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# On Edge Prime Index of a Graph

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**Abstract:**
Relatively Prime Edge labeling extends the notion of prime labeling by considering edges. Prime labeling requires adjacent vertices to possess relatively prime labels, while relatively prime edge labeling requires adjacent edges to have relatively prime labels. The transformation of a coprime edge-labeled graph into a relatively prime edge-labeled graph introduces the concept of Edge Prime Index (or Relatively Prime Index). This study focuses on cases where a coprime edge-labeled graph can be converted into a relatively prime edge-labeled graph by removing certain edges from graph G, thereby establishing the concept of Edge Prime Index. Finally, the Edge Prime Index of some graphs are found.

**Keywords**: Prime Labeling, Relatively Prime Edge Labeling, Coprime Edge Labeling, Prime Index, Edge Prime Index.

## 1. Introduction

Labeling plays a significant role in the field of graph theory. To meet out the current needs different types of labeling are emerging now a days (2, 9). One such labeling is prime labeling. In prime labeling, vertices are labeled from 1 to n, with the condition that any two adjacent vertices have relatively prime labels (1). From the knowledge attained from prime labeling, Relatively Prime Edge Labeling technique focuses on labeling the edges such that adjacent edges have relatively prime labels, unlike prime labeling, which assigns relatively prime labels to adjacent vertices. A graph that allows relatively prime edge labeling is known as a relatively prime edge-labeled graph.

Coprime labeling is another labeling technique that is derived from prime labeling (3, 4). In coprime labeling, the labels are not limited to 1 to n as in prime labeling. If the vertices are labeled from 1 to k, then the least k is called the minimum coprime number of G. Inspired by the above research, coprime edge labeling is employed when a graph does not have a relatively prime edge labeling. In relatively prime edge labeling, the edges are labeled using numbers from 1 to q. However, coprime edge labeling does not have any restrictions on the labels used for the edges. A prime graph G is a bijection $f : V \rightarrow \{1,2,3 \dots, p\}$ such that, for each edge $e = uv \in E$, we have $GCD(f(u), f(v)) = 1$ (2).

***Definition 1.1***

Let $G = (V, E)$ be a graph. A bijection $f : E \rightarrow \{1,2,3 \ldots \ldots q\}$ is called relatively prime edge labeling if, for each vertex $v \in V(G)$, the labels of the edge's incident on $v$ are pairwise relatively prime. A graph that admits a relatively prime edge labeling is called a relatively prime edge labeled graph. (8)

In other words, a graph with $p$ vertices and $q$ edges are said to be a relatively prime edge labeled graph, if the edges are labeled with the first q natural numbers with the condition that any two adjacent edges have relatively prime labels. Considering, edge incident on pendent vertex as relatively prime. (8)

***Definition 1.2***

For a graph, $G = (p, q)$, coprime edge labeling is defined to be a bijection $f: E \rightarrow \{1, 2, \ldots, k\}$ such that, for $k \geq q$ , and for each vertex $v \in V$, the labels of the edge's incident on $v$ are pairwise relatively prime. (10)

The minimum value of $k$, for which G is coprime edge labeling is called as minimum coprime edge labeling, with minimum coprime edge number, $p\tau_E(G) = k$.

***Definition 1.3***

For a coprime graph G, the prime index is the least number of edges removed from G to form a prime graph G*. And is denoted by, ε(G). In other words, (10)

$$\varepsilon(G) = \min \{|E(H)| : H \subseteq G \text{ and } G - E(H) \text{ is prime}\}$$

## 2. Edge Prime Index

In this section, the formal definition of Edge Prime Index is defined with an appropriate example.

### 2.1. Definition

**Edge Prime Index:** Let G be a coprime edge labeled graph. Edge Prime Index $\varepsilon_r(G)$ is defined to be the minimum number of edges removed from G to form a relatively prime edge labeled graph $G^*$. In other words,

$$\varepsilon_r(G) = min\{|E(H)| : H \subseteq G \;\&\; G - E(H) \; is \; relatively \; prime \; edge \; labeled \; graph\}$$

### 2.2.Illustration

For a complete graph $K_4$ , the edge prime index is explained in the given figure 1. That is, by removing an edge from $K_4$ , it becomes a relatively prime edge labeled graph.

