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Fuzzy Soft Paranormal Operator in Fuzzy Soft Hilbert Space

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Abstract:
This paper defines the fuzzy soft paranormal operator and discusses several fundamental fuzzy soft paranormal operator properties in fuzzy soft hilbert space. Some concepts relevant to the fuzzy soft paranormal operator have been defined in fuzzy soft Hilbert space.

Keywords: Fuzzy soft normal operator, fuzzy soft Hilbert space, fuzzy soft hyponormal operator, fuzzy soft paranormal operator

I INTRODUCTION

More than a century ago, the field of functional analysis was established to address a number of problems in pure mathematics. In addition to regularly presenting us with uncertainty, the phenomena under study's ambiguity also provides us with instruments for assessing faults in solutions to issues with both infinite and limited dimensions. In a variety of fields, including engineering, business, medicine, and economics, this kind of problem might be encountered. Our conventional mathematical methods frequently fall short in addressing such problems. Thus, L. Zadeh[3] provided an extension of set theory in 1965. Fuzzy set theory was the term given to the resulting theory. Fuzzy set theory quickly established itself as an effective method for dealing with ambiguous circumstances. The basis function from a set x to a set [0,1] defines the set x in classical set theory. In contrast, a set in fuzzy set theory is described by its membership function, which ranges from x to the closed range between 0 and 1.

In 1999, Molodtsov[4] also developed a fresh generalisation for dealing with uncertainty. Soft set theory was created as a result of this research. Since then, it has been applied to tackle difficult issues in a number of fields, including computer science, engineering, medicine, and others. A soft set is a collection of universal sets that has been parametrized. Soft set gave rise to the ideas of soft point, soft normed space, soft inner product space, and soft Hilbert space, which were later applied in functional analysis to tackle a number of different mathematical topics.

The concept of a fuzzy soft set was initially introduced in 2001 by Maji[5] et al. The idea was created by using a soft set and a fuzzy set. To provide more precise and thorough findings, it was necessary to merge the two concepts. Fuzzy soft point[6] and fuzzy soft normed space[7] were created as a result of the framework's expansion to include these new concepts. Faried

[10]et al. presented fuzzy soft Hilbert spaces in 2020. The fuzzy soft linear operators are also included. We introduce a brand-new class of fuzzy soft paranormal operator and establish a number of associated theorems in this article.

II PRELIMINARIES

This section serves as a preface to the topic that follows by providing specific notations, definitions, and preliminaries for fuzzy set, soft set, and fuzzy soft set.

Definition 2.1: [3] Fuzzy set

Let Ʈ be a universal set. A fuzzy set $\breve{Ⱥ}$ over Ʈ is a set characterized by a function $\eta_{\breve{Ⱥ}}: Ʈ \rightarrow [0,1]$. $\eta_{\breve{Ⱥ}}$ is called the membership, characteristic or indicator function of the fuzzy set $\breve{Ⱥ}$ and the value $\eta_{\breve{Ⱥ}}(\mathfrak{x})$ is termed the grade of membership of $\mathfrak{x} \in Ʈ$ in $\breve{Ⱥ}$.

Definition 2.2: [4, 10] Soft set

Assume that $\mathcal{P}(Ʈ)$ the power set of Ʈ and E be the collection of parameters and $\subseteq E$. The mapping $ɠ: \breve{Ⱥ} \rightarrow ƥ(Ʈ)$, where $(ɠ, \breve{Ⱥ}) = \{ɠ(l) \epsilon\, ƥ(Ʈ): l \in \breve{Ⱥ}\}$. As a result $(ɠ, \breve{Ⱥ})$ is called the soft set.

Definition 2.3: [5] Fuzzy soft set

LetƮ be a universal set, E be a set of parameters and $\breve{Ⱥ} \subseteq$ E. A pair $(ɠ, Ⱥ)$ is called a fuzzy soft set over Ʈ, where ɠ is a mapping given by $ɠ: Ⱥ \rightarrow \mathscr{F}(Ʈ)$, $\mathscr{F}(Ʈ)$ is the family of all fuzzy subsets of Ʈ and the fuzzy subset of Ʈ is defined as a map η from Ʈ to [0,1]. The family of all fuzzy soft sets $(ɠ, Ⱥ)$ over a universal set Ʈ, in which all the parameter sets $\breve{Ⱥ}$ are the same, is denoted by $FSS(Ʈ)_{\breve{Ⱥ}} = FSS(Ʈ)$

Definition 2.4: [9] Fuzzy soft Hilbert space

A fuzzy soft inner product space is defined as$\left(\widetilde{Ʈ}, \widetilde{\langle .,. \rangle}\right)$. This space, which is fuzzy soft complete in the induced fuzzy soft normed space called as a fuzzy soft Hilbert space and denoted by $\left(\tilde{\hbar}, \widetilde{\langle .,. \rangle}\right)$. Every fuzzy soft Hilbert space is obviously a fuzzy soft Banach space.

Definition 2.5: [2] Fuzzy soft linear operator in $\tilde{\hbar}$

Consider $\tilde{\hbar}$ to be a fuzzy soft Hilbert space. A fuzzy soft linear operator $\tilde{Ŧ}: \tilde{\hbar} \rightarrow \tilde{\hbar}$ is called a fuzzy soft linear operator in $\tilde{\hbar}$, then $\tilde{Ŧ}$ is a fuzzy soft linear operator on $\tilde{\hbar}$ which is denoted as $\tilde{Ŧ} \tilde{\in} \mathbb{L}(\tilde{\hbar})$. $\tilde{Ŧ}$ is fuzzy soft bounded if there exists $\tilde{\kappa} \tilde{\in} \Re(Ⱥ)$: $\left\|\tilde{Ŧ}\left(\widetilde{\tilde{l}_{\eta_{ɠ(e)}}}\right)\right\| \tilde{\leqq} \tilde{\kappa} \left\|\tilde{l}_{\eta_{ɠ(e)}}\right\| \quad \forall\, \tilde{l}_{\eta_{ɠ(e)}} \tilde{\in} \tilde{\hbar}$, then $\tilde{Ŧ} \tilde{\in} \tilde{\mathfrak{B}}(\tilde{\hbar})$

Definition 2.6: [2] Fuzzy soft adjoint operator in $\tilde{\hbar}$

The fuzzy soft adjoint operator $\tilde{Ŧ}^{\tilde{*}}$of a fuzzy soft linear operator $\tilde{Ŧ}$ is defined by $\langle \widetilde{\tilde{Ŧ}\tilde{l}^1{}_{\eta_{1_{ɠ(e_1)}}}, \tilde{l}^2{}_{\eta_{2_{ɠ(2)}}}} \rangle \cong \langle \widetilde{\tilde{l}^1{}_{\eta_{1_{ɠ(e_1)}}}, \tilde{Ŧ}^{\tilde{*}}\tilde{l}^2{}_{\eta_{2_{ɠ(2)}}}} \rangle$ for all $\tilde{l}^1{}_{\eta_{1_{ɠ(e_1)}}}, \tilde{l}^2{}_{\eta_{2_{ɠ(e_2)}}} \tilde{\in} \tilde{\hbar}$

Definition 2.7:[11] Fuzzy soft Normal Operator

Let $\tilde{\hbar}$ be an FS Hilbert space and $\tilde{Ŧ} \tilde{\in} \tilde{\mathfrak{B}}(\tilde{\hbar})$. Then, $\tilde{Ŧ}$ is said to be an FS normal operator if $\tilde{Ŧ}\tilde{Ŧ}^{*} \cong \tilde{Ŧ}^{*}\tilde{Ŧ}$

