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Some Applications of Fractional Derivative Operator

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Abstract:

Introduction: The aim of this paper is to introduce a new subclass $\mathrm{TS}(\omega, \sigma, \varsigma, \vartheta)$ of univalent functions with negative coefficients related to fractional derivative operator in the unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$. We obtain basic properties like coefficient inequality, distortion and covering theorem, radii of starlikeness, convexity and close-to-convexity, extreme points, Hadamard product, and closure theorems for functions belonging to our class

Keywords: Univalent , derivative opertator , Starlike, Extreme points, Hadamard product.

1. Introduction

Let A signify the class of all functions $u(z)$ of the type

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition $u(0) = u'(0) - 1 = 0$. We denote by S the subclass of A consisting of functions $u(z)$ which are all univalent in $\mathbb{U}$. A function $u \in A$ is a starlike function of the order $m, 0 \leq m < 1,$ if it satisfy

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > m, z \in \mathbb{U}. \tag{1.2}$$

We denote this class with $S^*(m)$.

A function $u \in A$ is a convex function of the order $m, 0 \leq m < 1,$ if it fulfil

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > m, z \in \mathbb{U}. \tag{1.3}$$

We denote this class with $K(m)$.

Note that $S^*(0) = S^*$ and $K(0) = K$ are the usual classes of starlike and convex functions in $\mathbb{U}$ respectively.

Let T denote the class of functions analytic in $\mathbb{U}$ that are of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \; z \in \mathbb{U} \tag{1.4}$$

and let $T^*(m) = T \cap S^*(m)$, $C(m) = T \cap K(m)$. The class $T^*(m)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [14].

Many basically equivalent definitions of fractional computation have been given in literature ((cf.)e.g.,[13] and ([15], p. 45)). We state the following definitions due to Owa and Srivastava [9] which have been used rather frequently in the theory of analytic functions (see also [4]).

Definition 1.1. The fractional integral of order ϑ is defined, for a function $u(z)$, by

$$D_z^{-\vartheta} u(z) = \frac{1}{\omega(\vartheta)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\vartheta}} d\zeta, \quad (\vartheta > 0) \tag{1.5}$$

and the fractional derivative of order ς is defined, for a function $u(z)$, by $D_z^{\vartheta} u(z) =$

$$\frac{1}{\omega(1-\vartheta)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\vartheta}} d\zeta, \quad (0 \leq \vartheta < 1) \tag{1.6}$$

where $u(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\vartheta-1}$ involved in (1.5) (and that of $(z-\zeta)^{-\vartheta}$ involved in (1.6) is removed by requiring $\log(z-\zeta)$ to be real when $(z-\zeta) > 0$.

Definition 1.2. Under the hypotheses of Definition1.1, the fractional derivative of order $n+\vartheta$ is defined by

$$D_z^{n+\vartheta} u(z) = \frac{d^n}{dz^n} D_z^{\vartheta} u(z), \quad (0 \leq \vartheta < 1; n \in N_0 = N \cup \{0\}). \tag{1.7}$$

With the aid of the above definitions, Owa and Srivastava [9] defined the fractional operator $\mathcal{J}_z^{\vartheta}$ by

$$\begin{aligned} \mathcal{J}_z^{\vartheta} u(z) &= \omega(2-\vartheta) z^{\vartheta} D_z^{\vartheta} u(z), \quad (\vartheta \neq 2,3,4,\cdots) \\ \mathcal{J}_z^{\vartheta} u(z) &= z + \sum_{n=2}^{\infty} \Theta(\vartheta, n) a_n z^n \\ \text{where } \Theta(\vartheta, n) &= \frac{\omega(n+1)\omega(2-\vartheta)}{\omega(n-\vartheta+1)} \\ \text{and } \Theta(\vartheta, 2) &= \frac{2}{(2-\vartheta)}. \end{aligned} \tag{1.8}$$

Now, by making use of the linear operator $\mathcal{J}_z^{\vartheta} u$, we define a new subclass of functions belonging to the class A.