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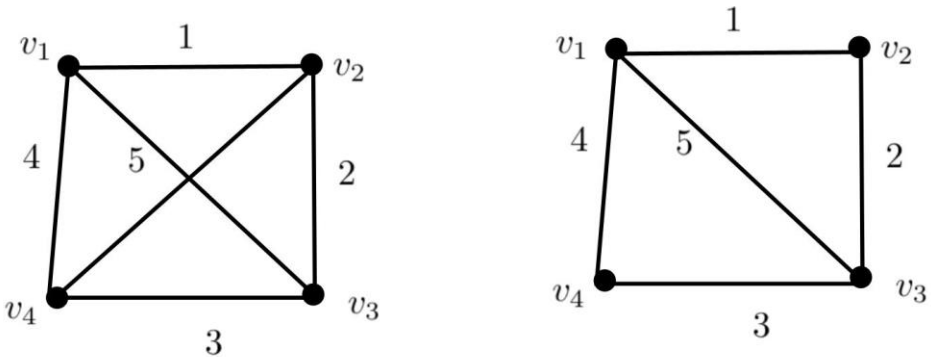


Figure 1: $\varepsilon_r(K_4) = 1$

## 3. Relation with Other Parameters

The next theorem helps to find the upper bound of the edge prime index of a graph having a Hamiltonian circuit (7).

**Theorem 3.1**

For a graph $G = (p, q)$ which contains a Hamiltonian circuit of length k, then $\varepsilon_r(G) \leq \ q - \ k$.

Proof.

Suppose, $\varepsilon_r(G) > q - \ k$, where k is the length of the Hamiltonian circuit and q is the number of edges. Let $v_1, v_2, \cdots, v_p$ be the p vertices. As G contains a Hamiltonian circuit of length k, say $v_1, v_2, \cdots, v_k$ then label the edges of the Hamiltonian circuit in such a way that, $L(v_i v_{i+1}) = i$ for $i = 1, 2, \ldots, k \ and \ v_{k+1} = v_1$. Since the prime index is greater than $q - k$ , which is the contradiction to the above labeling. Hence the maximum number of edges to be removed from G is less than or equal to $q - k$ .

**Corollary 3.2**

For a complete graph $K_4$ , $\varepsilon_r(K_4 ) \leq 2$.

Proof.

By the above theorem, Figure 2 shows that, $K_4$ contains a Hamiltonian circuit of length 4 and the number of edges in $K_4$ is 6. Hence $\varepsilon_r(G) \leq \ 6 - 4 = 2$.

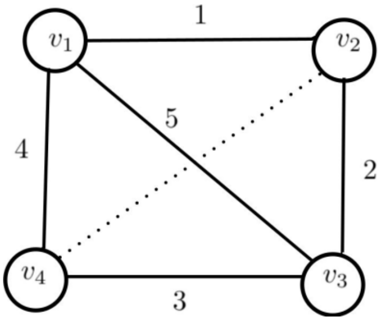


Figure 2: Complete graph with 4 vertices

## 4. Edge Prime Index of Some Class of Graphs

In this section, Edge Prime Index is found for specific classes of graphs, namely the complete graph, the corona product of graphs and so on (7).

### 4.1. Corona Product of Graph

In the following theorem, the relatively prime index of the corona product of $K_n$ and $K_1$ is determined. The corona product of two graphs G and H is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the $i^{th}$ vertex of G to every vertex in the $i^{th}$ copy of H.

**Theorem 4.1**

For a graph $K_n \odot K_1, \varepsilon_r(K_n \odot K_1) = \begin{cases} \frac{n(n-3)}{2} & , if\ 2n+1 \equiv 0 (mod\ 3) \\ \frac{n(n-3)}{2} - 1 & , if\ 2n+1 \not\equiv 0 (mod\ 3) \end{cases}$

*Proof*

Let G = $K_n \odot K_1$ be the graph with 2n vertices and $\frac{n(n+1)}{2}$ edges and let $v_1, v_2, \cdots\cdots, v_n, u_1, u_2, \cdots\cdots, u_n$ be the vertices of $K_n \odot K_1$.