Definition 2.8: [11] Fuzzy soft self adjoint operator

The FS-operator $\tilde{Ŧ}$ of FSH-space $\tilde{\hbar}$ is called fuzzy soft self adjoint (FS-self adjoint operator) if $\tilde{Ŧ} \cong \tilde{Ŧ}^{*}$

Definition 2.9: [14] Fuzzy soft isometry operator

Let $\tilde{\hbar}$ be an FS Hilbert space and $\tilde{Ŧ} \tilde{\in} \tilde{\mathfrak{B}}(\tilde{\hbar})$. Then, $\tilde{Ŧ}$ is said to be an FS isometry operator if $\langle \widetilde{\tilde{Ŧ}\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}}, \tilde{Ŧ}\tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}} \rangle \cong \langle \widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}}, \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(2)}}}} \rangle$ for all $\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}}, \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}} \tilde{\in} \tilde{\hbar}$

Definition 2.10: [13] Fuzzy soft projection operator

Consider $\tilde{\hbar}$ to be a fuzzy soft Hilbert space. A fuzzy soft linear operator $\tilde{Ŧ}: \tilde{\hbar} \to \tilde{\hbar}$ is called a fuzzy soft projection operator in $\tilde{\hbar}$ if $\tilde{Ŧ}^2 \cong \tilde{Ŧ}$ ie, $\tilde{Ŧ}$ is an idempotent.

Definition 2.11: [15] Fuzzy soft hyponormal operator

Consider $\tilde{\hbar}$ to be a fuzzy soft Hilbert space. $\tilde{Ŧ} \in \tilde{\mathfrak{B}}(\tilde{\hbar})$is called fuzzy soft hyponormal operator if $\left\|\tilde{Ŧ}^{*}\tilde{l}_{\eta_{\mathfrak{g}(e)}}\right\| \leq \left\|\tilde{Ŧ}\tilde{l}_{\eta_{\mathfrak{g}(e)}}\right\|$ for all $\tilde{l}_{\eta_{\mathfrak{g}(e)}} \tilde{\in} \tilde{\hbar}$ or equivalently $\tilde{Ŧ}^{*}\tilde{Ŧ} \geq \tilde{Ŧ}\tilde{Ŧ}^{*}$

Definition 2.12: [16] M-Fuzzy soft hyponormal operator

Let $\tilde{\hbar}$ be an FS Hilbert space and let $\tilde{Ŧ} \in \tilde{\mathfrak{B}}(\tilde{\hbar})$ is called M – fuzzy soft hyponormal operator if there exist a real number $\mathcal{M}$, such that $\left\|\widetilde{(\tilde{Ŧ} \tilde{-} \tilde{ƺ}\tilde{I})}^{*}\tilde{l}_{\eta_{\mathfrak{g}(e)}}\right\| \tilde{\leq} \tilde{\mathcal{M}} \left\|\widetilde{(\tilde{Ŧ} \tilde{-} \tilde{ƺ}\tilde{I})}\tilde{l}_{\eta_{\mathfrak{g}(e)}}\right\|$
for all $\tilde{l}_{\eta_{\mathfrak{g}(e)}} \tilde{\in} \tilde{\hbar}$ and for all $\tilde{ƺ} \tilde{\in} \tilde{\mathbb{C}}(Ⱥ)$

III Main results

The definition of the fuzzy soft paranormal operator in fuzzy soft Hilbert space is provided in this section.

Definition 3.1: Fuzzy Soft Paranormal Operator (FSPN)

Let $\tilde{\hbar}$ be an FS Hilbert space and let $\tilde{Ţ} \in \tilde{\mathfrak{B}}(\tilde{\hbar})$ then $\tilde{Ţ}$ is a FSPN operator if

$$\left\|\widetilde{\tilde{Ţ}^2\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \left\|\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \tilde{\geq} \left\|\widetilde{\tilde{Ţ}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \text{ for all } \tilde{l}_{\eta_{\mathfrak{g}(e)}} \tilde{\in} \tilde{\hbar}$$

Note:

An operator $\tilde{Ţ} \in \tilde{\mathfrak{B}}(\tilde{\hbar})$ and $\tilde{\hbar}$ be a FSHS then $\tilde{Ţ}$ is said to be an FSPN operator if $\left\|\widetilde{\tilde{Ţ}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \tilde{\leq} \left\|\widetilde{\tilde{Ţ}^2\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$, for every unit vector $\tilde{l}_{\eta_{\mathfrak{g}(e)}}$ in $\tilde{\hbar}$.

Remark:

Let $\widetilde{\mathfrak{T}} \in \widetilde{\mathfrak{B}}(\tilde{\hbar}), \tilde{\hbar} \cong l^2(\tilde{\mathcal{A}})$

ie) $l^2(\tilde{\mathcal{A}}) \cong \left\{\tilde{l}_{\eta_{\mathbb{G}(e)}} \cong \left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}}, \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}} \dots\right) : \sum_{i=1}^{\infty} \left|\widetilde{l_{\iota_{\eta_{\iota_{\mathfrak{g}(e_\iota)}}}}}\right|^2 < \infty,\ l_{\iota_{\eta_{\iota_{\mathfrak{g}(e_\iota)}}}} \widetilde{\in \mathcal{C}^n}(\mathcal{A})\right\}$

for $\tilde{l}_{\eta_{\mathfrak{g}(e)}} \mathrel{\tilde{\in}} l^2(\tilde{\mathcal{A}})$, defined

$$\left\|\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \cong \langle \tilde{l}_{\eta_{\mathfrak{g}(e)}}, \widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}} \rangle^{1/2} \cong \left(\sum_{\iota=1}^{\infty} \left|\widetilde{l_{\iota_{\eta_{\iota_{\mathfrak{g}(e_\iota)}}}}}\right|^2\right)^{1/2}$$

Let $\widetilde{\mathfrak{T}} : \tilde{\hbar} \to \tilde{\hbar}$ defined by $\widetilde{\mathfrak{T}}\left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}}, \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}} \dots\right) \cong \left(\theta,\ \widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}}, \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}} \dots\right)$
$\forall \left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}}, \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}} \dots\right) \mathrel{\tilde{\in}} l^2(\tilde{\mathcal{A}})$

a) To find $\widetilde{\mathfrak{T}}$ is linear

Take $\tilde{l}_{\eta_{\mathfrak{g}(e)}} \cong \left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}}, \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}} \dots\right)$