Definition 1.3. For $0 \leq \omega < 1, 0 \leq \sigma < 1, 0 < \varsigma < 1$, and $0 \leq \vartheta < 1$, we let $TS(\omega, \sigma, \varsigma, \vartheta)$ be the subclass of u consisting of functions of the form (1.4) and its geometrical condition satisfy

$$\left| \frac{\omega\left(\left(\mathcal{J}_z^{\vartheta} u(z)\right)' - \frac{\mathcal{J}_z^{\vartheta} u(z)}{z} \right)}{\sigma\left(\mathcal{J}_z^{\vartheta} u(z)\right)' + (1-\omega)\frac{\mathcal{J}_z^{\vartheta} u(z)}{z}} \right| < \varsigma, \quad z \in \mathbb{U} \textit{ where } \mathcal{J}_z^{\vartheta}, \textit{ is given by } (1.8).$$

2. Coefficient Inequality

In the following theorem, we obtain a necessary and sufficient condition for function to be in the class $TS(\omega,\sigma,\varsigma,\vartheta)$.

Theorem 2.1. Let the function u be defined by (1.4). Then $u \in TS(\omega,\sigma,\varsigma,\vartheta)$ if and only if

$$\sum_{n=2}^{\infty}[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\,\Theta(n,\vartheta)a_n \le \varsigma\big(\sigma+(1-\omega)\big), \qquad (2.1)$$

where $0<\varsigma<1, 0\le\omega<1, 0\le\sigma<1,$ and $0\le\vartheta<1.$

The result (2.1) is sharp for the function
$u(z)=z-\frac{\varsigma(\sigma+(1-\omega))}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\vartheta)}z^n,\ \ n\ge 2.$

Proof. Suppose that the inequality (2.1) holds true and $|z|=1$. Then we obtain

$$\begin{aligned}
&\left|\omega\left(\left(\mathcal{J}_z^{\vartheta}u(z)\right)'-\frac{\mathcal{J}_z^{\vartheta}u(z)}{z}\right)\right|-\varsigma\left|\sigma\left(\mathcal{J}_z^{\vartheta}u(z))'+(1-\omega)\frac{\mathcal{J}_z^{\vartheta}u(z)}{z}\right)\right|\\
=\ &\left|-\omega\sum_{n=2}^{\infty}(n-1)\,\Theta(n,\vartheta)a_n z^{n-1}\right|\\
&\quad-\varsigma\left|\sigma+(1-\omega)-\sum_{n=2}^{\infty}(n\sigma+1-\omega)\,\Theta(n,\vartheta)a_n z^{n-1}\right|\\
\le\ &\sum_{n=2}^{\infty}[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\,\Theta(n,\vartheta)a_n-\varsigma\big(\sigma+(1-\omega)\big)\\
\le\ &0.
\end{aligned}$$

Hence, by maximum modulus principle, $u\in TS(\omega,\sigma,\varsigma,\vartheta)$. Now assume that $u\in TS(\omega,\sigma,\varsigma,\vartheta)$ so that

$$\left|\frac{\omega\left(\left(\mathcal{J}_z^{\vartheta}u(z)\right)'-\frac{\mathcal{J}_z^{\vartheta}u(z)}{z}\right)}{\sigma\left(\mathcal{J}_z^{\vartheta}u(z)\right)'+(1-\omega)\frac{\mathcal{J}_z^{\vartheta}u(z)}{z}}\right|<\varsigma,\ \ z\in\mathbb{U}$$

Hence

$$\left|\omega\left(\left(\mathcal{J}_z^{\vartheta}u(z)\right)'-\frac{\mathcal{J}_z^{\vartheta}u(z)}{z}\right)\right|<\varsigma\left|\sigma\left(\mathcal{J}_z^{\vartheta}u(z))'+(1-\omega)\frac{\mathcal{J}_z^{\vartheta}u(z)}{z}\right)\right|.$$

Therefore, we get

$$\left|-\sum_{n=2}^{\infty}\omega\,(n-1)\Theta(n,\vartheta)a_n z^{n-1}\right|$$
$$< \quad \varsigma\left|\sigma+(1-\omega)-\sum_{n=2}^{\infty}(n\sigma+1-\omega)\,\Theta(n,\vartheta)a_n z^{n-1}\right|.$$

Thus

$$\sum_{n=2}^{\infty}[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\,\Theta(n,\vartheta)a_n \le \varsigma\big(\sigma+(1-\omega)\big)$$

and this completes the proof.