Case 1: For $2n+1 \equiv 0(mod\ 3)$.

It is enough to prove that, the removal of $\frac{n(n-3)}{2}$ edges results in a relatively prime edge labeled graph. Suppose the removal of $\frac{n(n-3)}{2} - 1$ edges in $K_n \odot K_1$ results in a relatively prime edge labeled graph. That is, remaining $\frac{n(n+1)}{2} - \frac{n(n-3)}{2} + 1 = 2n+1$ edges of $K_n$ can be labeled from $1$ to $2n+1$. Hence by removing $\frac{n(n-3)}{2} - 1$ interior edges of $K_n \odot K_1$, the resultant graph will be of the form $C_n$ with n edges, n pendent vertices $(u_1, u_2, \cdots\cdots, u_n)$ connecting to the each vertex of $C_n$ and an edge connecting any two non-adjacent vertices of $C_n$. Each vertex $v_1, v_2, \cdots\cdots, v_n$ of $C_n$ is of degree 3. By labeling the edges of $C_n$ with $1, 3, 5, \ldots. 2n-1$ , each edge incident on the pendant vertices is labeled with $2, 4, 6, \ldots, 2n$ and an edge connecting any two non-adjacent vertices of $C_n$ is labeled with

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$2n+1$. As $2n+1 \equiv 0(mod\ 3)$, that is $2n+1 = 3m$, the label incident on the vertices of the edge with label $2n+1$ fails to be relatively prime.

Case 2: For $2n+1 \not\equiv 0(mod\ 3)$.

It is enough to prove that, the removal of $\frac{n(n-3)}{2} - 1$ edges results in a relatively prime edge labeled graph. Suppose the removal of $\frac{n(n-3)}{2} - 2$ edges in $K_n \odot K_1$ results in a relatively prime edge labeled graph. That is, remaining $\frac{n(n+1)}{2} - \frac{n(n-3)}{2} + 2 = 2n + 2$ edges of $K_n$ can be labeled from 1 to $2n+2$. Hence by removing $\frac{n(n-3)}{2} - 2$ interior edges of $K_n \odot K_1$, the resultant graph will be of the form $C_n$ with n edges, n pendent vertices $(u_1,\ u_2,\ \cdots\cdots, u_n)$ connecting to the each vertex of $C_n$ and two edges connecting any two non-adjacent vertices of $C_n$. Each vertex $v_1,\ v_2,\ \cdots\cdots, v_n$ of $C_n$ is of degree 3. By labeling the edges of $C_n$ with $1, 3, 5, \ldots .\, 2n-1$, each edge incident on the pendant vertices is labeled with $2, 4, 6, \ldots, 2n$ and the edge connecting any two non-adjacent vertices of $C_n$ is labeled with $2n+1, 2n+2$. As $2n+1 \not\equiv 0(mod\ 3)$, the label incident on the vertices of the edge with label $2n+2$ fails to be relatively prime.

***Illustration***

As an illustration of above theorem, edge prime index of , $K_4 \odot K_1$ *and* , $K_5 \odot K_1$ is given in Figure – 3, 4 respectively.

For $2n+1 \equiv 0(mod\ 3)$, consider n = 4 that is, $K_4 \odot K_1$, the edge prime index is $\varepsilon_r(K_4 \odot K_1) = \frac{n(n-3)}{2} = \frac{4}{2} = 2$ , which is illustrated in Figure 3.

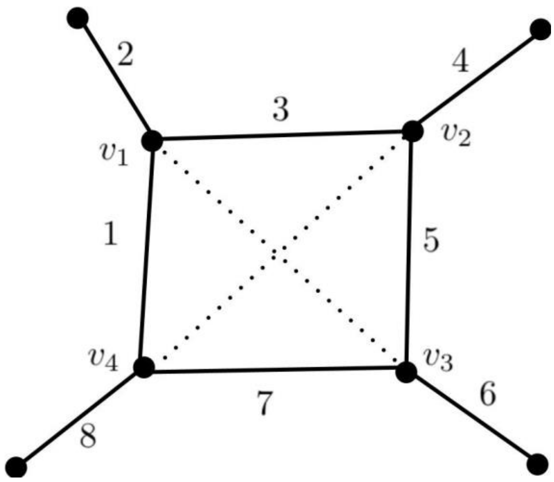


Figure 3: Edge Prime Index $\varepsilon_r(K_4 \odot K_1) = 2$

For $2n+1 \not\equiv 0(mod\ 3)$, consider n = 5, that is $K_5 \odot K_{1.}$ .