$\tilde{m}_{\gamma_{\mathfrak{g}(a)}} \cong \left(\widetilde{\tilde{m}^1{}_{\gamma_{1_{\mathfrak{g}(a_1)}}}, m^2{}_{\gamma_{2_{\mathfrak{g}(a_2)}}}} \dots\right) \mathrel{\tilde{\in}} l^2(\tilde{\mathcal{A}})$

$$\begin{aligned}
\widetilde{\mathfrak{T}}\left(\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}} + \tilde{m}_{\gamma_{\mathfrak{g}(a)}}}\right) &\cong \widetilde{\mathfrak{T}}\left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}} + \tilde{m}^1{}_{\gamma_{1_{\mathfrak{g}(a_1)}}}, \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}} + m^2{}_{\gamma_{2_{\mathfrak{g}(a_2)}}}}, \dots\right) \\
&\cong \left(\theta,\ \widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}} + \tilde{m}^1{}_{\gamma_{1_{\mathfrak{g}(a_1)}}}, \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}} + m^2{}_{\gamma_{2_{\mathfrak{g}(a_2)}}}}, \dots\right) \\
&\cong \left(\theta,\ \widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}}, \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}} \dots\right) \mathbin{\tilde{+}} \left(\widetilde{\tilde{m}^1{}_{\gamma_{1_{\mathfrak{g}(a_1)}}}, m^2{}_{\gamma_{2_{\mathfrak{g}(a_2)}}}} \dots\right)
\end{aligned}$$

$$\widetilde{\mathfrak{T}}\left(\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}} + \tilde{m}_{\gamma_{\mathfrak{g}(a)}}}\right) \cong \widetilde{\mathfrak{T}}\left(\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right) \mathbin{\tilde{+}} \widetilde{\mathfrak{T}}\left(\widetilde{\tilde{m}_{\gamma_{\mathfrak{g}(a)}}}\right)$$

$$\begin{aligned}
\widetilde{\mathfrak{T}}\left(\widetilde{\alpha\, \tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right) &\cong \left(\theta,\ \widetilde{\alpha \tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}}, \tilde{\alpha} l^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}} \dots\right) \\
&\cong \tilde{\alpha}\left(\theta,\ \widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}}, \tilde{l}^2{}_{\eta_{2_{v(e_2)}}}} \dots\right) \\
&\cong \tilde{\alpha}\, \widetilde{\mathfrak{T}}\left(\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right)
\end{aligned}$$

b) To find $\widetilde{\mathfrak{T}}$ is finite

Take $\left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}}, \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}} \dots\right) \mathrel{\tilde{\in}} l^2(\tilde{\mathcal{A}})$

$$\begin{aligned}
\left\|\widetilde{\mathfrak{T}}\left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}}, \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}} \dots\right)\right\|^2 &\cong \left\|\left(\theta,\ \widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}}, \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}} \dots\right)\right\|^2 \\
&\cong \sum_{i=1}^{\infty} \left|\widetilde{l_{\iota_{\eta_{\iota_{\mathfrak{g}(e_\iota)}}}}}\right|^2
\end{aligned}$$

$$\cong \left\|\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2$$

ie) $\left\|\tilde{\mathfrak{T}}\left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right)\right\|^2 \cong \left\|\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2$

$\left\|\widetilde{\tilde{\mathfrak{T}} l_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \cong \left\|\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2$ iff $\left\|\widetilde{\tilde{\mathfrak{T}} l_{\eta_{\mathfrak{g}(e)}}}\right\| \cong \left\|\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$ which implies $\tilde{\mathfrak{T}}$ is finite

Therefore, $\tilde{\mathfrak{T}} \in \tilde{\mathfrak{B}}(\tilde{\hbar})$

c) To find $\tilde{\mathfrak{T}}$ is FSPN

Take $\left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right) \tilde{\in}\ l^2(\tilde{\mathcal{A}})$

$$\left\|\tilde{\mathfrak{T}}\left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right)\right\|^2 \cong \left\|\left(\theta,\ \widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right)\right\|^2$$

$$\cong \sum_{i=1}^{\infty}\left|\widetilde{l_{\iota_{\eta_{\iota_{\mathfrak{g}(e_\iota)}}}}}\right|^2$$

$$\cong \left\|\left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right)\right\|^2$$

$$\Leftrightarrow \left\|\tilde{\mathfrak{T}}\left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right)\right\| \cong \left\|\left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right)\right\|$$

d) Take $\left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right) \tilde{\in}\ l^2(\tilde{\mathcal{A}})$

$$\left\|\tilde{\mathfrak{T}}\left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right)\right\| \cong \left\|\left(\theta,\ \widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right)\right\|$$

Let $\tilde{\mathfrak{T}}^2\left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right) \cong \tilde{\mathfrak{T}}\left(\tilde{\mathfrak{T}}\left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right)\right)$

$$\cong \tilde{\mathfrak{T}}\left(\theta,\ \widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right)$$

$$\tilde{\mathfrak{T}}^2\left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right) \cong \left(\theta,\ \widetilde{\theta\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right)$$

$$\left\|\tilde{\mathfrak{T}}^2\left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right)\right\| \cong \left\|\left(\theta,\ \widetilde{\theta\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right)\right\|$$

$$\left\|\tilde{\mathfrak{T}}^2\left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right)\right\| \cong \sum_{i=1}^{\infty}\left|\widetilde{l_{\iota_{\eta_{\iota_{\mathfrak{g}(e_\iota)}}}}}\right|$$

e) Taken any $\left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right) \tilde{\in}\ l^2(\tilde{\mathcal{A}})$

$$\tilde{\mathfrak{T}}\left(\widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right) \cong \left(\theta,\ \widetilde{\tilde{l}^1{}_{\eta_{1_{\mathfrak{g}(e_1)}}},\ \tilde{l}^2{}_{\eta_{2_{\mathfrak{g}(e_2)}}}}\ \ldots\right)$$

$$\left\|\widetilde{\mathfrak{T}}\left(\widetilde{\tilde{l}^{1}{}_{\eta_{1_{\mathfrak{g}(e_1)}}}, \tilde{l}^{2}{}_{\eta_{2_{\mathfrak{g}(e_2)}}}} \dots\right)\right\|^{2} \cong \left\|\left(\theta, \widetilde{\tilde{l}^{1}{}_{\eta_{1_{\mathfrak{g}(e_1)}}}, \tilde{l}^{2}{}_{\eta_{2_{\mathfrak{g}(e_2)}}}} \dots\right)\right\|^{2}$$
$$\cong \sum_{i=1}^{\infty}\left|\widetilde{l_{\iota_{\eta_{\iota_{\mathfrak{g}(e_\iota)}}}}}\right|^{2}$$

From d) and e), we get

$$\left\|\widetilde{\mathfrak{T}}(\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}})\right\|^{2} \lesseqqgtr \left\|\widetilde{\mathfrak{T}}^{2}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$$

Therefore, $\widetilde{\mathfrak{T}}$ is FSPN operator

Theorem 3.3:

$\left\|\widetilde{\mathfrak{T}}^{3}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \gtreqqless \left\|\widetilde{\mathfrak{T}}^{2}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \left\|\widetilde{\mathfrak{T}}\,\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$ for every unit vector $\tilde{l}_{\eta_{\mathfrak{g}(e)}}$ in $\tilde{\hbar}$

Proof:

For every unit vector $\tilde{l}_{\eta_{\mathfrak{g}(e)}} \;\widetilde{\in}\; \tilde{\hbar}$