Corollary 2.1. Let the function $u \in TS(\omega,\sigma,\varsigma,\vartheta)$.*Then*

$$a_n \le \frac{\varsigma\big(\sigma+(1-\omega)\big)}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\vartheta)} z^n, \;\; n \ge 2.$$

3. Distortion and Covering Theorem

We introduce the growth and distortion theorems for the functions in the class $TS(\omega,\sigma,\varsigma,\vartheta)$

Theorem 3.1. Let the function $u \in TS(\omega,\sigma,\varsigma,\vartheta)$. Then

$$|z|-\frac{\varsigma\big(\sigma+(1-\omega)\big)}{\Theta(2,\vartheta)[\omega+\varsigma(2\sigma+1-\omega)]}|z|^2 \quad \le |u(z)|$$
$$\le \quad |z|+\frac{\varsigma\big(\sigma+(1-\omega)\big)}{\Theta(2,\vartheta)[\omega+\varsigma(2\sigma+1-\omega)]}|z|^2.$$

The result is sharp and attained

$$u(z)=z-\frac{\varsigma\big(\sigma+(1-\omega)\big)}{\Theta(2,\vartheta)[\omega+\varsigma(2\sigma+1-\omega)]}z^2.$$

Proof.

$$|u(z)| \quad = \left|z-\sum_{n=2}^{\infty}a_n\,z^n\right| \le |z|+\sum_{n=2}^{\infty}a_n\,|z|^n$$
$$\le |z|+|z|^2\sum_{n=2}^{\infty}a_n\,.$$

By Theorem 2.1, we get

$$\sum_{n=2}^{\infty}a_n \le \frac{\varsigma\big(\sigma+(1-\omega)\big)}{[\omega+\varsigma(2\sigma+1-\omega)]\Theta(n,\vartheta)} \qquad (3.1).$$

Thus

$$|u(z)| \leq |z| + \frac{\varsigma\big(\sigma + (1-\omega)\big)}{\Theta(2,\vartheta)[\omega + \varsigma(2\sigma + 1 - \omega)]}|z|^2.$$

Also

$$\begin{aligned} |u(z)| &\geq |z| - \sum_{n=2}^{\infty} a_n |z|^n \\ &\geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n \\ &\geq |z| - \frac{\varsigma\big(\sigma + (1-\omega)\big)}{\Theta(2,\vartheta)[\omega + \varsigma(2\sigma + 1 - \omega)]}|z|^2. \end{aligned}$$

Theorem 3.2. *Let* $u \in TS(\omega,\sigma,\varsigma,\vartheta)$. *Then*

$$1 - \frac{2\varsigma\big(\sigma + (1-\omega)\big)}{\Theta(2,\vartheta)[\omega + \varsigma(2\sigma + 1 - \omega)]}|z| \leq |u'(z)| \leq 1 + \frac{2\varsigma\big(\sigma + (1-\omega)\big)}{\Theta(2,\vartheta)[\omega + \varsigma(2\sigma + 1 - \omega)]}|z|$$

with equality for

$$u(z) = z - \frac{2\varsigma\big(\sigma + (1-\omega)\big)}{\Theta(2,\vartheta)[\omega + \varsigma(2\sigma + 1 - \omega)]}z^2.$$

