Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 2 (2024)

Oscillatory Properties of Second Order Half-Linear Delay Difference Equations

K. Masaniammal [1], I. Mohammed Ali Jaffer [2]

[1, 2] Department of Mathematics, Government Arts College, Udumalpet-642126, Tamilnadu, India.

[1] reka.maths@gmail.com [2] jaffermathsgac@gmail.com

Article History:

Received: *25-01-2024*

Revised: *02-04-2024*

Accepted: *25-04-2024*

Abstract:
This study explores, some necessary and sufficient conditions that are established for oscillatory properties of second order half-linear delay difference equations of the form

$$\Delta(p(\xi)(\Delta x(\xi))^r) + q(\xi)x^s(\sigma(\xi)) = 0, \text{ for } \xi \geq \xi_0 .$$

Under the assumption $\sum_{t=\xi_0}^{\xi-1} \frac{1}{p^{\frac{1}{r}}(t)} = \infty$. Two cases are considered for $r < s$ and $r > s$, where r and s are the quotients of two positive odd integers. The effectiveness and applicability of the result are illustrated through few examples.

Keywords: Half-Linear, Delay Difference Equation, Oscillation.

1. Introduction

We consider the second order half-linear Delay difference equations of the form

$$\Delta(p(\xi)(\Delta x(\xi))^r) + q(\xi)x^s(\sigma(\xi)) = 0, \qquad \text{for } \xi \geq \xi_0. \tag{1.1}$$

where r and s are the quotient of two positive odd integers, and Δ is the forward difference operator defined by $\Delta x(\xi) = x(\xi+1) - x(\xi)$.

The following assumptions are used in this paper to obtain the result:

H1) $\{p(\xi)\}$ is sequence of positive real numbers, $0 < p < 1, \sigma(\xi) < \xi,\ \lim_{\xi\to\infty}\sigma(\xi) = \infty$.

H2) $\{q(\xi)\}$ is a sequence of nonnegative real numbers and $q(\xi)$ is not identically zero for sufficiently large values of ξ.

H3) $v(\xi) = \sum_{t=\xi_1}^{\xi-1} p^{-\frac{1}{r}}(t)$ with $\lim_{\xi\to\infty} v(\xi) = \infty$.

H4) $0 < \sigma_0(\xi) \leq \sigma(\xi)$, for $\Delta\sigma_0(\xi) \geq \sigma_0 > 0$, for $\xi \geq \xi_0$.

2. Preliminary Results

In this section, we provide useful lemma that will be essential in the analysis of the oscillation behavior of (1.1).

Lemma 2.1. Assuming $(H1) - (H3)$ hold and that $x(\xi)$ is an eventually positive solution of (1.1). Then, there exists $\xi_1 \geq \xi_0$ and $d > 0$ such that

$$0 < x(\xi) \le dv(\xi), \tag{2.1}$$

$$v(\xi)\left[\sum_{\zeta=\xi}^{\infty} q(\zeta)x^s(\sigma(\zeta))\right]^{\frac{1}{r}} \le x(\xi), \text{ for } \xi \ge \xi_1. \tag{2.2}$$

Proof. Assume that $x(\xi)$ be an eventually positive solution of (1.1). Then, by (H1), there exists a ξ^* such that $x(\xi) > 0$ and $x(\sigma(\xi)) > 0$ for all $\xi \ge \xi^*$

It follows from (1.1) that

$$\Delta(p(\xi)(\Delta x(\xi))^r) = -q(\xi)x^s(\sigma(\xi)) \le 0. \tag{2.3}$$

Consequently, $p(\xi)(\Delta x(\xi))^r$ is nonincreasing for $\xi \ge \xi^*$. Next, we establish that $p(\xi)(\Delta x(\xi))^r$ is positive. By contradiction, let $p(\xi)(\Delta x(\xi))^r \le 0$ at a certain time $\xi \ge \xi^*$.