The edge prime index of $K_5 \odot K_1$ is $\varepsilon_r(K_5 \odot K_1) = \frac{n(n-3)}{2} - 1 = \frac{5\times 2}{2} - 1 = 4$, which is illustrated in Figure 4.

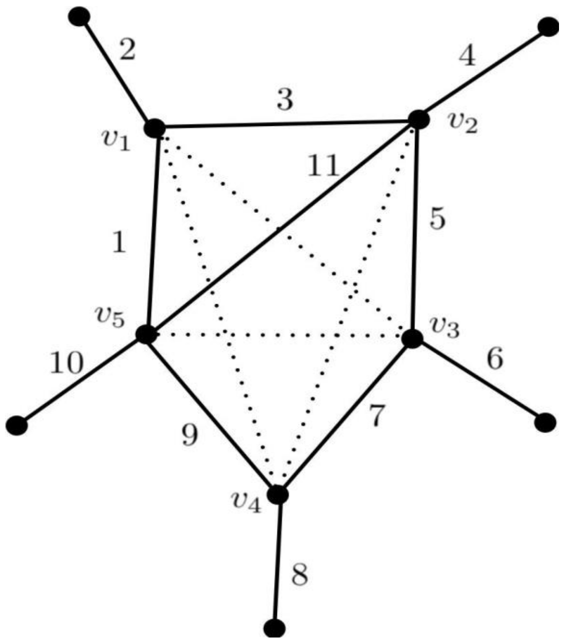


Figure 4: Edge Prime Index $\varepsilon_r(K_5 \odot K_1) = 4$

### 4.2. Complete Bipartite Graph

The following theorem finds the edge prime index of the complete bipartite graph $K_{2,t}$ and $K_{3,t}$ with proper illustration (7) .

**Theorem 4.2**

For a graph $G = K_{2,t}$ , $t > 2$ then, $\varepsilon_r(G) = 2t - 5$.

*Proof.*

We know that, $G = K_{2,t}$ is not RPEL graph. Now, it is enough to find the minimum number of edges to be removed from G to make G as a RPEL. The number of vertices and edges in $G = K_{2,t}$ is $t + 2$ $and$ $2t$.

Label the edges of G in such a way that, $L(u_1v_1) = 1, L(u_1v_2) = 2$ , $L(u_1v_3) = 3$ , $L(u_1v_4) = 5$ $and$ $L(u_2v_1) = 4$ as in Figure 5.

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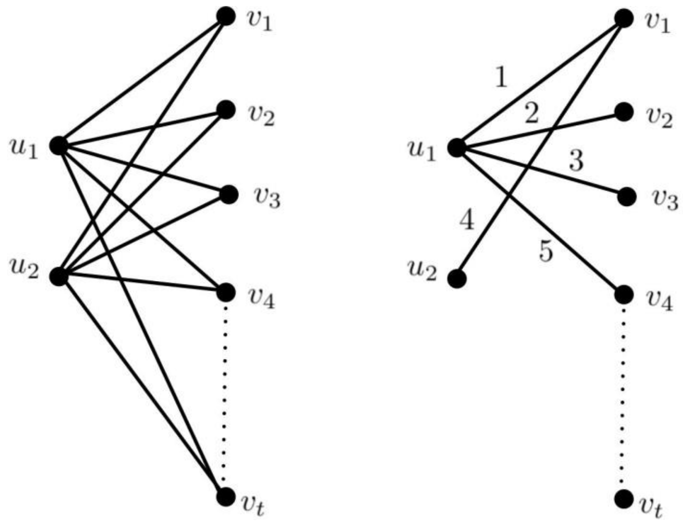


Figure 5 Edge Prime Index Illustration

Also, the label 6 cannot be labeled in any of the edge's incident on $u_1$ and $u_2$. Thus, it is needed to remove $2t-5$ edges from G to make it as a relatively prime edge labeled graph.