Let $\left\|\widetilde{\mathfrak{T}}^{3}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^{2} \cong \langle \widetilde{\widetilde{\mathfrak{T}}^{3}\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \widetilde{\mathfrak{T}}^{3}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\rangle$

$$\cong \langle \widetilde{\widetilde{\mathfrak{T}}\,\widetilde{\mathfrak{T}}^{2}\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \widetilde{\mathfrak{T}}\widetilde{\mathfrak{T}}^{2}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}\rangle$$
$$\cong \langle \widetilde{\mathfrak{T}}^{*}\widetilde{\mathfrak{T}}\,\widetilde{\mathfrak{T}}^{2}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}, \widetilde{\mathfrak{T}}^{2}\tilde{l}_{\eta_{\mathfrak{g}(e)}}\rangle$$
$$\cong \langle \widetilde{\mathfrak{T}}^{2}\,\widetilde{\mathfrak{T}}^{2}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}, \widetilde{\mathfrak{T}}^{2}\tilde{l}_{\eta_{\mathfrak{g}(e)}}\rangle$$
$$\cong \langle \widetilde{\widetilde{\mathfrak{T}}^{4}\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \widetilde{\mathfrak{T}}}^{2}\tilde{l}_{\eta_{\mathfrak{g}(e)}}\rangle$$
$$\lesseqqgtr \left\|\widetilde{\mathfrak{T}}^{4}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \left\|\widetilde{\mathfrak{T}}^{2}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$$

$\left\|\widetilde{\mathfrak{T}}^{3}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^{2} \gtreqqless \left\|\widetilde{\mathfrak{T}}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^{4} \left\|\widetilde{\mathfrak{T}}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^{2}$ (Since $\widetilde{\mathfrak{T}}$ is FSPN operator)

$$\Rightarrow \left\|\widetilde{\mathfrak{T}}^{3}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \gtreqqless \left\|\widetilde{\mathfrak{T}}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^{2} \left\|\widetilde{\mathfrak{T}}\,\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$$

Hence $\left\|\widetilde{\mathfrak{T}}^{3}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \gtreqqless \left\|\widetilde{\mathfrak{T}}^{2}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \left\|\widetilde{\mathfrak{T}}\,\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$

Theorem 3.4:

Let $\tilde{\hbar}$ be a FS Hilbert space and let $\widetilde{\mathfrak{T}} \in \widetilde{\mathfrak{B}}(\tilde{\hbar})$ be a FSPN operator. Then

$\left\|\widetilde{\mathfrak{T}}^{k+1}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^{2} \gtreqqless \left\|\widetilde{\mathfrak{T}}^{k}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^{2} \left\|\widetilde{\mathfrak{T}}^{2}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$ for every positive integer $k \geq 1$ and for every unit vector $\tilde{l}_{\eta_{\mathfrak{g}(e)}}$ in $\tilde{\hbar}$.

Proof:

Let $\widetilde{\mathfrak{T}} \in \widetilde{\mathfrak{B}}(\tilde{\hbar})$ be a FSPN operator

By using the induction hypothesis, we will prove the theorem. For the case $k = 1$,

$$\left\|\widetilde{\widetilde{T}^2\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \gtrapprox \left\|\widetilde{\widetilde{T}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \left\|\widetilde{\widetilde{T}^2\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$$

Now suppose that $\left\|\widetilde{\widetilde{T}^{k+1}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \gtrapprox \left\|\widetilde{\widetilde{T}^{k}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \left\|\widetilde{\widetilde{T}^2\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$ is valid for k.

Then $k = k + 1$

$$\left\|\widetilde{\widetilde{T}^{k+2}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \cong \langle \widetilde{\widetilde{T}^{k+2}\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \widetilde{T}^{k+2}}\tilde{l}_{\eta_{\mathfrak{g}(e)}} \rangle$$

$$\cong \langle \left(\widetilde{T}^{k}\right)^{\tilde{*}} \widetilde{\widetilde{T}^{k+2}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}, \ \widetilde{T}^2\tilde{l}_{\eta_{\mathfrak{g}(e)}} \rangle$$

$$\cong \langle \left(\widetilde{T}^{\tilde{*}}\right)^{k} \widetilde{\widetilde{T}^{k+2}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}, \ \widetilde{T}^2\tilde{l}_{\eta_{\mathfrak{g}(e)}} \rangle$$

$$\cong \langle \widetilde{\widetilde{T}^{2k+2}\tilde{l}_{\eta_{\mathfrak{g}(e)}},} \ \widetilde{T}^2\tilde{l}_{\eta_{\mathfrak{g}(e)}} \rangle$$

$$\cong \langle \widetilde{\widetilde{T}^{2(k+1)}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}, \ \widetilde{T}^2\tilde{l}_{\eta_{\mathfrak{g}(e)}} \rangle$$

$$\lessapprox \left\|\widetilde{\widetilde{T}^{2(k+1)}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \left\|\widetilde{\widetilde{T}^2\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$$

Since $\left\|\widetilde{T^2\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \gtrapprox \left\|\widetilde{\widetilde{T}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \left\|\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \quad \forall\, \tilde{l}_{\eta_{\mathfrak{g}(e)}} \ \tilde{\in} \ \tilde{\hbar}$,

$$\left\|\widetilde{\widetilde{T}^{k+2}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \gtrapprox \left\|\widetilde{\widetilde{T}^{k+1}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \left\|\widetilde{T}^2\tilde{l}_{\eta_{\mathfrak{g}(e)}}\right\|$$

So $k = k + 1$ is valid and the proof is complete by the mathematical induction.

Lemma 3.5:

Let $\widetilde{T} \in \widetilde{\mathfrak{B}}(\tilde{\hbar})$ be a FSPN operator. Then $\widetilde{T}^n$ is also FSPN for every integer $n \geq 1$

Proof:

It is sufficient to show that if $\widetilde{T}$ and $\widetilde{T}^k$ is a FSPN then $\widetilde{T}^{k+1}$ is also FSPN

For every unit vector $\tilde{l}_{\eta_{\mathfrak{g}(e)}}$ in $\tilde{\hbar}$

Let $\left\|\widetilde{\widetilde{T}^{2(k+1)}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \cong \langle \widetilde{\widetilde{T}^{2(k+1)}\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \widetilde{T}^{2(k+1)}}\tilde{l}_{\eta_{\mathfrak{g}(e)}} \rangle$

$$\cong \langle \left(\widetilde{T}^{2(k+1)}\right)^{\tilde{*}} \widetilde{\widetilde{T}^{2(k+1)}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \ \tilde{l}_{\eta_{\mathfrak{g}(e)}} \rangle$$

$$\cong \langle \left(\widetilde{T}^{\tilde{*}}\right)^{2(k+1)} \widetilde{\widetilde{T}^{2(k+1)}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \ \tilde{l}_{\eta_{\mathfrak{g}(e)}} \rangle$$

$$\cong \langle \widetilde{\widetilde{T}^{4k+4}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}, \ \tilde{l}_{\eta_{\mathfrak{g}(e)}} \rangle$$

$$\cong \langle \widetilde{\widetilde{T}^{4(k+1)}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}, \ \tilde{l}_{\eta_{\mathfrak{g}(e)}} \rangle$$

$$\lessapprox \left\|\widetilde{\mathfrak{T}}^{4(k+1)}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \left\|\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$$

$$\lessapprox \left\|\widetilde{\mathfrak{T}}^{2(k+1)}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \left\|\widetilde{\mathfrak{T}}^{2(k+1)}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \left\|\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$$

ie) $\left\|\widetilde{\mathfrak{T}}^{2(k+1)}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \gtrapprox \left\|\widetilde{\mathfrak{T}}^{k+1}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^4 \left\|\tilde{l}_{\eta_{\mathfrak{g}(e)}}\right\|$ implies that

$$\left\|\widetilde{\mathfrak{T}}^{2(k+1)}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \gtrapprox \left\|\widetilde{\mathfrak{T}}^{k+1}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2$$

By the above lemma, so $\widetilde{\mathfrak{T}}^{(k+1)}$ is also FSPN.