In accordance to q is not identically zero and by (2.3),

there exists $\xi_1 \ge \xi^*$ such that

$$p(\xi)(\Delta x(\xi))^r \le p(\xi_1)\big(\Delta x(\xi_1)\big)^r < 0, \quad \xi \ge \xi_1 . \tag{2.4}$$

Remember that r is the quotient of two positive odd integers. Then,

$$\Delta x(\xi) \le \left(\frac{p(\xi_1)}{p(\xi)}\right)^{\frac{1}{r}} \Delta x(\xi_1), \quad \text{for } \xi \ge \xi_1. \tag{2.5}$$

Summing (2.5) from ξ_1 to $\xi - 1$, we arrive at the result

$$x(\xi) \le x(\xi_1) + \big(p(\xi_1)\big)^{\frac{1}{r}}\Delta x(\xi_1)v(\xi). \tag{2.6}$$

By $(H3)$, the approach of the right hand side is $-\infty$ then, $\lim_{\xi\to\infty} v(\xi) = -\infty$.

This is a contradiction to the fact that $x(\xi) > 0$.

Thus,

$$p(\xi)(\Delta x(\xi))^r > 0, \quad \text{for all } \xi \ge \xi^*.$$

From $p(\xi)(\Delta x(\xi))^r$ being nonincreasing, we have

$$\Delta x(\xi) \le \left(\frac{p(\xi_1)}{p(\xi)}\right)^{\frac{1}{r}} \Delta x(\xi_1), \text{ for } \xi \ge \xi_1. \tag{2.7}$$

Summing (2.7) from ξ_1 to $\xi - 1$, we obtain

$$x(\xi) \le x(\xi_1) + \big(p(\xi_1)\big)^{\frac{1}{r}}\Delta x(\xi_1)v(\xi). \tag{2.8}$$

Since $\lim_{\xi\to\infty} v(\xi) = \infty$, there exists a positive constant d such that (2.1) holds.

Since $p(\xi)(\Delta x(\xi))^r$ is positive and nonincreasing, $\lim_{\xi\to\infty} p(\xi)(\Delta x(\xi))^r$ exists and is nonnegative.

Summing (1.1) from ξ to $b-1$, we get

$$p(b)(\Delta x(b))^r - p(\xi)(\Delta x(\xi))^r + \sum_{t=\xi}^{b-1} q(t)x^s(\sigma(t)) = 0. \quad (2.9)$$

Letting limit as $b\to\infty$, we obtain

$$p(\xi)(\Delta x(\xi))^r \geq \sum_{t=\xi}^{\infty} q(t)x^s(\sigma(t)). \quad (2.10)$$

Then,

$$\Delta x(\xi) \geq \left[\frac{1}{p(\xi)}\sum_{t=\xi}^{\infty} q(t)x^s(\sigma(t))\right]^{\frac{1}{r}}. \quad (2.11)$$

Since $x(\xi_1) > 0$, summing (2.11) from ξ_1 to $\xi-1$, we have

$$x(\xi) \geq \sum_{t=\xi_1}^{n-1}\left[\frac{1}{p(t)}\sum_{\zeta=t}^{\infty} q(\zeta)x^s(\sigma(\zeta))\right]^{\frac{1}{r}}. \quad (2.12)$$

Use the definition of $v(\xi)$ to obtain

$$x(\xi) \geq v(\xi)\left[\sum_{\zeta=\xi}^{\infty} q(\zeta)x^s(\sigma(\zeta))\right]^{\frac{1}{r}}. \quad (2.13)$$

This yields (2.2).

3. Main Results

Theorem 3.1. Assume that there exists a constant β_1, the quotient of two positive odd integers, such that $0 < s < \beta_1 < r$. If $(H1)-(H3)$ hold, then each solution of (1.1) is oscillatory if and only if

$$\sum_{\zeta=0}^{\infty} q(\zeta)v^s(\sigma(\zeta)) = \infty. \quad (3.1)$$

Proof. On the contrary, let $x(\xi)$ be an eventually positive solution. So Lemma 2.1 holds, and then there exists $\xi_1 \geq \xi_0$ such that

$$x(\xi) \geq v(\xi)w^{\frac{1}{r}}(\xi) \geq 0, \quad \text{for } \xi \geq \xi_1, \quad (3.2)$$

where

$$w(\xi) = \sum_{\zeta=\xi}^{\infty} q(\zeta)x^s(\sigma(\zeta))\,. \tag{3.3}$$

Computing we have ,

$$\Delta w(\xi) = -q(\xi)x^s(\sigma(\xi))\,. \tag{3.4}$$

Thus, w is nonnegative and nonincreasing. Since $x > 0$, by $(H2)$, in continuation $q(\xi)x^s(\sigma(\xi))$ cannot be identically zero. Thus, Δw cannot be identically zero, and w cannot be constant.

