\|y^* - S(y^*)\| = 0$

$$\implies S(y^*) = y^*$$

(i.e), y^* is a fixed point of S.

To prove uniqueness:

Let if possible, z^* be any other near fixed point of S. Then $S(z^*) = z^*$. From eqn 1, we have

where, $0 \leq \beta < \frac{1}{2}$ $\quad \|S(y^*) \ominus S(z^*)\| \leq \beta \left[\|y^* \ominus S(y^*)\| + \|z^* \ominus S(z^*)\|\right]$

$$\begin{aligned}
\|y^* \ominus z^*\| &\leq \beta \left[\|y^* \ominus y^*\| + \|z^* \ominus z^*\|\right] \\
\implies \|y^* \ominus z^*\| &= 0 \\
\implies y^* &= z^*
\end{aligned}$$

Hence the near fixed point of S is unique.

Theorem 3.1.3:

Suppose there is a BHS $(K(S), \|.\|)$. Let H be the graph connected to $K(S)$, and allow $S : K(S) \to K(S)$. Then S satisfies,

$$\|S(x) \ominus S(y)\| \leq \beta\|x \ominus S(y)\| + \|y \ominus S(x)\| \tag{2}$$

for all $x, y \in K(S)$ where, $\beta \in [0, \frac{1}{2})$. Then S has a unique fixed point.

Proof :

Let y_0 be a random point in $K(S)$. These apply to H and H_0 as in Definitions 3.1.1 and

3.1.2 and refer [20].

Consider, the iterated sequence,

$$\{S(y_0), S^2(y_0), S^3(y_0), \ldots\} \in K(S)$$

According to Lemma 3.1 of [20],

to demonstrate the cauchy nature of this sequence. It suffices to demonstrate that the graph H_0's w-sequence is non-increasing.

From eqn 2, we have,

$$
\begin{aligned}
w_{n+1} &= \|S^n y_0 \ominus S^{n+1} y_0\| \\
&\leq \beta \, \|S^{n-1} y_0 \ominus S^{n+1} y_0\| \oplus \|S^n y_0 \ominus S^n y_0\| \\
w_{n+1} &\leq \beta \, \|S^{n-1} y_0 \ominus S^n y_0\| \oplus \|S^n y_0 \ominus S^{n+1} y_0\| \\
w_{n+1} &\leq \beta [w_n + w_{n+1}] \\
w_{n+1} &\leq \frac{\beta}{1 - \beta} w_n < w_n \qquad \text{Since } 0 \leq \beta < \frac{1}{2}
\end{aligned}
$$

Hence the w-sequence associated with H_0 is non-increasing.

From Lemma 3.1 of [20], the iterated sequence is cauchy sequence in $K(S)$. But $K(S)$ is complete.

$\therefore$ The sequence converges in $K(S)$.

Let $y^* = \lim_{n \to \infty} S^n(y_0)$

Consider,

$$
\begin{aligned}
\|y^* \ominus S(y^*)\| &\leq \|y^* \ominus S^n(y_0)\| + \|S^n(y_0) - S(y^*)\| \\
&\leq \beta \, \|y^* - S^n(y_0)\| + \beta \|S^{n-1}(y_0) - S(y^*)\| + \|y^* \ominus S^n(y_0)\|
\end{aligned}
$$

Allow $n \to \infty$ on both sides. Then we have,

$$
\begin{aligned}
\|y^* \ominus S(y^*)\| &\leq \|y^* - y^*\| + \beta \, \|y^* - S(y^*)\| + \|y^* - y^*\| \\
(1 - \beta)\|y^* - S(y^*)\| &\leq 0
\end{aligned}
$$

Hence $\|y^* - S(y^*)\| = 0$

$\implies S(y^*) = y^*$

$\therefore \ y^*$ is a fixed point of S.

To prove uniqueness:

Let if possible, z^* be any other fixed point of S. Then $S(z^*) = z^*$. From eqn 2, we have

$$\|S(y^*) \ominus S(z^*)\| \leq \beta \, \|y^* \ominus S(z^*)\| + \|z^* \ominus S(y^*)\|$$

where, $0 \leq \alpha < \frac{1}{2}$

$$\|y^* \ominus z^*\| \leq \beta^{\cdot} \|y^* \ominus z^*\| + \|z^* \ominus y^*\|^{\cdot}$$

$$\| ^* \quad ^* \qquad ^* \quad \mathring{y}_{-}$$

$$=\!\Rightarrow \beta \geq \frac{1}{2}$$

This is a contradiction. Hence $y^* = z^*$.

(i.e), The near fixed point of T is unique.