Proof. Notice that

$$\begin{aligned} & \Theta(2,\vartheta)[\omega + \varsigma(2\sigma + 1 - \omega)] \sum_{n=2}^{\infty} n\, a_n \\ \leq \quad & \sum_{n=2}^{\infty} n\,[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\Theta(n,\vartheta) a_n \\ \leq \quad & \varsigma\big(\sigma + (1-\omega)\big), \end{aligned} \tag{3.2}$$

from Theorem 2.1. Thus

$$\begin{aligned} |u'(z)| &= \left|1 - \sum_{n=2}^{\infty} n\, a_n z^{n-1}\right| \\ &\leq 1 + \sum_{n=2}^{\infty} n\, a_n |z|^{n-1} \\ &\leq 1 + |z| \sum_{n=2}^{\infty} n\, a_n \\ &\leq 1 + |z| \frac{2\varsigma\big(\sigma + (1-\omega)\big)}{\Theta(2,\vartheta)[\omega + \varsigma(2\sigma + 1 - \omega)]}. \end{aligned} \tag{3.3}$$

On the other hand

$$|u'(z)| = \left|1-\sum_{n=2}^{\infty} n\, a_n z^{n-1}\right|$$
$$\geq 1-\sum_{n=2}^{\infty} n\, a_n |z|^{n-1}$$
$$\geq 1-|z|\sum_{n=2}^{\infty} n\, a_n$$
$$\geq 1-|z|\frac{2\varsigma\big(\sigma+(1-\omega)\big)}{\Theta(2,\vartheta)[\omega+\varsigma(2\sigma+1-\omega)]} \qquad (3.4).$$

Combining (3.3) and (3.4), we get the result.

4. Radii of Starlikeness, Convexity and Close-to-Convexity

In the following theorems, we obtain the radii of starlikeness, convexity and close-to-convexity for the class $TS(\omega,\sigma,\varsigma,\vartheta)$.

Theorem 4.1. *Let $u \in TS(\omega,\sigma,\varsigma,\vartheta)$. Then u is starlike in $|z| < R_1$ of order δ, $0 \leq \delta < 1$, where*

$$R_1 = \inf_n \left\{\frac{(1-\delta)\big(\omega(n-1)+\varsigma(n\sigma+1-\omega)\big)\Theta(n,\vartheta)}{(n-\delta)\varsigma\big(\sigma+(1-\omega)\big)}\right\}^{\frac{1}{n-1}}, \quad n \geq 2 \qquad (4.1).$$

Proof. u is starlike of order $\delta, 0 \leq \delta < 1$ if

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \delta.$$

Thus it is enough to show that

$$\left|\frac{zu'(z)}{u(z)}-1\right| = \left|\frac{-\sum_{n=2}^{\infty}(n-1)\, a_n z^{n-1}}{1-\sum_{n=2}^{\infty} a_n\, z^{n-1}}\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)\, a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty} a_n\, |z|^{n-1}}. \qquad (4.2)$$

Thus

$$\left|\frac{zu'(z)}{u(z)}-1\right| \leq 1-\delta \;\; if \;\; \sum_{n=2}^{\infty}\frac{(n-\delta)}{(1-\delta)} a_n|z|^{n-1} \leq 1.$$

Hence by Theorem 2.1, (4.2) will be true if

$$\frac{n-\delta}{1-\delta}|z|^{n-1} \leq \frac{\big(\omega(n-1)+\varsigma(n\sigma+1-\omega)\big)\Theta(n,\vartheta)}{\varsigma(\sigma+(1-\omega)}$$

or if

$$|z| \leq \left[\frac{(1-\delta)\big(\omega(n-1)+\varsigma(n\sigma+1-\omega)\big)\Theta(n,\vartheta)}{(n-\delta)\varsigma\big(\sigma+(1-\omega)\big)}\right]^{\frac{1}{n-1}}, n \geq 2.$$

The theorem is proved.

Theorem 4.2. Let $u \in TS(\omega,\sigma,\varsigma,\vartheta)$. Then u is convex in $|z| < R_2$ of order $\delta, 0 \leq \delta < 1$, where $R_2 = \inf_n\left\{\frac{(1-\delta)\big(\omega(n-1)+\varsigma(n\sigma+1-\omega)\big)\Theta(n,\vartheta)}{n(n-\delta)\varsigma\big(\sigma+(1-\omega)\big)}\right\}^{\frac{1}{n-1}}, \ n \geq 2.$ (4.3)