Therefore, $w(\xi) > 0$ for $\xi \geq \xi_1$.

Computing we get,

$$\Delta w^{1-\frac{\beta_1}{r}}(\xi) \geq \left(1 - \frac{\beta_1}{r}\right) w^{\frac{-\beta_1}{r}}(\xi)\Delta w(\xi).$$

(3.5)

Summing (3.5) from ξ_2 to $\xi - 1$ and using that $w > 0$, we have

$$w^{1-\frac{\beta_1}{r}}(\xi_2) \geq \left(1 - \frac{\beta_1}{r}\right)\left[-\sum_{\zeta=\xi_2}^{\xi-1} w^{\frac{-\beta_1}{r}}(\zeta)\Delta w(\zeta)\right]$$

$$\geq \left(1 - \frac{\beta_1}{r}\right)\left[\sum_{\zeta=\xi_2}^{\xi-1} w^{\frac{-\beta_1}{r}}(\zeta)(q(\zeta)x^s(\sigma(\zeta)))\right].$$

(3.6)

By (2.1) and (3.2), we obtain

$$\begin{aligned} x^s(\xi) &= x^{s-\beta_1}(\xi)x^{\beta_1}(\xi) \\ &\geq (dv(\xi))^{s-\beta_1}x^{\beta_1}(\xi) \\ &\geq (dv(\xi))^{s-\beta_1}\left(v(\xi)w^{\frac{1}{r}}(\xi)\right)^{\beta_1} \\ &= d^{s-\beta_1}v^s(\xi)w^{\frac{\beta_1}{r}}(\xi), \text{ for } \xi \geq \xi_2. \end{aligned}$$

Since w is nonincreasing , $\frac{\beta_1}{r} > 0$, and $\sigma(t) < t$, it follows that

$$x^s(\sigma(\mathrm{t})) \geq d^{s-\beta_1}v^s(\sigma(t))w^{\frac{\beta_1}{r}}\sigma((t))$$

$$\geq d^{s-\beta_1}v^s(\sigma(t))w^{\frac{\beta_1}{r}}(t). \tag{3.7}$$

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 2 (2024)

Going back to (3.6), we obtain

$$w^{1-\frac{\beta_1}{r}}(\xi_2) \geq \left(1-\frac{\beta_1}{r}\right) d^{s-\beta_1}\left[\sum_{t=\xi_2}^{\xi-1} q(t)x^s(\sigma(t))\right] . \qquad (3.8)$$

Since $\left(1-\frac{\beta_1}{r}\right) > 0$, by (3.1) the right-hand side approaches $+\infty$ as $\xi \to \infty$.

In contradiction with (3.8), this completes the sufficiency proof for eventually positive solutions.

Similar to this the eventually negative solution can be dealt by introducing the variables $\sigma = -x$. Then, the necessary part can be shown by the contrapositive argument. If (3.1) is not hold, then for each $\alpha > 0$ there exists $\xi_1 \geq \xi_0$ such that

$$\sum_{\zeta=t}^{\infty} q(\zeta)v^s(\sigma(\zeta)) \leq \frac{\alpha^{\left(1-\frac{s}{r}\right)}}{2}, \quad \text{for all } \xi \geq \xi_1 .$$

(3.9)

We define

$$T = \left\{x : \left(\frac{\alpha}{2}\right)^{\frac{1}{r}} v(\xi) \leq x(\xi) \leq \alpha^{\frac{1}{r}} v(\xi),\ \xi \geq \xi_1\right\}.$$

(3.10)

