Oscillatory Properties of Second Order Half-Linear Delay Difference Equations

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Abstract:
This study explores, some necessary and sufficient conditions that are established for oscillatory properties of second order half-linear delay difference equations of the form
\[\Delta(p(\xi)(\Delta x(\xi)))^r + q(\xi)x^s(\sigma(\xi)) = 0, \quad \text{for} \quad \xi \geq \xi_0.\]
Under the assumption \[\sum_{t=\xi_0}^{\xi-1} \frac{1}{p^r(t)} = \infty.\] Two cases are considered for \(r < s\) and \(r > s\), where \(r\) and \(s\) are the quotients of two positive odd integers. The effectiveness and applicability of the result are illustrated through few examples.

Keywords: Half-Linear, Delay Difference Equation, Oscillation.

1. Introduction

We consider the second order half-linear Delay difference equations of the form
\[\Delta(p(\xi)(\Delta x(\xi)))^r + q(\xi)x^s(\sigma(\xi)) = 0, \quad \text{for} \quad \xi \geq \xi_0. \quad (1.1)\]
where \(r\) and \(s\) are the quotient of two positive odd integers, and \(\Delta\) is the forward difference operator defined by \(\Delta x(\xi) = x(\xi + 1) - x(\xi)\).

The following assumptions are used in this paper to obtain the result:

H1) \(\{p(\xi)\}\) is sequence of positive real numbers, \(0 < p < 1, \sigma(\xi) < \xi, \lim_{\xi \to \infty} \sigma(\xi) = \infty.\)

H2) \(\{q(\xi)\}\) is a sequence of nonnegative real numbers and \(q(\xi)\) is not identically zero for sufficiently large values of \(\xi.\)

H3) \(v(\xi) = \sum_{t=\xi_0}^{\xi-1} \frac{1}{p^r(t)}\) with \(\lim_{\xi \to \infty} v(\xi) = \infty.\)

H4) \(0 < \sigma_0(\xi) \leq \sigma(\xi), \lim_{\xi \to \infty} \sigma(\xi) = \sigma_0 > 0, \text{ for } \xi \geq \xi_0.\)

2. Preliminary Results

In this section, we provide useful lemma that will be essential in the analysis of the oscillation behavior of (1.1).

Lemma 2.1. Assuming \((H1) - (H3)\) hold and that \(x(\xi)\) is an eventually positive solution of (1.1). Then, there exists \(\xi_1 \geq \xi_0\) and \(d > 0\) such that
Summing (2.7) from \( \xi \) to \( \xi - 1 \), we obtain

\[
x(\xi) \leq x(\xi_1) + \left( p(\xi_1) \right)^{\frac{1}{r}} \Delta x(\xi_1) v(\xi).
\]

(2.8)

Since \( \lim_{\xi \to \infty} v(\xi) = \infty \), there exists a positive constant \( d \) such that (2.1) holds.

\[0 < x(\xi) \leq d v(\xi), \quad (2.1)\]

\[
v(\xi) \left[ \sum_{\xi=\xi}^{\infty} q(\xi) x^{s}(\sigma(\xi)) \right]^{\frac{1}{p'}} \leq x(\xi), \quad \text{for} \ \xi \geq \xi_1.
\]

(2.2)

**Proof.** Assume that \( x(\xi) \) be an eventually positive solution of (1.1). Then, by (H1), there exists a \( \xi^* \) such that \( x(\xi) > 0 \) and \( x(\sigma(\xi)) > 0 \) for all \( \xi \geq \xi^* \).

It follows from (1.1) that

\[
\Delta(p(\xi)(\Delta x(\xi))^r) = -q(\xi)x^{s}(\sigma(\xi)) \leq 0.
\]

(2.3)

Consequently, \( p(\xi)(\Delta x(\xi))^r \) is nonincreasing for \( \xi \geq \xi^* \). Next, we establish that \( p(\xi)(\Delta x(\xi))^r \) is positive. By contradiction, let \( p(\xi)(\Delta x(\xi))^r \leq 0 \) at a certain time \( \xi \geq \xi^* \).