Definition 3.1.4:

Let the mapping $F : T^+ \to T$ satisfies the following conditions,

1. F is strictly increasing

2. For each sequence $s_n \subset T^+$, $\lim\limits_{n \to +\infty} s_n = 0$ iff $\lim\limits_{n \to +\infty} F(s_n) = -\infty$

3. There exists $m \in (0, 1)$ provided that $\lim\limits_{\lambda \to 0+} \lambda^m F(\lambda) = 0$. The collection of all such mappings is denoted by Ω.

Definition 3.1.5:

Let $(K(S), \|.\|)$ be a BHS. A map $S : K(S) \to K(S)$ is F−contraction if there exist $F \in \Omega$ and $\tau > 0$ provided that $\|\gamma_l \ominus \gamma_n\| > 0 \Rightarrow \tau \oplus F(\|\gamma_l \oplus \gamma_n\|) \leq F(\|l \oplus n\|) \dots (1)$, $\forall\, l, n \in K(S)$.

Example 3.1.2:

Let $F \in \Omega$ be $F(\beta) = I_n \beta$ for any $m \in (0, 1)$ here, every map $S : K(S) \to K(S)$ satisfying (1) is an F- contraction such that $\|\gamma_l \oplus \gamma_n\| \leq e^{-\tau} \|l \oplus n\|$, for every $l, n \in K(S), \gamma_l \neq \gamma_n$

Example 3.1.3:

Consider $F \in \Omega$ be $F(\beta) = \frac{1}{\sqrt{\beta}}, \beta > 0$. In this case, for any $m \in \left(\frac{1}{2}, 1\right)$ every F-contraction γ satisfies,

$$\|\gamma_l \ominus \gamma_n\| \leq \frac{1}{\left(1 \oplus \tau \sqrt{\frac{1}{\|l \ominus n\|}}\right)^2} \|l \ominus n\|, \forall\, l, n \in K(X), \gamma_l \neq \gamma_n$$

Theorem 3.1.4:

Let $S : K(S) \to K(X)$ be an F-contraction and $(K(S), \|.\|)$ be a banach hyperspace. Then S has a unique near fixed point $l^* \in K(S)$ and for every $l \in K(S)$ the sequence $(S^n l), n \in \mathbb{N}$ converges to $l*$.

Remark :

Every F-contraction γ is a contractive map. (i.e) $\|\gamma_l \ominus \gamma_n\| \leq \|l \ominus n\|, \forall l, n \in K(S)$ and $\gamma_l \neq \gamma_n$. Thus every F-contraction is continuous map.

3.2 Near Fixed Point Theorems in BHS endowed with a graph

We provide fixed point theorems for mappings in Banach hyperspace endowed with a graph by utilising the idea of F-contraction [3][4][5][8][13][19][21]. A particular kind of contraction mapping that is defined in the context of the investigation is referred to as a F-contraction. Regarding a specific set or function class indicated by F, it suggests a contractive feature. A measure of the "nearness" between the pictures of distinct locations is provided by the contraction property that the mappings under consideration under F-contractions display inside the designated function class. A graph is present in the Banach hyperspace, indicating a visual depiction of connections among compact sets. This incorporation of graph theory into the context of Banach hyperspace probably offers a more illustrative and possibly enlightening viewpoint [6].

THEOREM 3.2.1:

Suppose $(K(S), d, H)$ be a BHS with a weakly connected and directed graph H holds the following property, for any sequence $\{S_n\}_{n=1}^{\infty} \subset K(X)$ with $S_n \to S$ as $n \to \infty$ and $(S_n, S_{n+1}) \in E(H), \forall n \in \mathbb{N}$, there exist a subsequence $\{S_{\setminus}\}_{n=1}^{\infty}$, satisfying $(S_{m_n}, S) \in E(G)), \forall n \in \mathbb{N}$

Let $S : K(S) \to K(S)$ be a H F-contraction, if the set $S_\gamma = \{S \in K(S); (S, \gamma \gamma_l) \in E(H)\}$ is non-empty, then S has a unique near fixed point in $K(S)$.

Proof :

Let $Sy_0 \in S_\gamma$, therefore $(Sy_0, \gamma Sy_0) \in E(H)$, we get $(\gamma^n Sy_0, \gamma^{n+1} Sy_0) \in E(H), \forall n \in \mathbb{N}$

Denote $S_n = \gamma^n Sy_0, \forall n \in \mathbb{N}$. By the fact that γ is a H F-contraction and using selfF-contraction case, we get,

$$F\|S_n \ominus S_{n+1}\| \le F\|S_{n-1}, S_n\| - \tau, \forall\, n \in \mathbb{N}$$

Denote $\beta_n = \|S_n \ominus S_{n+1}\|, n = 0, 1, 2, \dots\dots$ Take $S_{n+1} \ne S_n, \forall n \in \mathbb{N} \cup 0$.