An operator ϕ is defined on T by

$$(\phi x)(\xi) = \begin{cases} 0, & \text{if } \xi \leq \xi_1, \\ \sum_{t=\xi_1}^{\xi-1}\left[\frac{1}{p(t)}\left[\frac{\alpha}{2}+\sum_{\zeta=t}^{\infty} q(\zeta)x^s(\sigma(\zeta))\right]\right]^{\frac{1}{r}}, & \text{if } \xi > \xi_1. \end{cases} \qquad (3.11)$$

If x is a fixed point of ϕ, i.e., $\phi x = x$, then x is a solution of (1.1). First, we estimate $(\phi x)(\xi)$. By $(H3)$, we have

$$(\phi x)(\xi) \geq \sum_{t=\xi_1}^{\xi-1}\left[\frac{1}{p(t)}\left(\frac{\alpha}{2}+0\right)\right]^{\frac{1}{r}} = \left(\frac{\alpha}{2}\right)^{\frac{1}{r}} v(\xi) . \qquad (3.12)$$

Now, we establish $(\phi x)(\xi)$ from above. For x in T, as we have $x^s(\sigma(\zeta)) \leq \left(\alpha^{\frac{1}{\alpha}} v(\sigma(\zeta))\right)^s$.

Then, by (3.9),

$$(\phi x)(\xi) \leq \sum_{t=\xi_1}^{\xi-1} \left[\frac{1}{p(t)}\left[\frac{\alpha}{2} + \sum_{\zeta=t}^{\infty} p(\zeta)x^s(\sigma(\zeta))\right]\right]^{\frac{1}{r}}$$

$$\leq \alpha^{\frac{1}{r}} v(\xi). \tag{3.13}$$

Therefore, ϕ maps T to T,

Next, we find a fixed point for ϕ in T.

Let us define a sequence of functions in T by the recurrence relation

$$\sigma_0(\xi) = 0, \ \text{for } \xi \geq \xi_0,$$
$$\sigma_1(\xi) = (\phi\sigma_0)(\xi) = \begin{cases} 0, & \text{if } \xi < \xi_1, \\ \alpha^{\frac{1}{r}} v(\xi), & \text{if } \xi \geq \xi_1, \end{cases} \tag{3.14}$$
$$\sigma_{\mathrm{n}+1}(\xi) = (\phi\sigma_{\mathrm{n}})(\xi), \ \text{for } \mathrm{n} \geq 1, \ \xi \geq \xi_1.$$

Note that for each fixed ξ, we have $\sigma_1(\xi) \geq \sigma_0(\xi)$.

Using Mathematical induction, we can show that $\sigma_{\mathrm{n}+1}(\xi) \geq \sigma_{\mathrm{n}}(\xi)$. Therefore, the sequence $\{\sigma_{\mathrm{n}}\}$ converges pointwise to a sequence σ.

Using the Lebesgue dominated convergence theorem, we can show that σ is a fixed point of ϕ in T. This shows under assumption (3.9), there is a nonoscillatory solution that dose not converge to zero.

This concludes the proof.

Theorem 3.2. Assume that there exists a constant β_2, the quotient of two positive odd integers, such that $0 < r < \beta_2 < s$. If $(H1) - (H4)$ hold and $p(\xi)$ is nondecreasing, then each solution of (1.1) is oscillatory if and only if

$$\sum_{s=\xi_1}^{\infty} \left[\frac{1}{p(s)} \sum_{\zeta=s}^{\infty} q(\zeta)\right]^{\frac{1}{r}} = \infty. \tag{3.15}$$

Proof. On the contrary, consider that $x(\xi)$ is an eventually positive solution that does not converge to zero. Using the same argument as in Lemma 2.1, there exits $\xi_1 \geq \xi_0$ such that $x(\sigma(\xi)) > 0$ and $p(\xi)(\Delta x(\xi))^r$ is positive and nonincreasing.