Proof. u is convex of order $\delta, 0 \leq \delta < 1$ if

$$\Re\left\{1+\frac{zu''(z)}{u'(z)}\right\} > \delta.$$

Thus it is enough to show that

$$\left|\frac{zu''(z)}{u'(z)}\right| = \left|\frac{-\sum_{n=2}^{\infty} n\,(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} n\,a_n z^{n-1}}\right| \leq \frac{\sum_{n=2}^{\infty} n\,(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty} n\,a_n|z|^{n-1}}. \tag{4.4}$$

Thus

$$\left|\frac{zu''(z)}{u'(z)}\right| \leq 1-\delta \ \ if \ \ \sum_{n=2}^{\infty}\frac{n(n-\delta)}{(1-\delta)}a_n|z|^{n-1} \leq 1.$$

Hence by Theorem 2.1, (4.4)will be true if

$$\frac{n(n-\delta)}{1-\delta}|z|^{n-1} \leq \frac{\big(\omega(n-1)+\varsigma(n\sigma+1-\omega)\big)\Theta(n,\vartheta)}{\varsigma(\sigma+(1-\omega)}$$

or if

$$|z| \leq \left[\frac{(1-\delta)\big(\omega(n-1)+\varsigma(n\sigma+1-\omega)\big)\Theta(n,\vartheta)}{n(n-\delta)\varsigma\big(\sigma+(1-\omega)\big)}\right]^{\frac{1}{n-1}}, n \geq 2.$$

The theorem proved.

Theorem 4.3. *Let* $u \in TS(\omega,\sigma,\varsigma,\vartheta)$. Then u is close-to-convex in $|z| < R_3$ of order δ, $0 \leq \delta < 1$, where $R_3 = \inf_n\left\{\frac{(1-\delta)\big(\omega(n-1)+\varsigma(n\sigma+1-\omega)\big)\Theta(n,\vartheta)}{n\varsigma\big(\sigma+(1-\omega)\big)}\right\}^{\frac{1}{n-1}}, \ n \geq 2.$ (4.5)

Proof. u is close-to-convex of order $\delta, 0 \leq \delta < 1$ if

$$\Re\{u'(z)\} > \delta.$$

Thus it is enough to show that

$$|u'(z)-1| = \left|-\sum_{n=2}^{\infty} n\,a_n z^{n-1}\right| \leq \sum_{n=2}^{\infty} n\,a_n |z|^{n-1}.$$

Thus

$$|u'(z)-1| \leq 1-\delta \;\; if \;\; \sum_{n=2}^{\infty} \frac{n}{(1-\delta)} a_n |z|^{n-1} \leq 1. \qquad (4.6)$$

Hence by Theorem 2.1, (4.6) will be true if

$$\frac{n}{1-\delta}|z|^{n-1} \leq \frac{\big(\omega(n-1)+\varsigma(n\sigma+1-\omega)\big)\Theta(n,\vartheta)}{\varsigma(\sigma+(1-\omega)}$$

or if

$$|z| \leq \left[\frac{(1-\delta)\big(\omega(n-1)+\varsigma(n\sigma+1-\omega)\big)\Theta(n,\vartheta)}{n\varsigma\big(\sigma+(1-\omega)\big)}\right]^{\frac{1}{n-1}}, n \geq 2.$$

The theorem follows.

5. Extreme Points

In the following theorem, we obtain extreme points for the class $TS(\omega,\sigma,\varsigma,\vartheta)$.

Theorem 5.1. Let $u_1(z)=z$ and $u_n(z) = z - \frac{\varsigma\left(\sigma+(1-\omega)\right)}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\vartheta)} z^n,\; for\; n=2,3,\cdots$. Then $u \in TS(\omega,\sigma,\varsigma,\vartheta)$ if and only if it can be expressed in the form
$u(z) = \sum_{n=1}^{\infty} \theta_n\, u_n(z),\;\; where\;\; \theta_n \geq 0\;\; and\;\; \sum_{n=1}^{\infty}\theta_n = 1.$

Proof. Assume that $u(z) = \sum_{n=1}^{\infty} \theta_n\, u_n(z)$, hence we get

$$u(z) = z - \sum_{n=2}^{\infty} \frac{\varsigma\big(\sigma+(1-\omega)\big)\theta_n}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\vartheta)} z^n.$$