In accordance to \( q \) is not identically zero and by (2.3), there exists \( \xi_1 \geq \xi^* \) such that

\[
p(\xi)(\Delta x(\xi))^r \leq p(\xi_1)(\Delta x(\xi_1))^r < 0, \quad \xi \geq \xi_1.
\]

(2.4)

Remember that \( r \) is the quotient of two positive odd integers. Then,

\[
\Delta x(\xi) \leq \left( \frac{p(\xi_1)}{p(\xi)} \right)^{\frac{1}{r}} \Delta x(\xi_1), \quad \text{for} \ \xi \geq \xi_1.
\]

(2.5)

Summing (2.5) from \( \xi_1 \) to \( \xi - 1 \), we arrive at the result

\[
x(\xi) \leq x(\xi_1) + \left( p(\xi_1) \right)^{\frac{1}{r}} \Delta x(\xi_1) v(\xi).
\]

(2.6)

By (H3), the approach of the right hand side is \( -\infty \) then, \( \lim_{\xi \to \infty} v(\xi) = -\infty \).

This is a contradiction to the fact that \( x(\xi) > 0 \).

Thus,

\[
p(\xi)(\Delta x(\xi))^r > 0, \quad \text{for all} \ \xi \geq \xi^*.
\]

From \( p(\xi)(\Delta x(\xi))^r \) being nonincreasing, we have

\[
\Delta x(\xi) \leq \left( \frac{p(\xi_1)}{p(\xi)} \right)^{\frac{1}{r}} \Delta x(\xi_1), \quad \text{for} \ \xi \geq \xi_1.
\]

(2.7)

Summing (2.7) from \( \xi_1 \) to \( \xi - 1 \), we obtain

\[
x(\xi) \leq x(\xi_1) + \left( p(\xi_1) \right)^{\frac{1}{r}} \Delta x(\xi_1) v(\xi).
\]

These results provide the expected behavior of the solution and establish the desired inequality for all \( \xi \geq \xi_1 \).
Since \( p(\xi)(\Delta x(\xi))^r \) is positive and nonincreasing, \( \lim_{\xi \to \infty} p(\xi)(\Delta x(\xi))^r \) exists and is nonnegative.

Summing (1.1) from \( \xi \) to \( b - 1 \), we get
\[
(p(b)(\Delta x(b))^r - p(\xi)(\Delta x(\xi))^r) + \sum_{t=\xi}^{b-1} q(t)x^s(\sigma(t)) = 0. \tag{2.9}
\]

Letting limit as \( b \to \infty \), we obtain
\[
p(\xi)(\Delta x(\xi))^r \geq \sum_{t=\xi}^{\infty} q(t)x^s(\sigma(t)). \tag{2.10}
\]

Then,
\[
\Delta x(\xi) \geq \left[ \frac{1}{p(\xi)} \sum_{t=\xi}^{\infty} q(t)x^s(\sigma(t)) \right]^{\frac{1}{r}}. \tag{2.11}
\]

Since \( x(\xi_1) > 0 \), summing (2.11) from \( \xi_1 \) to \( \xi - 1 \), we have
\[
x(\xi) \geq \sum_{t=\xi_1}^{\xi-1} \left[ \frac{1}{p(t)} \sum_{\zeta=t}^{\infty} q(\zeta)x^s(\sigma(\zeta)) \right]^{\frac{1}{r}}. \tag{2.12}
\]

Use the definition of \( v(\xi) \) to obtain
\[
x(\xi) \geq v(\xi) \left[ \sum_{\zeta=\xi_1}^{\infty} q(\zeta)x^s(\sigma(\zeta)) \right]^{\frac{1}{r}}. \tag{2.13}
\]

This yields (2.2).

3. Main Results

**Theorem 3.1.** Assume that there exists a constant \( \beta_1 \), the quotient of two positive odd integers, such that \( 0 < s < \beta_1 < r \). If \( (H1) \) − \( (H3) \) hold, then each solution of (1.1) is oscillatory if and only if
\[
\sum_{\zeta=0}^{\infty} q(\zeta)v^s(\sigma(\zeta)) = \infty. \tag{3.1}
\]

**Proof.** On the contrary, let \( x(\xi) \) be an eventually positive solution. So Lemma 2.1 holds, and then there exists \( \xi_1 \geq \xi_0 \) such that
\[
x(\xi) \geq v(\xi)w^r(\xi) \geq 0, \quad \text{for } \xi \geq \xi_1, \tag{3.2}
\]
where
\[ w(\xi) = \sum_{\zeta=\xi}^{\infty} q(\zeta)x^s(\sigma(\zeta)). \]  

(3.3)

Computing we have,
\[ \Delta w(\xi) = -q(\xi)x^s(\sigma(\xi)). \]  

(3.4)

Thus, \( w \) is nonnegative and nonincreasing. Since \( x > 0 \), by \((H2)\), in continuation \( q(\xi)x^s(\sigma(\xi)) \) cannot be identically zero. Thus, \( \Delta w \) cannot be identically zero, and \( w \) cannot be constant.