Theorem 3.6:

Let $\widetilde{\mathfrak{T}} \in \widetilde{\mathfrak{B}}(\tilde{\hbar})$ is a self-adjoint fuzzy soft operator then $\widetilde{\mathfrak{T}}$ is FSPN.

Proof:

For any $\tilde{l}_{\eta_{\mathbb{G}(e)}}$ in $\tilde{\hbar}$ with $\left\|\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \cong 1$, we know that $\widetilde{\mathfrak{T}}$ is a self-adjoint fuzzy soft operator

ie) $\widetilde{\mathfrak{T}} \cong \widetilde{\mathfrak{T}}^{\tilde{*}}$

Let $\left\|\widetilde{\mathfrak{T}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \cong \langle \widetilde{\mathfrak{T}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \widetilde{\widetilde{\mathfrak{T}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\rangle$

$$\cong \langle \widetilde{\mathfrak{T}}^{\tilde{*}}\widetilde{\mathfrak{T}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\rangle$$

$$\cong \langle \widetilde{\mathfrak{T}}\,\widetilde{\mathfrak{T}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\rangle$$

$$\cong \langle \widetilde{\mathfrak{T}}^2\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\rangle$$

$$\lessapprox \left\|\widetilde{\mathfrak{T}}^2\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \left\|\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$$

$\left\|\widetilde{\mathfrak{T}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \lessapprox \left\|\widetilde{\mathfrak{T}}^2\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \left\|\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$ implies that $\left\|\widetilde{\mathfrak{T}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \lessapprox \left\|\widetilde{\mathfrak{T}}^2\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$

So $\widetilde{\mathfrak{T}}$ is FSPN.

Theorem 3.7:

Let $\widetilde{\mathfrak{T}} \in \widetilde{\mathfrak{B}}(\tilde{\hbar})$ be FSPN and fuzzy soft self adjoint operator then $\widetilde{\mathfrak{T}}^{\tilde{*}}$ is FSPN.

Proof:

For any $\tilde{l}_{\eta_{\mathfrak{g}(e)}} \mathrel{\tilde{\in}} \tilde{\hbar}$, $\left\|\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \cong 1$

Let $\left\|\widetilde{\mathfrak{T}}^{\tilde{*}}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \cong \langle \widetilde{\mathfrak{T}}^{\tilde{*}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \widetilde{\widetilde{\mathfrak{T}}^{\tilde{*}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\rangle$

$$\cong \langle \widetilde{\mathfrak{T}}\widetilde{\mathfrak{T}}^{\tilde{*}}\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}, \tilde{l}_{\eta_{\mathfrak{g}(e)}}\rangle$$

$$\cong \langle (\tilde{T}^{\tilde{*}})^2 \widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \rangle$$

$$\tilde{\leq} \left\| \widetilde{(\tilde{T}^{\tilde{*}})^2 \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \right\| \left\| \widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}} \right\|$$

$\left\| \widetilde{\tilde{T}^{\tilde{*}} \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \right\|^2 \tilde{\leq} \left\| \widetilde{(\tilde{T}^{\tilde{*}})^2 \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \right\| \left\| \widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}} \right\|$ implies that $\left\| \widetilde{\tilde{T}^{\tilde{*}} \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \right\|^2 \tilde{\leq} \left\| \widetilde{(\tilde{T}^{\tilde{*}})^2 \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \right\|$

ie) $\left\| \widetilde{(\tilde{T}^{\tilde{*}})^2 \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \right\| \tilde{\geq} \left\| \widetilde{\tilde{T}^{\tilde{*}} \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \right\|^2$

Therefore, $\tilde{T}$ is FSPN.

Theorem 3.8:

Let $\tilde{S}$ and $\tilde{T} \tilde{\in} \tilde{\hbar}$ is a FSPN operator and fuzzy soft self adjoint operator. Then

a) $\tilde{S} \tilde{+} \tilde{T}$ **b)** $\tilde{S} . \tilde{T}$ are also as FSPN.

Proof:

For every unit vector $\tilde{l}_{\eta_{\mathfrak{g}(e)}} \tilde{\in} \tilde{\hbar}$

We know that $\left\| \widetilde{\tilde{T}^2 \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \right\| \tilde{\geq} \left\| \widetilde{\tilde{T} \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \right\|^2$

$$\left\| \widetilde{\tilde{S}^2 \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \right\| \tilde{\geq} \left\| \widetilde{\tilde{S} \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \right\|^2 \text{ and } \tilde{S} \cong \tilde{S}^{\tilde{*}}, \tilde{T} \cong \tilde{T}^{\tilde{*}}$$

a) To prove that $\tilde{S} \tilde{+} \tilde{T}$ is a FSPN operator

Let $\left\| \widetilde{(\tilde{S} \tilde{+} \tilde{T}) \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \right\|^2 \cong \langle \widetilde{(\tilde{S} \tilde{+} \tilde{T}) \tilde{l}_{\eta_{\mathfrak{g}(e)}}, (\tilde{S} \tilde{+} \tilde{T}) \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \rangle$

$$\cong \langle (\tilde{S} \tilde{+} \tilde{T})^{\tilde{*}} \widetilde{(\tilde{S} \tilde{+} \tilde{T}) \tilde{l}_{\eta_{\mathfrak{g}(e)}}, \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \rangle$$

$$\cong \langle \left(\tilde{S}^{\tilde{*}} \tilde{+} \tilde{T}^{\tilde{*}}\right) \widetilde{(\tilde{S} \tilde{+} \tilde{T}) \tilde{l}_{\eta_{\mathfrak{g}(e)}}, \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \rangle$$

$$\cong \langle (\tilde{S} \tilde{+} \tilde{T}) \widetilde{(\tilde{S} \tilde{+} \tilde{T}) \tilde{l}_{\eta_{\mathfrak{g}(e)}}, \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \rangle$$

$$\tilde{\leq} \left\| \widetilde{(\tilde{S} \tilde{+} \tilde{T})^2 \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \right\| \left\| \widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}} \right\| \text{ implies that}$$

$$\left\| \widetilde{(\tilde{S} \tilde{+} \tilde{T}) \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \right\|^2 \tilde{\leq} \left\| \widetilde{(\tilde{S} \tilde{+} \tilde{T})^2 \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \right\|$$

Therefore, $\tilde{S} \tilde{+} \tilde{T}$ is a FSPN operator.