Hence, $\varepsilon_r(G) = 2t-5$.

**Theorem 4.3.**

For a graph $G = K_{3,t}, t > 3,$ then, $\varepsilon_r(G) = 3t-7$.

*Proof.*

Since $G = K_{3,t}$ is not RPEL graph. It is enough to find the minimum number of edges to be removed from G to make it as a RPEL graph. The number of vertices and edges in $G = K_{3,t}$ is $t+3$ *and* $3t$.

Label the edges of G in such a way that, $L(u_1v_1) = 1, L(u_1v_2) = 2, L(u_1v_3) = 3, L(u_1v_4) = 5, L(u_2v_1) = 4, L(u_2v_2) = 7$ and , $L(u_3v_4) = 6$ as in Figure 6.

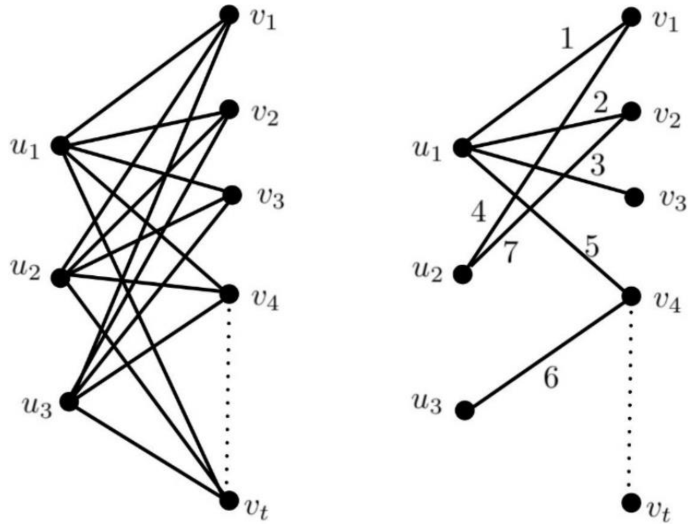


Figure 6 Edge Prime Index

Also, the label 8 cannot be labeled in any of the edge's incident on $u_1, u_2$ and $u_3$. Thus, it is needed to remove $3t - 7$ edges from G to make it as a relatively prime edge labeled graph. Hence, $\varepsilon_r(G) = 3t - 7$.

### 4.3. Generalized Petersen Graph

The next theorem finds the edge prime index of the Generalized Petersen Graph $G = P(n, k)$. The Generalized Petersen graphs $P(n, k)$ with $n \geq 3$ and $1 \leq k \leq \frac{n}{2}$ are defined to be a graph with $V(P(n, k)) = \{u_j, v_j : 1 \leq j \leq n\}$ and $E(P(n, k)) = \{v_1 v_{j+1}, v_j u_j, u_j u_{j+k} : 1 \leq j \leq n, \text{subscript mod n}\}$. (5,6)

**Theorem 4.4.**

For even n, the generalized Petersen graph $G = P(n, 2)$ then, $\varepsilon_r(G) = n - 1$.

*Proof*

The vertices of $P(n, 2)$ are $\{v_1, v_2 \ldots. v_n, u_1, u_2, \ldots, u_n\}$, where $v_1, v_2 \ldots. v_n$ represents the outer vertices and $u_1, u_2, \ldots, u_n$ represents the inner vertices and the edges of $P(n, 2)$ are $\{v_1 v_{i+1}, v_i u_i, u_i u_{i+2} : 1 \leq i \leq n, \text{ subscript mod } n\}$.

Thus, there are 2n vertices and 3n edges in $P(n, 2)$. As the $\gcd(n, 2) = 2$, for even n, there exists 2 disjoint inner cycles of length $\frac{n}{2}$ each and there exists an outer cycle $\{v_1 v_2 \ldots. v_n v_1\}$ of length n.