Therefore, \( w(\xi) > 0 \) for \( \xi \geq \xi_1 \).

Computing we get,
\[ \Delta w^{1-\frac{\beta_1}{r}}(\xi) \geq \left(1 - \frac{\beta_1}{r}\right) w^{\frac{-\beta_1}{r}}(\xi)\Delta w(\xi). \]  

(3.5)

Summing (3.5) from \( \xi_2 \) to \( \xi - 1 \) and using that \( w > 0 \), we have
\[ w^{1-\frac{\beta_1}{r}}(\xi_2) \geq \left(1 - \frac{\beta_1}{r}\right) \left[ \sum_{\zeta=\xi_2}^{\xi-1} w^{\frac{-\beta_1}{r}}(\zeta)\Delta w(\zeta) \right] \]
\[ \geq \left(1 - \frac{\beta_1}{r}\right) \left[ \sum_{\zeta=\xi_2}^{\xi-1} w^{\frac{-\beta_1}{r}}(\zeta)(q(\zeta)x^s(\sigma(\zeta))) \right]. \]  

(3.6)

By (2.1) and (3.2), we obtain
\[ x^s(\xi) = x^{s-\beta_1}(\xi)x^{\beta_1}(\xi) \]
\[ \geq (dv(\xi))^{s-\beta_1}x^{\beta_1}(\xi) \]
\[ \geq (dv(\xi))^{s-\beta_1} \left(v(\xi)w^{\frac{1}{r}}(\xi)\right)^{\beta_1} \]
\[ = d^{s-\beta_1}v^s(\xi)w^{\frac{\beta_1}{r}}(\xi), \text{ for } \xi \geq \xi_2. \]

Since \( w \) is nonincreasing, \( \frac{\beta_1}{r} > 0 \), and \( \sigma(t) < t \), it follows that
\[ x^s(\sigma(t)) \geq d^{s-\beta_1}v^s(\sigma(t))w^{\frac{\beta_1}{r}}(\sigma(t)) \]
\[ \geq d^{s-\beta_1}v^s(\sigma(t))w^{\frac{\beta_1}{r}}(t). \]  

(3.7)
Going back to (3.6), we obtain
\[ w^{1 - \beta_1 / r}(\xi_2) \geq \left( 1 - \frac{\beta_1}{r} \right) (\xi_2)^{d^{x-\beta_1}} \sum_{t=\xi_2}^{\xi-1} q(t)x^\sigma(\sigma(t)) \quad . \quad (3.8) \]

Since \( 1 - \frac{\beta_1}{r} > 0 \), by (3.1) the right-hand side approaches \(+\infty\) as \( \xi \to \infty \).

In contradiction with (3.8), this completes the sufficiency proof for eventually positive solutions.

Similar to this the eventually negative solution can be dealt by introducing the variables \( \sigma = -x \). Then, the necessary part can be shown by the contrapositive argument. If (3.1) is not hold, then for each \( \alpha > 0 \) there exists \( \xi_1 \geq \xi_0 \) such that
\[ \sum_{\zeta=\xi_1}^{\infty} q(\zeta)v^\sigma(\sigma(\zeta)) \leq \frac{\alpha(1 - \frac{\tau}{\gamma})}{2}, \quad \text{for all} \quad \xi \geq \xi_1 . \quad (3.9) \]

We define
\[ T = \left\{ x : \left( \frac{\alpha}{\sigma} \right)^{\frac{1}{2}} v^\sigma(\zeta) \leq x(\zeta) \leq \frac{1}{\alpha^\gamma} v(\xi), \ \zeta \geq \xi_1 \right\} . \quad (3.10) \]

An operator \( \phi \) is defined on \( T \) by
\[
(\phi x)(\xi) = \begin{cases} 
0, & \text{if } \xi \leq \xi_1, \\
\sum_{t=\xi_1}^{\xi-1} \frac{1}{p(t)} \left[ \frac{\sigma^x(\sigma(t))}{2} + \sum_{\zeta=t}^{\infty} q(\zeta)v^\sigma(\sigma(\zeta)) \right]^{\frac{1}{\tau}} & , & \text{if } \xi > \xi_1 . \quad (3.11) 
\end{cases}
\]