b) To prove that $\tilde{S} . \tilde{T}$ is a FSPN operator

Let $\left\| \widetilde{(\tilde{S} . \tilde{T}) \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \right\|^2 \cong \langle \widetilde{(\tilde{S} . \tilde{T}) \tilde{l}_{\eta_{\mathfrak{g}(e)}}, (\tilde{S} . \tilde{T}) \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \rangle$

$$\cong \langle (\tilde{S} . \tilde{T})^{\tilde{*}} \widetilde{(\tilde{S} . \tilde{T}) \tilde{l}_{\eta_{\mathfrak{g}(e)}}, \tilde{l}_{\eta_{\mathfrak{g}(e)}}} \rangle$$

$$\cong \langle \left(\tilde{\mathfrak{T}}^{\tilde{*}}\,\tilde{\mathfrak{S}}^{\tilde{*}}\right)\,\widetilde{(\tilde{\mathfrak{S}}.\tilde{\mathfrak{T}})}\,\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \tilde{l}_{\eta_{\mathfrak{g}(e)}}\rangle$$

$$\cong \langle (\tilde{\mathfrak{T}}\,\tilde{\mathfrak{S}})\,\widetilde{(\tilde{\mathfrak{S}}.\tilde{\mathfrak{T}})\,\tilde{l}_{\eta_{\mathfrak{g}(e)}}}, \tilde{l}_{\eta_{\mathfrak{g}(e)}}\rangle$$

$$\cong \langle (\tilde{\mathfrak{S}}.\tilde{\mathfrak{T}})\,\widetilde{(\tilde{\mathfrak{S}}.\tilde{\mathfrak{T}})\,\tilde{l}_{\eta_{\mathfrak{g}(e)}}}, \tilde{l}_{\eta_{\mathfrak{g}(e)}}\rangle$$

$$\lesseqqgtr \left\|\widetilde{(\tilde{\mathfrak{S}}.\tilde{\mathfrak{T}})^2}\tilde{l}_{\eta_{\mathfrak{g}(e)}}\right\|\,\left\|\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$$

$$\left\|\widetilde{(\tilde{\mathfrak{S}}.\tilde{\mathfrak{T}})\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \lesseqqgtr \left\|\widetilde{(\tilde{\mathfrak{S}}.\tilde{\mathfrak{T}})^2}\tilde{l}_{\eta_{\mathfrak{g}(e)}}\right\|\,\left\|\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \text{ implies that}$$

$$\left\|\widetilde{(\tilde{\mathfrak{S}}.\tilde{\mathfrak{T}})\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \lesseqqgtr \left\|\widetilde{(\tilde{\mathfrak{S}}.\tilde{\mathfrak{T}})^2}\tilde{l}_{\eta_{\mathfrak{g}(e)}}\right\|$$

Therefore, $\tilde{\mathfrak{S}}.\tilde{\mathfrak{T}}$ is a FSPN operator.

Theorem 3.9:

Let $\tilde{\mathfrak{T}} \in \tilde{\mathfrak{B}}(\tilde{\hbar})$ is a FSN operator then $\tilde{\mathfrak{T}}$ is a FSPN operator

Proof:

For every unit vector $\tilde{l}_{\eta_{\mathfrak{g}(e)}} \,\tilde{\in}\, \tilde{\hbar}$

Let $\left\|\widetilde{\tilde{\mathfrak{T}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \cong \langle \widetilde{\tilde{\mathfrak{T}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \tilde{\mathfrak{T}}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}\rangle$

$$\cong \langle \tilde{\mathfrak{T}}^{\tilde{*}}\,\widetilde{\tilde{\mathfrak{T}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \tilde{l}}_{\eta_{\mathfrak{g}(e)}}\rangle$$

$$\cong \langle (\tilde{\mathfrak{T}}\tilde{\mathfrak{T}}^{\tilde{*}})\,\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}, \tilde{l}_{\eta_{\mathfrak{g}(e)}}\rangle$$

$$\cong \langle \tilde{\mathfrak{T}}^2\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \tilde{l}}_{\eta_{\mathfrak{g}(e)}}\rangle$$

$$\lesseqqgtr \left\|\widetilde{\tilde{\mathfrak{T}}^2\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|\,\left\|\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$$

ie) $\left\|\widetilde{\tilde{\mathfrak{T}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \lesseqqgtr \left\|\widetilde{\tilde{\mathfrak{T}}^2\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|\,\left\|\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$ implies that $\left\|\widetilde{\tilde{\mathfrak{T}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \lesseqqgtr \left\|\widetilde{\tilde{\mathfrak{T}}^2\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$

Therefore, $\tilde{\mathfrak{T}}$ is FSPN.

Theorem 3.10:

Let $\tilde{\mathfrak{T}} \in \tilde{\mathfrak{B}}(\tilde{\hbar})$ is a FSPN operator and FSHN. Then $\|\tilde{\mathfrak{T}}\| \gtreqqless \|\tilde{\mathfrak{T}}^{\tilde{*}}\|$ is a FSPN operator.

Proof:

For every unit vector $\tilde{l}_{\eta_{\mathfrak{g}(e)}} \,\tilde{\in}\, \tilde{\hbar}$

Let $\left\|\widetilde{\tilde{\mathfrak{T}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \cong \langle \widetilde{\tilde{\mathfrak{T}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \tilde{\mathfrak{T}}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}\rangle$

$$\cong \langle \tilde{\mathfrak{T}}^{\tilde{*}}\,\widetilde{\tilde{\mathfrak{T}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \tilde{l}}_{\eta_{\mathfrak{g}(e)}}\rangle$$

$$\gtreqqless \langle (\tilde{\mathfrak{T}}\tilde{\mathfrak{T}}^{\tilde{*}})\,\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}}, \tilde{l}_{\eta_{\mathfrak{g}(e)}}\rangle$$

$$\gtreqqless \langle \tilde{\mathfrak{T}}^{\tilde{*}}\,\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \tilde{\mathfrak{T}}^{\tilde{*}}}\,\tilde{l}_{\eta_{\mathfrak{g}(e)}}\rangle$$

Since $\left\|\widetilde{\tilde{\mathfrak{T}}^2\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \gtreqqless \left\|\widetilde{\tilde{\mathfrak{T}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2$ and $\tilde{\mathfrak{T}}^{\tilde{*}}\tilde{\mathfrak{T}} - \tilde{\mathfrak{T}}\tilde{\mathfrak{T}}^{\tilde{*}} \gtreqqless \tilde{0} \quad \forall\, \tilde{l}_{\eta_{\mathfrak{g}(e)}} \,\tilde{\in}\, \tilde{\hbar}$

$$\left\|\widetilde{\widetilde{\mathfrak{T}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \geqq \left\|\widetilde{\widetilde{\mathfrak{T}}^{\tilde{*}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2$$

$\Rightarrow \left\|\widetilde{\widetilde{\mathfrak{T}}\,\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\| \geqq \left\|\widetilde{\widetilde{\mathfrak{T}}^{\tilde{*}}\,\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|$ implies that $\|\widetilde{\mathfrak{T}}\,\| \geqq \|\widetilde{\mathfrak{T}}^{\tilde{*}}\|$

Theorem 3.11:

Let $\widetilde{\mathfrak{T}} \in \widetilde{\mathfrak{B}}(\tilde{\hbar}\,)$ is a FSPN operator and $\tilde{\mathfrak{S}}$ is unitarily equivalent to $\widetilde{\mathfrak{T}}$ then $\tilde{\mathfrak{S}}$ is a FSPN.