Since $p(\xi) > 0, x(\xi)$ is increasing for $\xi \geq \xi_1$. Using $x(\xi) \geq x(\xi_1)$, we have

$$x^s(\xi) \geq x^{s-\beta_2}(\xi)x^{\beta_2}(\xi) \geq x^{s-\beta_2}(\xi_1)x^{\beta_2}(\xi), \tag{3.16}$$

and hence

$$x^s(\sigma(\xi)) \geq x^{s-\beta_2}(\xi_1)x^{\beta_2}(\sigma(\xi)), \ \text{for } \xi \geq \xi_2 \tag{3.17}$$

Using (3.17) and $\sigma(\xi) \geq \sigma_0(\xi)$, from (2.10), we have

$$p(\xi)(\Delta x(\xi))^r \geq x^{s-\beta_2}(\xi_1)x^{\beta_2}(\sigma_0(\xi))\sum_{t=\xi}^{\infty} q(t), \text{ for } \xi \geq \xi_2. \quad (3.18)$$

From $p(\xi)(\Delta x(\xi))^r$ being nonincreasing and $\sigma_0(\xi) \leq \xi$, we have

$$p(\sigma_0(\xi))\big(\Delta x(\sigma_0(\xi))\big)^r \geq p(\xi)(\Delta x(\xi))^r. \quad (3.19)$$

We apply this in the left-hand side of (3.18). Then, dividing by $p(\sigma_0(\xi))x^{\beta_2}(\sigma_0(\xi)) > 0$ and raising both side to the $\frac{1}{r}$ power, we get

$$\frac{\Delta x(\sigma_0(\xi))}{x^{\frac{\beta_2}{r}}(\sigma_0(\xi))} \geq \left[\frac{x^{s-\beta_2}(\xi_1)}{p(\sigma_0(\xi))}\sum_{t=\xi}^{\infty} q(t)\right]^{\frac{1}{r}}, \quad \text{for } \xi \geq \xi_2. \quad (3.20)$$

Multiplying the left - hand side by $\frac{\Delta\sigma_0(\xi)}{\sigma_0} \geq 1$ and summing from ξ_2 to $\xi - 1$, we have

$$\frac{1}{\sigma_0}\sum_{t=\xi_2}^{\xi-1}\frac{\Delta x(\sigma_0(\mathrm{t}))\Delta\sigma_0(\mathrm{t})}{x^{\frac{\beta_2}{r}}(\sigma_0(\mathrm{t}))} \geq x^{s-\beta_2}(\xi_1)\left[\sum_{t=\xi_2}^{\xi-1}\frac{1}{p(\sigma_0(\mathrm{t}))}\sum_{\zeta=t}^{\infty} q(\zeta)\right]^{\frac{1}{\mathrm{r}}}. \quad (3.21)$$

On the left-hand side, since $r < \beta_2$, using summation by parts, we have

$$x^{\frac{-\beta_2}{r}}\sigma_0(\xi)x(\sigma_0(\xi)) - \mathrm{x}^{\frac{-\beta_2}{r}}\sigma_0(\xi_2)x\big(\sigma_0(\xi_2)\big)$$
$$\leq \sum_{s=\xi_2}^{\xi-1} x(\sigma_0(s+1))\left[\frac{-\left(\frac{\beta_2}{r}\right)x^{\frac{\beta_2}{r}-1}\sigma_0(s)}{x^{\frac{\beta_2}{r}}(\sigma_0(s))x^{\frac{\beta_2}{r}}(\sigma_0(s+1))}\right] < \infty.$$
(3.22)

On the right-hand side of (3.21), we use that $p(\sigma_0(t)) \leq p(t)$ to conclude that (3.15) implies the right hand side approaching $+\infty$ as $y \longrightarrow \infty$, which is a contradiction.

Hence, the solution $x(\xi)$ cannot be eventually positive. For eventually negative solutions, the same change of variables is used as in Theorem 3.1 and is proceed above.

In order to prove the necessity part, we assume that (3.15) does not hold and obtain an eventually positive solution that does not converge to zero.