If \( x \) is a fixed point of \( \phi \), i.e., \( \phi x = x \), then \( x \) is a solution of (1.1). First, we estimate \( (\phi x)(\xi) \). By \( (H3) \), we have
\[
(\phi x)(\xi) \geq \sum_{t=\xi_1}^{\xi-1} \left[ \frac{1}{p(t)} \left( \frac{\sigma^x(\sigma(t))}{2} + 0 \right) \right]^{\frac{1}{\tau}} \\
= \left( \frac{\sigma^x(\sigma(t))}{2} \right)^{\frac{1}{\tau}} v(\xi) . \quad (3.12)
\]

Now, we establish \( (\phi x)(\xi) \) from above. For \( x \) in \( T \), as we have \( x^\sigma(\sigma(\zeta)) \leq \left( \frac{1}{\alpha^\gamma} v(\sigma(\zeta)) \right)^s \).
Then, by (3.9),

\[
(\phi x)(\xi) \leq \sum_{t=\xi+1}^{\xi-1} \left[ \frac{1}{p(t)} \left( \frac{\alpha}{2} + \sum_{\zeta=t}^{\infty} p(\zeta)x^{s}(\sigma(\zeta)) \right) \right]^{\frac{1}{r}} \leq \alpha \nu(\xi).
\]  

(3.13)

Therefore, \( \phi \) maps \( T \) to \( T \).

Next, we find a fixed point for \( \phi \) in \( T \).

Let us define a sequence of functions in \( T \) by the recurrence relation

\[
\sigma_{0}(\xi) = 0, \quad \text{for } \xi \geq \xi_{0},
\]

\[
\sigma_{1}(\xi) = (\phi \sigma_{0})(\xi) = \begin{cases} 
0, & \text{if } \xi < \xi_{1}, \\
\frac{1}{\alpha \nu(\xi)}, & \text{if } \xi \geq \xi_{1}, 
\end{cases}
\]

\[
\sigma_{n+1}(\xi) = (\phi \sigma_{n})(\xi), \quad \text{for } n \geq 1, \ \xi \geq \xi_{1}.
\]  

(3.14)

Note that for each fixed \( \xi \), we have \( \sigma_{1}(\xi) \geq \sigma_{0}(\xi) \).

Using Mathematical induction, we can show that \( \sigma_{n+1}(\xi) \geq \sigma_{n}(\xi) \). Therefore, the sequence \( \{\sigma_{n}\} \) converges pointwise to a sequence \( \sigma \).

Using the Lebesgue dominated convergence theorem, we can show that \( \sigma \) is a fixed point of \( \phi \) in \( T \). This shows under assumption (3.9), there is a nonoscillatory solution that dose not converge to zero.

This concludes the proof.

**Theorem 3.2.** Assume that there exists a constant \( \beta_{2} \), the quotient of two positive odd integers, such that \( 0 < r < \beta_{2} < s \). If (H1) – (H4) hold and \( p(\xi) \) is nondecreasing, then each solution of (1.1) is oscillatory if and only if

\[
\sum_{\zeta=\xi+1}^{\infty} \left[ \frac{1}{p(\zeta)} \sum_{\zeta=\xi}^{\infty} q(\zeta) \right]^{\frac{1}{r}} = \infty.
\]  

(3.15)

**Proof.** On the contrary, consider that \( x(\xi) \) is an eventually positive solution that does not converge to zero. Using the same argument as in Lemma 2.1, there exists \( \xi_{1} \geq \xi_{0} \) such that \( x(\sigma(\xi)) > 0 \) and \( p(\xi)(\Delta x(\xi))^{r} \) is positive and nonincreasing.

Since \( p(\xi) > 0 \), \( x(\xi) \) is increasing for \( \xi \geq \xi_{1} \). Using \( x(\xi) \geq x(\xi_{1}) \), we have

\[
x^{s}(\xi) \geq x^{s-\beta_{2}}(\xi)x^{\beta_{2}}(\xi) \geq x^{s-\beta_{2}}(\xi_{1})x^{\beta_{2}}(\xi),
\]  

(3.16)

and hence

\[
x^{s}(\sigma(\xi)) \geq x^{s-\beta_{2}}(\xi_{1})x^{\beta_{2}}(\sigma(\xi)), \quad \text{for } \xi \geq \xi_{2}
\]  

(3.17)
We define

\[ p(\xi)(\Delta x(\xi))^\gamma \geq x^{s-\beta z}(\xi_1) x^{\beta z}(\sigma_0(\xi)) \sum_{t=\xi}^{\infty} q(t), \quad \text{for} \quad \xi \geq \xi_2. \quad (3.18) \]