Then $\beta_n > 0, \forall n \in \mathbb{N} \cup 0$ and by using the known result, we get

$$F(\beta_n) = F(\beta_{n-1}) \ominus \tau$$
$$\le F(\beta_{n-2}) - 2\tau \le \dots\dots \le F(\beta_0) - n\tau$$
$$\therefore \lim_{n \to \infty} F(\beta_n) = -\infty, \text{ obtain } \beta_n \to 0 \text{ as } n \to \infty$$

There exist $m \in (0, 1)$ such that $\lim_{n \to \infty} \beta_n^k F(\beta_n) = 0$.

$$\beta_n^k\, F(\beta_n) \ominus \beta_n^k\, F(\beta_0) \le \beta_n^k(F(\beta_0) \ominus n\tau)$$
$$\ominus \beta_n^k F(\beta_0) = -\beta_n^k\, n\tau$$

holds for all $n \in \mathbb{N}$. Take $n \to \infty, \lim_{n \to \infty} n\beta_n^k = 0$.

Observe that there exist $n' \in \mathbb{N}$ such that $n\beta_n^k \le 1, \forall n \ge n'$ we have,

$$\beta_n \le \frac{1}{n^{1/k}} \forall n \ge n'$$

Choose $l, n \in \mathsf{N}$, such that $l \geq n \geq n'$, we get

$$||\mathsf{S}_l \ominus \mathsf{S}_n|| \leq \beta_{l\ 1} + \ldots \beta_n \overset{\infty}{\underset{j=n}{X}} \beta_n \leq \overset{\infty}{\underset{j=n}{X}} \frac{1}{j^{1/k}}$$

The convergence of the above series that $\{\mathsf{S}_n\}$ is a cauchy sequence, it is convergent in $(K(S), d, H)$.

$$\therefore \lim_{n \to \infty} \mathsf{S}_n = \mathsf{S}*$$

The subsequence $\{\mathsf{S}_{mn}\}$ satisfying $(\mathsf{S}_{mn}, \mathsf{S}*) \in E(H)$, $\forall n \in \mathsf{N}$, we get

$$\mathsf{F}||\gamma\mathsf{S}_{mn} \ominus \gamma\mathsf{S}*|| \quad \leq \quad \mathsf{F}||\mathsf{S}_{mn} \ominus \mathsf{S}*|| - \tau$$

$$< \quad \mathsf{F}||\gamma_{mn} \ominus \mathsf{S}*||$$

$$||\gamma\mathsf{S}_{mn} \ominus \gamma\mathsf{S}*|| \quad \leq \quad ||\mathsf{S}_{mn} \ominus \mathsf{S}*||$$

By triangle inequality, we have

$$||\mathsf{S}* \ominus \gamma\mathsf{S}*|| = ||\mathsf{S}* \ominus \mathsf{S}_{mn}|| \oplus ||\gamma\mathsf{S}_{mn} \ominus \gamma\mathsf{S}*||, \forall\ n \geq 1$$

assuming $n \to \infty$ and using the above results we get, $||\mathsf{S}*, \gamma\mathsf{S}*|| = 0$.

$\Rightarrow \mathsf{S}* = \gamma\mathsf{S}*$

$\Rightarrow \mathsf{S}*$ is a fixed point of γ we can extend this to non-Self-F-contraction also.

IV   CONCLUSION

The detailed research seems to present a novel way of conceptualising Banach hyperspace through graph associations. Furthermore, the development of the w-sequence is discussed, which quantifies the edge intensities in the graph. It appears that this sequence is essential to showing how a set of iterated functions converges to a Cauchy sequence. Moreover, the approach is said to be useful in illustrating different contraction concepts. Below is a summary of the essential components: Banach Hyperspace Graphs: Graphs are associated with the Banach hyperspace, indicating that the structure and connections among compact sets are being portrayed visually. The inquiry may revolve around the nature of these graphs and how they represent the Banach hyperspace.

w-**Sequence:** The w-sequence appears to be a novel concept introduced to measure the intensities of the edges in the associated graph. It's likely that the *w*-sequence plays a significant role in quantifying the relationships or properties of the sets in the hyperspace.

Convergence of Iterated Functions: The investigation demonstrates that a series of iterated functions form a Cauchy sequence, and this is done using the *w*-sequence.

This suggests a connection between the properties of the associated graph and the convergence behavior of the iterated functions.

Application to Contraction Principles: The method is claimed to be applicable in demonstrating various contraction principles. This indicates that the insights gained from the w-sequence and the associated graph are useful in proving contraction properties, potentially extending the classical Banach contraction principle.

Overall, the investigation seems to introduce a new perspective by incorporating graph theory concepts, w-sequences, and iterated functions to study the properties of Banach hyperspace. It's an interesting integration of different mathematical ideas to explore the convergence and contraction properties of mappings in this particular setting.

Finding : No external findings were obtained for this study.

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