If (3.15) does not hold, then for each $\alpha > 0$ there exists $\xi_1 \geq \xi_0$ such that

$$\sum_{t=\xi_1}^{\infty}\left[\frac{1}{p(t)}\sum_{\zeta=t}^{\infty} q(\zeta)\right]^{\frac{1}{r}} \leq \frac{\alpha^{\left(1-\frac{s}{r}\right)}}{2}, \text{ for all } \xi \geq \xi_1. \quad (3.23)$$

We define

$$T = \left\{x{:}\ \frac{\alpha}{2} \leq x(\xi) \leq \alpha, \text{ for } \xi \geq \xi_1\right\}. \quad (3.24)$$

we define an operator ϕ on T by

$$(\phi x)(\xi) = \begin{cases} 0, & \text{if } \xi \leq \xi_{1,} \\ \frac{\alpha}{2} + \sum_{t=\xi_1}^{\xi-1} \frac{1}{p(t)} \left[\sum_{\zeta=t}^{\infty} q(\zeta) x^s(\sigma(\zeta))\right]^{\frac{1}{r}}, & \text{if } \xi > \xi_{1.} \end{cases} \quad (3.25)$$

If x is a fixed point of ϕ, i.e., $\phi x = x$, then, x is a solution of (1.1). First, we estimate $(\phi x)(\xi)$. Let $x \in M$, we have

$$(\phi x)(\xi) \geq \frac{\alpha}{2} + 0,$$

Now, we estimate $(\phi x)(\xi)$ from above. Let $x \in M$. Then $x \leq \alpha$ and by (3.23), we have

$$(\phi x)(\xi) \leq \frac{\alpha}{2} + \alpha^{\frac{s}{r}} \sum_{t=\xi_1}^{\xi-1} \left[\frac{1}{p(t)} \sum_{\zeta=t}^{\infty} q(\zeta)\right]^{\frac{1}{r}}$$

$$\leq \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha. \quad (3.26)$$

Therefore, ϕ maps T to T,

We find a fixed point for ϕ in T.

Let us define a sequence of functions in T by the recurrence relation

$$\sigma_0(\xi) = 0, \quad \text{for } \xi \geq \xi_0,$$

$$\sigma_1(\xi) = (\phi\sigma_0)(\xi) = 1, \quad \text{for } \xi \geq \xi_0,$$

$$\sigma_{n+1}(\xi) = (\phi\sigma_n)(\xi), \quad \text{for } n \geq 1, \xi \geq \xi_1. \quad (3.27)$$

Note that for each fixed ξ, we have $\sigma_1(\xi) \geq \sigma_0(\xi)$.

Using Mathematical induction, we can show that $\sigma_{n+1}(\xi) \geq \sigma_n(\xi)$. Therefore, the sequence $\{\sigma_n\}$ converges pointwise to a sequence σ in T.

Then, σ is a fixed point of ϕ and a positive solution of (1.1).

This complete the proof.

4. Example

Example 4.1. Consider the second order half-linear delay difference equation

$$\Delta\left[\frac{1}{\xi}(\Delta x(\xi))^{\frac{7}{3}}\right] + 2^{\frac{7}{3}}\left[\frac{2\xi+1}{\xi^2+\xi}\right](x(7\xi-3))^{\frac{1}{3}} = 0. \quad (4.1)$$

where, $p(\xi) = \frac{1}{\xi}$, $q(\xi) = 2^{\frac{7}{3}}\left[\frac{2\xi+1}{\xi^2+\xi}\right]$, $\sigma(\xi) = 7\xi - 3$, $s = \frac{1}{3}$, $r = \frac{7}{3}$,

$\beta_1 = \frac{5}{3}$ we have $0 < s < \beta_1 < r$.

$$\sum_{\zeta=0}^{\infty} q(\zeta) v^s(\sigma(\zeta)) = \sum_{\zeta=0}^{\infty} 2^{\frac{7}{3}}\left[\frac{2\zeta+1}{\zeta^2+\zeta}\right] \sum_{t=\xi}^{\xi-1} t^{\frac{3}{7}} = \infty.$$

Hence all the conditions of Theorem 3.1 are satisfied. Hence every solution of (4.1) is oscillatory. One of such solution of equation (1.1) is $x(\xi) = (-1)^{\xi+1}$.