From \( p(\xi)(\Delta x(\xi))^\gamma \) being nonincreasing and \( \sigma_0(\xi) \leq \xi \), we have

\[ p(\sigma_0(\xi))(\Delta x(\sigma_0(\xi)))^\gamma \geq p(\xi)(\Delta x(\xi))^\gamma. \quad (3.19) \]

We apply this in the left-hand side of (3.18). Then, dividing by \( p(\sigma_0(\xi))x^{\beta z}(\sigma_0(\xi)) > 0 \) and raising both side to the \( \frac{1}{\gamma} \) power, we get

\[ \frac{\Delta x(\sigma_0(\xi))}{x^{\gamma}(\sigma_0(\xi))} \geq \left[ \frac{x^{s-\beta z}(\xi_1)}{p(\sigma_0(\xi))} \sum_{t=\xi}^{\infty} q(t) \right]^{\frac{1}{\gamma}}, \quad \text{for} \quad \xi \geq \xi_2. \quad (3.20) \]

Multiplying the left-hand side by \( \frac{\Delta \sigma_0(\xi)}{\sigma_0} \geq 1 \) and summing from \( \xi_2 \) to \( \xi - 1 \), we have

\[ \frac{1}{\sigma_0} \sum_{t=\xi_2}^{\xi-1} \Delta x(\sigma_0(t)) \Delta \sigma_0(t) \geq x^{s-\beta z}(\xi_1) \left[ \sum_{t=\xi_2}^{\xi-1} \frac{1}{\sigma_0(t)} \sum_{\xi=t}^{\infty} q(\xi) \right]^{\frac{1}{\gamma}}. \quad (3.21) \]

On the left-hand side, since \( r < \beta_2 \), using summation by parts, we have

\[
\begin{align*}
&x^{-\beta z} \sigma_0(\xi)x(\sigma_0(\xi)) - x^{-\beta z} \sigma_0(\xi_2)x(\sigma_0(\xi_2)) \\
&\leq \sum_{s=\xi_2}^{\xi-1} x(\sigma_0(s+1)) \left[ -\frac{\beta z}{x^{\gamma}(\sigma_0(s))x^{\gamma}(\sigma_0(s+1))} \right] < \infty.
\end{align*}
\]

(3.22)

On the right-hand side of (3.21), we use that \( p(\sigma_0(t)) \leq p(\xi) \) to conclude that (3.15) implies the right hand side approaching \( +\infty \) as \( y \to \infty \), which is a contradiction.

Hence, the solution \( x(\xi) \) cannot be eventually positive. For eventually negative solutions, the same change of variables is used as in Theorem 3.1 and is proceed above.

In order to prove the necessity part, we assume that (3.15) does not hold and obtain an eventually positive solution that does not converge to zero.

If (3.15) does not hold, then for each \( \alpha > 0 \) there exists \( \xi_1 \geq \xi_0 \) such that

\[ \sum_{t=\xi}^{\infty} \left[ \frac{1}{p(t)} \sum_{\xi=t}^{\infty} q(\xi) \right]^{\frac{1}{\gamma}} \leq \frac{\alpha^{(\frac{1}{\gamma})}}{2}, \quad \text{for} \quad \xi \geq \xi_1. \quad (3.23) \]

We define

\[ T = \left\{ x: \frac{\alpha}{2} \leq x(\xi) \leq \alpha, \quad \text{for} \quad \xi \geq \xi_1 \right\}. \quad (3.24) \]

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we define an operator $\phi$ on $T$ by

$$
(\phi x)(\xi) = \begin{cases} 
0, & \text{if } \xi \leq \xi_1, \\
\alpha/2 + \sum_{\xi=\xi_1}^{\xi-1} \frac{1}{p(t)} \sum_{t=\xi}^{\infty} q(t) x^s(\sigma(t)) \right)^{1/s} , & \text{if } \xi > \xi_1. 
\end{cases}
$$

(3.25)

If $x$ is a fixed point of $\phi$, i.e., $\phi x = x$, then, $x$ is a solution of (1.1). First, we estimate $(\phi x)(\xi)$. Let $x \in M$, we have

$$(\phi x)(\xi) \geq \frac{\alpha}{2} + 0,$$

Now, we estimate $(\phi x)(\xi)$ from above. Let $x \in M$. Then $x \leq \alpha$ and by (3.23), we have

$$(\phi x)(\xi) \leq \frac{\alpha}{2} + \alpha^s \sum_{t=\xi_1}^{\xi-1} \left[ \frac{1}{p(t)} \sum_{t=\xi}^{\infty} q(t) \right]^{1/s}$$

$$
\leq \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha.
$$

(3.26)

Therefore, $\phi$ maps $T$ to $T$.