Now, label the edges of $P(n, 2)$ in such a way that, the two inner cycles receive the label $\left\{1, 2, 3, \ldots., \frac{n}{2}\right\} and \left\{\frac{n+2}{2}, \ldots., n\right\}$ and the outer cycle with $\{n + 1, n + 2, \ldots\ldots 2n\}$. As $2n + 1$ is odd, label any of the edge $\{v_i u_i\}$ as $2n + 1$.

Also, $2n + 2$ cannot be labeled on any edge because $2n + 2$ is an even number that violates the relative prime property. Thus, the relatively prime index of $P(n, 2)$ for even n, is $\varepsilon_r(P(n, 2)) = 3n - 2n - 1 = n - 1$ . Hence the proof.

***Illustration:***

Figure 7 shows the illustration of the above theorem. Consider a generalized Petersen Graph with 16 vertices, P(8,2) . Then the edge prime index is given by, $\varepsilon_r(P(8, 2)) = n - 1 = 8 - 1 = 7$

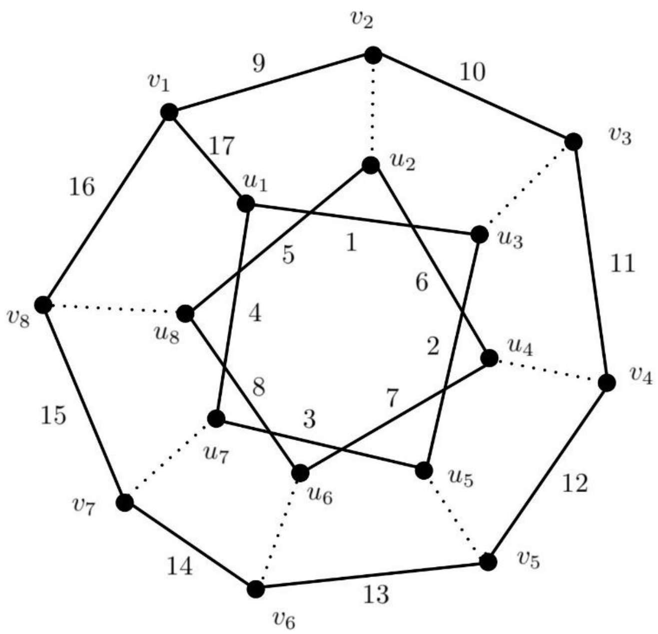


Figure 7: $\varepsilon_r(P(8,2)) = 7$

**Theorem 4.5.**

For even n, k, the generalized Petersen graph $G = P(n,k)$ then, $\varepsilon_r(G) = n - 1$.

*Proof*:

The vertices of $P(n,k)$ are $\{ v_1, v_2 \ldots.\ v_n, u_1, u_2, \ldots, u_n \}$, where $v_1, v_2 \ldots.\ v_n$ represents the outer vertices and $u_1, u_2, \ldots, u_n$ represents the inner vertices and the edges of $P(n,k)$ are $\{ v_1 v_{i+1}, v_i\, u_i, u_i u_{i+k} : 1 \le i \le n ,\ \text{subscript mod}\, n \}$. We know that, there exist $\gcd(n,k) = d$ inner cycles of length $\frac{n}{k}$ each.

Label the edges of $P(n,k)$ in such a way that, the d inner cycles receive the label $\{1, 2, 3, \ldots., n \}$ consecutively and the outer cycle with $\{n+1, n+2, \ldots..2n\}$. As $2n+1$ is odd, label any of the edge $\{v_i\, u_i\}$ as $2n\ + 1$.

Also, $2n + 2$ cannot be labeled on any edge because $2n + 2$ is an even number that violates the relative prime property. Thus, the edge prime index of $P(n,k)$ for even n, k , is $\varepsilon_r\big(P(n,k)\big) = 3n - 2n - 1 = n - 1$ . Hence the proof.

### 4.4. Ladder Graph

The ladder graph L(n) consists of two parallel paths, each with n vertices, and these paths are connected by n-1 edges known as "rails." The first and last vertices of each path are also connected by an additional edge called the "rung." Therefore, a ladder graph L(n) has a total of 2n vertices and 3n-2 edges. In this section, the edge prime index of the ladder graph L(n) is discussed.

**Theorem 4.6.**

For a ladder graph $G = L_n, n > 3$ then, $\varepsilon_r(G) = n - 3$.