Example 4.2. Consider the second order half-linear delay difference equation

$$\Delta\left[\frac{1}{\xi^2}(\Delta x(\xi))^{\frac{1}{3}}\right] + 2^{\frac{1}{3}}\left[\frac{2\xi^2+2\xi+1}{\xi^4+2\xi^3+\xi^2}\right](x(\xi-2))^{\frac{7}{3}} = 0. \tag{4.2}$$

where, $p(\xi) = \frac{1}{\xi^2}$, $q(\xi) = 2^{\frac{1}{3}}\left[\frac{2\xi^2+2\xi+1}{\xi^4+2\xi^3+\xi^2}\right]$, $\sigma(\xi) = \xi - 2, s = \frac{7}{3},\ r = \frac{1}{3}$,

$\beta_1 = \frac{5}{3}$ we have $s > \beta_1 > r$.

$$\sum_{s=\xi_1}^{\infty}\left[\frac{1}{p(s)}\sum_{\zeta=s}^{\infty} q(\zeta)\right]^{\frac{1}{r}} = \sum_{s=\xi_1}^{\infty}\left[s^2\sum_{\zeta=s}^{\infty} 2^{\frac{1}{3}}\left[\frac{2\zeta^2+2\zeta+1}{\zeta^4+2\zeta^3+\zeta^2}\right]\right]^3 = \infty.$$

Hence all the conditions of Theorem 3.2 are satisfied. Hence every solution of (4.2) is oscillatory. One of such solution of equation (1.1) is $x(\xi) = (-1)^{\xi+1}$.

5. Conclusion

In this paper, we established necessary and sufficient conditions for the oscillation of solution to second order half-linear delay difference equation. The above discussed examples illustrate the significance and relevance of the proven results.

References

[1] P. GOPALAKRISHNAN, A. MURUGESAN, C. JAYAKUMAR, Oscillation conditions of the second-order noncanonical difference equations, Journal of Mathematics and Computer Science, 25(2022), 351-360.

[2] C. SOUNDARA RAJAN AND A. MURUGESAN, Oscillatory and Asymptotic behavior of solutions to second-order non-linear neutral difference equations of advanced type, Malaya Journal of Matematik, Vol. 9, No. 1, 1160 - 1166, 2021.

[3] A. MURUGESAN AND C. JAYAKUMAR, Oscillation condition for second order half-linear advanced difference equation with variable coefficients, Malaya Journal of Matematik, Vol. 8, No. 4, 1872 - 1879, 2020.

[4] SHYAM SUNDAR SANTRA, Necessary and Sufficient conditions for oscillatory and asymptotic behaviour of solutions to second -order nonlinear Neutral differential equations with several delays, Tatra Mountains Mathematical publications, Vol. 75, (2020), 121 - 134.

[5] SHYAM SUNDAR SANTRA, Necessary and Sufficient condition for Oscillatory and Asymptotic behaviour of second-order functional differential equations, Kragujevac Journal of Mathematics, Vol. 44 (3) (2020), pp. 459-473.

[6] BASAK KARPUZ, SHYAM S. SANTRA, Oscillation theorems for second-order nonlinear delay differential equations of neutral type, Hacettepe Journal of Mathematics and Statistics, Vol. 48 (3) (2019), 633-643.

[7] P. DINAKAR, S. SELVARANGAM, E. THANDAPANI, New Oscillation Conditions for Second Order Half-Linear Advanced Difference Equations, International Journal of Mathematical Engineering and Management Sciences, Vol. 4, No. 6, 1459-1470, 2019.

[8] MARTIN BOHNER, SAID R. GRACE AND IRENA JADLOVSKA, Oscillation criteria for second-order neutral delay differential equations, Electronic Journal of Qualitative Theory of Differential Equations, 2017, No. 60, 1-12.

[9] HONGWU WU, LYNN ERBE, ALLAN PETERSON, Oscillation of solution to second order Half-Linear delay dynamic equations on time scales, Electronic Journal of Differential Equations, Vol. 2016, No. 71, pp. 1-15.

[10] E. THANDAPANI AND K. RAVI, J. R. GRAEF, Oscillation and Comparison Theorems for Half-Linear Second-Order Difference Equations, Computers and Mathematics with Applications 42 (2001), 953-960.