We find a fixed point for $\phi$ in $T$.

Let us define a sequence of functions in $T$ by the recurrence relation

$$
\sigma_0(\xi) = 0,
$$

for $\xi \geq \xi_0$.

$$
\sigma_1(\xi) = (\phi \sigma_0)(\xi) = 1, \text{ for } \xi \geq \xi_0,
$$

$$
\sigma_{n+1}(\xi) = (\phi \sigma_n)(\xi), \text{ for } n \geq 1, \xi \geq \xi_1.
$$

(3.27)

Note that for each fixed $\xi$, we have $\sigma_1(\xi) \geq \sigma_0(\xi)$.

Using Mathematical induction, we can show that $\sigma_{n+1}(\xi) \geq \sigma_n(\xi)$. Therefore, the sequence $\{\sigma_n\}$ converges pointwise to a sequence $\sigma$ in $T$.

Then, $\sigma$ is a fixed point of $\phi$ and a positive solution of (1.1).

This complete the proof.

4. Example

Example 4.1. Consider the second order half-linear delay difference equation

$$
\Delta \left[ \frac{1}{\xi} (\Delta x(\xi))^2 \right] + 2^{\frac{7}{3}} \left[ \frac{2^{\xi+1}}{\xi^{2+\xi}} \right] (x(7\xi - 3))^{1/3} = 0.
$$

(4.1)

where, $p(\xi) = \frac{1}{\xi}$, $q(\xi) = 2^{7/3} \left[ \frac{2^{\xi+1}}{\xi^{2+\xi}} \right]$, $\sigma(\xi) = 7\xi - 3$, $s = \frac{1}{3}$, $r = \frac{7}{3}$, $\beta_1 = \frac{5}{3}$, we have $0 < s < \beta_1 < r$.

$$
\sum_{\xi=0}^{\infty} q(\xi) x^{s}(\sigma(\xi)) = \sum_{\xi=0}^{\infty} 2^{\frac{7}{3}} \left[ \frac{2^{\xi+1}}{\xi^{2+\xi}} \right] \sum_{t=\xi}^{\xi-1} t^{\frac{3}{3}} = \infty.
$$

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Hence all the conditions of Theorem 3.1 are satisfied. Hence every solution of (4.1) is oscillatory. One of such solution of equation (1.1) is \( x(\xi) = (-1)^{\xi+1} \).

**Example 4.2.** Consider the second order half-linear delay difference equation

\[
\Delta \left[ \frac{1}{\xi^2} (\Delta x(\xi))^3 \right] + 2 \frac{1}{3} \left( \frac{2\xi^2 + 2\xi + 1}{\xi^4 + 2\xi^3 + \xi^2} \right) (x(\xi - 2))^3 = 0.
\]

where, \( p(\xi) = \frac{1}{\xi^2} \), \( q(\xi) = 2 \frac{1}{3} \left( \frac{2\xi^2 + 2\xi + 1}{\xi^4 + 2\xi^3 + \xi^2} \right) \), \( \sigma(\xi) = \xi - 2, s = \frac{7}{3}, r = \frac{2}{3}, \)

\( \beta_1 = \frac{5}{3} \) we have \( s > \beta_1 > r \).

\[ \sum_{s=1}^{\infty} \left( \frac{1}{p(s)} \sum_{\zeta=s}^{\infty} q(\zeta) \right)^{\frac{1}{3}} = \sum_{s=1}^{\infty} \left( s^2 \sum_{s=1}^{\infty} \frac{1}{s^3} \left( \frac{2\xi^2 + 2\xi + 1}{\xi^4 + 2\xi^3 + \xi^2} \right) \right)^{\frac{1}{3}} = \infty. \]

Hence all the conditions of Theorem 3.2 are satisfied. Hence every solution of (4.2) is oscillatory. One of such solution of equation (1.1) is \( x(\xi) = (-1)^{\xi+1} \).

**5. Conclusion**

In this paper, we established necessary and sufficient conditions for the oscillation of solution to second order half-linear delay difference equation. The above discussed examples illustrate the significance and relevance of the proven results.

**References**


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