Proof:

For $\tilde{\mathfrak{S}}$ is unitarily equivalent to $\widetilde{\mathfrak{T}}$, we have $\tilde{\mathfrak{S}} \cong \widetilde{\mathfrak{U}}\widetilde{\mathfrak{T}}\widetilde{\mathfrak{U}}^{\tilde{*}}$

For some unitarily equivalent to $\tilde{\mathfrak{S}}^2 \cong \widetilde{\mathfrak{U}}\widetilde{\mathfrak{T}}^2\widetilde{\mathfrak{U}}^{\tilde{*}} \Rightarrow \left\|\widetilde{\tilde{\mathfrak{S}}^2\tilde{l}_{\eta_{\mathbb{G}(e)}}}\right\| \cong \left\|\widetilde{\mathfrak{U}}\widetilde{\mathfrak{T}}^2\widetilde{\mathfrak{U}}^{\tilde{*}}\,\tilde{l}_{\eta_{\mathbb{G}(e)}}\right\|$

Let $\left\|\widetilde{\tilde{\mathfrak{S}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \cong \left\|\widetilde{(\widetilde{\mathfrak{U}}\widetilde{\mathfrak{T}}\widetilde{\mathfrak{U}}^{\tilde{*}})\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2$

$$\begin{aligned}
\langle\widetilde{\tilde{\mathfrak{S}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \tilde{\mathfrak{S}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\rangle &\cong \langle\widetilde{(\widetilde{\mathfrak{U}}\widetilde{\mathfrak{T}}\widetilde{\mathfrak{U}}^{\tilde{*}})\tilde{l}_{\eta_{\mathfrak{g}(e)}},\ (\widetilde{\mathfrak{U}}\widetilde{\mathfrak{T}}\widetilde{\mathfrak{U}}^{\tilde{*}})\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\rangle \\
&\cong \langle(\widetilde{\mathfrak{T}}\widetilde{\mathfrak{U}}^{\tilde{*}})\tilde{l}_{\eta_{\mathfrak{g}(e)}},\ \widetilde{\widetilde{\mathfrak{U}}^{\tilde{*}}\widetilde{\mathfrak{U}}}\,(\widetilde{\mathfrak{T}}\widetilde{\mathfrak{U}}^{\tilde{*}})\tilde{l}_{\eta_{\mathfrak{g}(e)}}\rangle \\
&\cong \langle\widetilde{(\widetilde{\mathfrak{T}}\widetilde{\mathfrak{U}}^{\tilde{*}})\tilde{l}_{\eta_{\mathfrak{g}(e)}},\ (\widetilde{\mathfrak{T}}\widetilde{\mathfrak{U}}^{\tilde{*}})\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\rangle \qquad \text{[since } \widetilde{\mathfrak{U}} \text{ is FS isometry]} \\
&\cong \langle(\widetilde{\mathfrak{T}}\widetilde{\mathfrak{U}}^{\tilde{*}})^{\tilde{*}}\widetilde{(\widetilde{\mathfrak{T}}\widetilde{\mathfrak{U}}^{\tilde{*}})\tilde{l}_{\eta_{\mathfrak{g}(e)}}},\ \tilde{l}_{\eta_{\mathfrak{g}(e)}}\rangle \\
&\cong \langle\widetilde{\mathfrak{U}}\,\widetilde{\mathfrak{T}}^{\tilde{*}}\widetilde{(\widetilde{\mathfrak{T}}\widetilde{\mathfrak{U}}^{\tilde{*}})\tilde{l}_{\eta_{\mathfrak{g}(e)}}},\ \tilde{l}_{\eta_{\mathfrak{g}(e)}}\rangle \\
&\cong \langle\widetilde{\mathfrak{U}}\widetilde{\mathfrak{T}}^2\widetilde{\mathfrak{U}}^{\tilde{*}}\,\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}},\ \tilde{l}_{\eta_{\mathfrak{g}(e)}}\rangle \\
&\leqq \left\|\widetilde{\mathfrak{U}}\widetilde{\mathfrak{T}}^2\widetilde{\mathfrak{U}}^{\tilde{*}}\,\tilde{l}_{\eta_{\mathfrak{g}(e)}}\right\|\left\|\,\tilde{l}_{\eta_{\mathfrak{g}(e)}}\right\|
\end{aligned}$$

$\left\|\widetilde{\tilde{\mathfrak{S}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \leqq \left\|\widetilde{\mathfrak{U}}\widetilde{\mathfrak{T}}^2\widetilde{\mathfrak{U}}^{\tilde{*}}\,\tilde{l}_{\eta_{\mathfrak{g}(e)}}\right\|\left\|\,\tilde{l}_{\eta_{\mathfrak{g}(e)}}\right\|$ implies that

$$\left\|\widetilde{\tilde{\mathfrak{S}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 \leqq \left\|\tilde{\mathfrak{S}}^2\tilde{l}_{\eta_{\mathfrak{g}(e)}}\right\|\left\|\,\tilde{l}_{\eta_{\mathfrak{g}(e)}}\right\|$$

Hence $\tilde{\mathfrak{S}}$ is a FSPN

Theorem 3.12:

Let $\widetilde{\mathfrak{T}} \in \widetilde{\mathfrak{B}}(\tilde{\hbar}\,)$ is an invertible and FSPN operator. Then $\widetilde{\mathfrak{T}}^{-1}$ is also a FSPN.

Proof:

For every unit vector $\tilde{l}_{\eta_{\mathbb{G}(e)}} \,\tilde{\in}\, \tilde{\hbar}$

$$\begin{aligned}
\text{Let } \left\|\widetilde{\widetilde{\mathfrak{T}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\right\|^2 &\cong \langle\widetilde{\widetilde{\mathfrak{T}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \widetilde{\mathfrak{T}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}}\rangle \\
&\cong \langle\widetilde{\mathfrak{T}}^{\tilde{*}}\,\widetilde{\widetilde{\mathfrak{T}}\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \tilde{l}_{\eta_{\mathfrak{g}(e)}}}\rangle \\
&\cong \langle\widetilde{\mathfrak{T}}\,\widetilde{\mathfrak{T}}^{\tilde{*}}\,\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \tilde{l}_{\eta_{\mathfrak{g}(e)}}}\rangle \\
&\cong \langle\widetilde{\mathfrak{T}}^2\,\widetilde{\tilde{l}_{\eta_{\mathfrak{g}(e)}}, \tilde{l}_{\eta_{\mathfrak{g}(e)}}}\rangle \\
&\leqq \left\|\widetilde{\mathfrak{T}}^2\,\tilde{l}_{\eta_{\mathfrak{g}(e)}}\right\|\left\|\,\tilde{l}_{\eta_{\mathfrak{g}(e)}}\right\|
\end{aligned}$$

$\tilde{l}_{\eta_{\mathfrak{g}(e)}}$ is replaced by $\widetilde{\mathfrak{T}}^{-2}\,\tilde{l}_{\eta_{\mathfrak{g}(e)}}$

$\left\|\widetilde{T}\widetilde{\widetilde{T}^{-2}\,\tilde{l}_{\eta_{g(e)}}}\right\|^2 \lesseqq \left\|\widetilde{T}^2\,\widetilde{T}^{-2}\,\tilde{l}_{\eta_{g(e)}}\right\|\left\|\widetilde{T}^{-2}\,\tilde{l}_{\eta_{g(e)}}\right\|$

$\left\|\widetilde{T}^{-1}\widetilde{\tilde{l}_{\eta_{g(e)}}}\right\|^2 \lesseqq \left\|\tilde{l}_{\eta_{g(e)}}\right\|\left\|\widetilde{T}^{-2}\,\tilde{l}_{\eta_{g(e)}}\right\|$

$\left\|\widetilde{T}^{-1}\widetilde{\tilde{l}_{\eta_{g(e)}}}\right\|^2 \lesseqq \left\|\widetilde{T}^{-2}\,\tilde{l}_{\eta_{g(e)}}\right\|\left\|\tilde{l}_{\eta_{g(e)}}\right\|$ implies that