*Proof:*

Let the vertices in the Ladder graph are $\{v_1, v_2 \ldots.\ v_n, u_1, u_2, \ldots, u_n\}$. Then by the definition of ladder graph, there are 2n and 3n-2 vertices and edges respectively. Now, it is enough to find the minimum number of edges to be removed from the ladder graph to make it as a relatively prime edge labeled graph. The maximal cycle formed by the vertices of the ladder graph is $v_1, v_2 \ldots.\ v_n, u_n, u_{n-1}, \ldots, u_{1,}, v_1$.

Now, label the edges of G in such a way that, the maximal cycle is labeled with $\{2, 3, \ldots., 2n+1\}$ and the edge $u_2v_2$ with the label 1 as shown in figure 8.

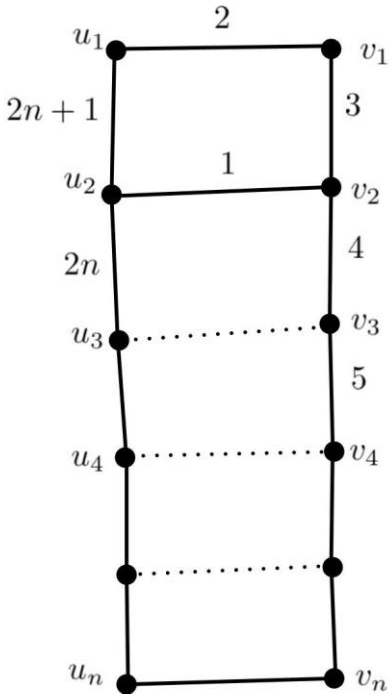


Figure 8: $\varepsilon_r(G) = n - 3$

Also, 2n+2 is an even number and to make the edges incident of each vertex as pairwise relatively prime, it cannot be labeled in any of the edges.

Thus, $\varepsilon_r(G) = 3n - 2 - 2n - 1 = n - 3$. Hence the proof.

## 4.5. Powers of Paths and Cycle

For the powers of path and cycle graph, the edge prime index is determined in this section. The m-th power of a simple graph G can be defined as the graph $G^m$, where the set of vertices in $G^m$ is the same as that in G. In $G^m$ , two vertices are adjacent if and only if their distance in G is not more than m. That is, their distance in G is at most $m$.

**Theorem 4.7.**

For even n, the power graph of a path G = $P_n^2$, then, $\varepsilon_r(G) = \begin{cases} n-3 & if\ n\ is\ odd \\ n-4 & if\ n\ is\ even \end{cases}$

*Proof:*

Let the number of vertices and edges in $P_n^2$ are $n$ and $2n-3$ respectively. From the definition of $P_n^2$, there exists a maximal cycle of length n. Now, label the maximal cycle with $\{1, 2, 3, \dots\dots, n\}$.

Case 1: n is odd

As n is odd, $n+1$ is even. Hence it is not possible to label $n+1$ in any of the remaining edges. Because the remaining edges associated with a vertex already contains the label of even multiple. Thus, by labeling $n+1$ will violate the relatively prime property. Therefore, it is necessary to remove $2n - 3 - n = n - 3$ edges from $P_n^2$ to make it as a relatively prime labeled graph. That is, $\varepsilon_r(G) = n-3$. (Example for $n = 7$ is illustrated in Figure 9)

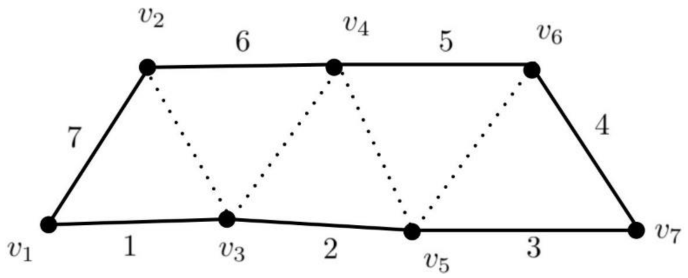


Figure 9: $\varepsilon_r(G) = n - 3 = 4$

Case 2: n is even

As n is even, $n+1$ is odd. Hence it is possible to label $n+1$ in any of the remaining edges. And, it is not possible to label $n+2$ in the remaining edges. Therefore, it is necessary to remove $2n - 3 - n - 1 = n - 4$ edges from $P_n^2$ to make it as a relatively prime labeled graph. That is, $\varepsilon_r(G) = n-4$. (Example for $n = 8$ is illustrated in Figure 10)