$\left\|\widetilde{T}^{-2}\,\tilde{l}_{\eta_{g(e)}}\right\|\left\|\tilde{l}_{\eta_{g(e)}}\right\| \gtreqqless \left\|\widetilde{T}^{-1}\widetilde{\tilde{l}_{\eta_{g(e)}}}\right\|^2$

ie) $\left\|\left(\widetilde{T}^{-1}\right)^2\tilde{l}_{\eta_{g(e)}}\right\|\left\|\tilde{l}_{\eta_{g(e)}}\right\| \geqq \left\|\widetilde{T}^{-1}\widetilde{\tilde{l}_{\eta_{g(e)}}}\right\|^2$

Hence $\widetilde{T}^{-1}$ is also a FSPN

Theorem 3.13:
If $\widetilde{T}^{\tilde{*}\mathbf{2}}\widetilde{T}^2 \geqq \left(\widetilde{T}^{\tilde{*}}\widetilde{T}\right)^2$, then $\widetilde{T}$ is FSPN operator

Proof:

For every unit vector $\tilde{l}_{\eta_{g(e)}} \tilde{\in} \tilde{\hbar}$

Let $\widetilde{T}^{\tilde{*}\mathbf{2}}\widetilde{T}^2 \geqq \left(\widetilde{T}^{\tilde{*}}\widetilde{T}\right)^2$

$\widetilde{T}^{\tilde{*}\mathbf{2}}\widetilde{T}^2 - \left(\widetilde{T}^{\tilde{*}}\widetilde{T}\right)^2 \geqq \tilde{0}$

$\langle\left(\widetilde{T}^{\tilde{*}\mathbf{2}}\widetilde{T}^2 \;\widetilde{-}\; \left(\widetilde{T}^{\tilde{*}}\widetilde{T}\right)^2\right)\tilde{l}_{\eta_{g(e)}}, \tilde{l}_{\eta_{g(e)}}\rangle \geqq \tilde{0}$

$\langle(\widetilde{T}^{\tilde{*}\mathbf{2}}\widetilde{T}^2)\,\tilde{l}_{\eta_{g(e)}}, \tilde{l}_{\eta_{g(e)}}\rangle \;\widetilde{-}\; \langle\left(\left(\widetilde{T}^{\tilde{*}}\widetilde{T}\right)^2\right)\tilde{l}_{\eta_{g(e)}}, \tilde{l}_{\eta_{g(e)}}\rangle \geqq \tilde{0}$

$\langle(\widetilde{T}^{\tilde{*}\mathbf{2}}\widetilde{T}^2)\,\tilde{l}_{\eta_{g(e)}}, \tilde{l}_{\eta_{g(e)}}\rangle \;\widetilde{\geqq}\; \langle\left(\left(\widetilde{T}^{\tilde{*}}\widetilde{T}\right)^2\right)\tilde{l}_{\eta_{g(e)}}, \tilde{l}_{\eta_{g(e)}}\rangle$

$\langle\widetilde{T}^2\,\tilde{l}_{\eta_{g(e)}}, \widetilde{T}^2\tilde{l}_{\eta_{g(e)}}\rangle \;\widetilde{\geqq}\; \langle\widetilde{T}^{\tilde{*}}\widetilde{T}\tilde{l}_{\eta_{g(e)}}, \widetilde{T}^{\tilde{*}}\widetilde{T}\,\tilde{l}_{\eta_{g(e)}}\rangle$ since $\left\|\widetilde{\widetilde{T}^{\tilde{*}}\widetilde{T}}\right\| \cong \left\|\widetilde{T}\right\|^2$

$\left\|\widetilde{T}^2\tilde{l}_{\eta_{g(e)}}\right\|^2 \geqq \left\|\widetilde{T}\widetilde{\tilde{l}_{\eta_{g(e)}}}\right\|^4 \Rightarrow \left\|\widetilde{T}^2\tilde{l}_{\eta_{g(e)}}\right\| \geqq \left\|\widetilde{T}\widetilde{\tilde{l}_{\eta_{g(e)}}}\right\|^2$

Hence $\widetilde{T}$ is FSPN operator

Theorem 3.14:
Let $\widetilde{T} \in \widetilde{\mathfrak{B}}(\tilde{\hbar})$ is a FSN then $\widetilde{T}^{\tilde{*}}$ is a FSPN

Proof:

Since $\widetilde{T}$ is fuzzy soft normal operator

We know that $\widetilde{T}^{\tilde{*}}\widetilde{T} \cong \widetilde{T}\,\widetilde{T}^{\tilde{*}}$ if and only if $\left\|\widetilde{T}^{\tilde{*}}\widetilde{\tilde{l}_{\eta_{g(e)}}}\right\| \cong \left\|\widetilde{T}\widetilde{\tilde{l}_{\eta_{g(e)}}}\right\|$

For every unit vector $\tilde{l}_{\eta_{g(e)}} \tilde{\in} \tilde{\hbar}$

Let $\left\|\widetilde{T}^{\tilde{*}}\widetilde{\tilde{l}_{\eta_{g(e)}}}\right\| \cong \langle\widetilde{T}^{\tilde{*}}\,\tilde{l}_{\eta_{g(e)}}, \widetilde{\widetilde{T}^{\tilde{*}}}\,\tilde{l}_{\eta_{g(e)}}\rangle$

$\cong \langle\widetilde{T}^{\tilde{*}}\,\tilde{l}_{\eta_{g(e)}}, \widetilde{\tilde{l}_{\eta_{g(e)}}}\rangle$

$\cong \langle\widetilde{T}\,\widetilde{T}^{\tilde{*}}\,\tilde{l}_{\eta_{g(e)}}, \widetilde{\tilde{l}_{\eta_{g(e)}}}\rangle$

$$\cong \langle (\tilde{\mathfrak{T}}^{\tilde{*}}\tilde{\mathfrak{T}})\widetilde{\tilde{l}_{\eta_{\tilde{g}(e)}}}, \tilde{l}_{\eta_{\tilde{g}(e)}} \rangle$$

$$\cong \langle \tilde{\mathfrak{T}}^2 \tilde{l}_{\eta_{\tilde{g}(e)}}, \tilde{l}_{\eta_{\tilde{g}(e)}} \rangle$$

$$\lessapprox \left\| \widetilde{(\tilde{\mathfrak{T}}^{\tilde{*}})^2 \tilde{l}_{\eta_{\tilde{g}(e)}}} \right\| \left\| \widetilde{\tilde{l}_{\eta_{\tilde{g}(e)}}} \right\|$$

$$\left\| \widetilde{\tilde{\mathfrak{T}}^{\tilde{*}} \tilde{l}_{\eta_{\tilde{g}(e)}}} \right\|^2 \lessapprox \left\| \widetilde{(\tilde{\mathfrak{T}}^{\tilde{*}})^2 \tilde{l}_{\eta_{\tilde{g}(e)}}} \right\| \left\| \widetilde{\tilde{l}_{\eta_{\tilde{g}(e)}}} \right\| \text{ implies that } \left\| \widetilde{\tilde{\mathfrak{T}}^{\tilde{*}} \tilde{l}_{\eta_{\tilde{g}(e)}}} \right\|^2 \lessapprox \left\| \widetilde{(\tilde{\mathfrak{T}}^{\tilde{*}})^2 \tilde{l}_{\eta_{\tilde{g}(e)}}} \right\|$$

Therefore, $\tilde{\mathfrak{T}}^{\tilde{*}}$ is FSPN

IV Conclusion

The ideas of normed space, metric space, and Hilbert space provide the soft and fuzzy updates. There are many uses for combining fuzzy and soft ideas. The fuzzy soft paranormal operator has been defined and explained in this article.

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