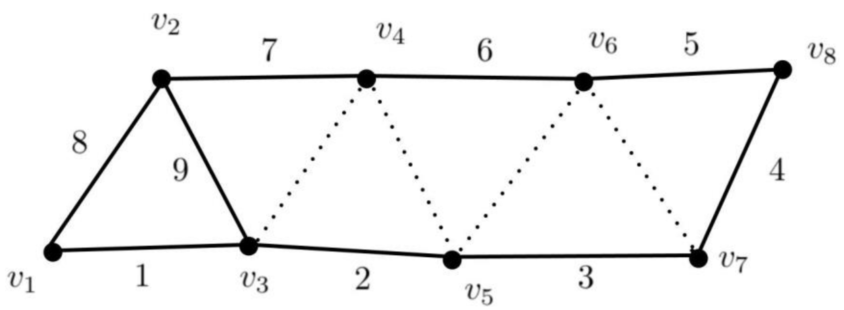


Figure 10: $\varepsilon_r(G) = n - 4 = 4$

**Theorem 4.8.**

For a power graph of a cycle $G = C_n^2$, then, $\varepsilon_r(G) = \begin{cases} n & if\ n\ is\ odd \\ n-1, & if\ n\ is\ even \end{cases}$

*Proof:*

Let the number of vertices and edges in $C_n^2$ are $n$ and $2n$ respectively. From the definition of $C_n^2$, there exists a cycle of length n. Now, label the cycle with $\{1, 2, 3, \ldots\ldots, n\}$.

Case 1: n is odd

As n is odd, $n + 1$ is even. Hence it is not possible to label $n + 1$ in any of the remaining edges. Because the remaining edges associated with a vertex already contains the label of even multiple. Thus, by labeling $n + 1$ will violate the relatively prime property. Therefore, it is necessary to remove $2n - n = n$ edges from $C_n^2$ to make it as a relatively prime labeled graph. That is, $\varepsilon_r(G) = n$ (Example for $n = 5$ is illustrated in Figure 11)

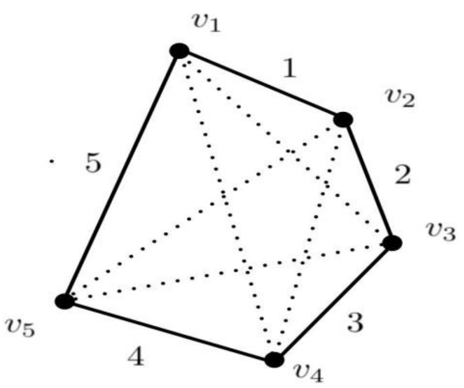


Figure 11: $\varepsilon_r(G) = n = 5$

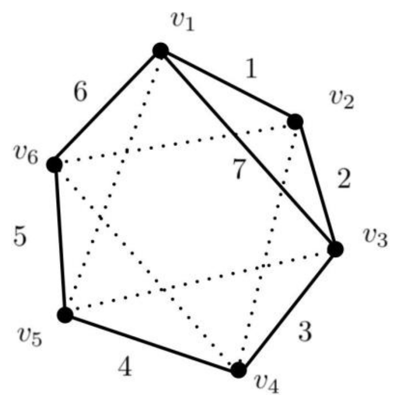


Figure 12: $\varepsilon_r(G) = n - 1 = 5$

Case 2: n is even

As n is even, $n + 1$ is odd. Hence it is possible to label $n + 1$ in any of the remaining edges. And, it is not possible to label $n + 2$ in the remaining edges. Therefore, it is necessary to remove $2n - n - 1 = n - 1$ edges from $C_n^2$ to make it as a relatively prime labeled graph. That is, $\varepsilon_r(G) = n - 1$. (Example for $n = 6$ is illustrated in Figure 12)

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