Now, $u \in TS(\omega,\sigma,\varsigma,\vartheta)$, since

$$\sum_{n=2}^{\infty} \frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\vartheta)}{\varsigma\big(\sigma+(1-\omega)\big)} \times \frac{\varsigma\big(\sigma+(1-\omega)\big)\theta_n}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\vartheta)}$$

$$= \sum_{n=2}^{\infty} \theta_n = 1-\theta_1 \leq 1.$$

Conversely, suppose $u \in TS(\omega, \sigma, \varsigma, \vartheta)$. Then we show that u can be written in the form $\sum_{n=1}^{\infty} \theta_n u_n(z)$.
Now $u \in TS(\omega, \sigma, \varsigma, \vartheta)$ implies from Theorem 2.1,

$$a_n \leq \frac{\varsigma\big(\sigma + (1-\omega)\big)}{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\Theta(n, \vartheta)}.$$

Setting $\theta_n = \frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\vartheta)}{\varsigma(\sigma+(1-\omega))} a_n, n = 2,3, \cdots$
and $\theta_1 = 1 - \sum_{n=2}^{\infty} \theta_n$, we obtain $u(z) = \sum_{n=1}^{\infty} \theta_n u_n(z)$.

6. Hadamard product

In the following theorem, we obtain the convolution result for functions belongs to the class $TS(\omega, \sigma, \varsigma, \vartheta)$.

Theorem 6.1. Let $u, g \in TS(\omega, \sigma, \varsigma, \vartheta)$. Then $u * g \in TS(\omega, \sigma, \zeta, \vartheta)$ for

$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ *and* $(u * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n$, where

$$\zeta \geq \frac{\varsigma^2\big(\sigma + (1-\omega)\big)\omega(n-1)}{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]^2 \Theta(n, \vartheta) - \varsigma^2\big(\sigma + (1-\omega)\big)(n\sigma + 1 - \omega)}.$$

Proof. $u \in TS(\omega, \sigma, \varsigma, \vartheta)$ and so

$$\sum_{n=2}^{\infty} \frac{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\Theta(n, \vartheta)}{\varsigma\big(\sigma + (1-\omega)\big)} a_n \leq 1,$$

and

$$\sum_{n=2}^{\infty} \frac{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\Theta(n, \vartheta)}{\varsigma\big(\sigma + (1-\omega)\big)} b_n \leq 1.$$

We have to find the smallest number ζ such that

$$\sum_{n=2}^{\infty} \frac{[\omega(n-1) + \zeta(n\sigma + 1 - \omega)]\Theta(n, \vartheta)}{\zeta\big(\sigma + (1-\omega)\big)} a_n b_n \leq 1.$$

By Cauchy-Schwarz inequality

$$\sum_{n=2}^{\infty} \frac{[\omega(n-1) + \varsigma(n\sigma + 1 - \omega)]\Theta(n, \vartheta)}{\varsigma\big(\sigma + (1-\omega)\big)} \sqrt{a_n b_n} \leq 1. \qquad (6.1)$$

Therefore it is enough to show that

$$\frac{[\omega(n-1)+\zeta(n\sigma+1-\omega)]\Theta(n,\vartheta)}{\zeta\big(\sigma+(1-\omega)\big)}a_nb_n$$
$$\leq \quad \frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\vartheta)}{\varsigma\big(\sigma+(1-\omega)\big)}\sqrt{a_nb_n}.$$

That is

$$\sqrt{a_nb_n}\leq\frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\zeta}{[\omega(n-1)+\zeta(n\sigma+1-\omega)]\varsigma}.$$

From (6.1)

$$\sqrt{a_nb_n}\leq\frac{\varsigma\big(\sigma+(1-\omega)\big)}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\vartheta)}.$$

Thus it is enough to show that

$$\frac{\varsigma\big(\sigma+(1-\omega)\big)}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\vartheta)}\leq\frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\zeta}{[\omega(n-1)+\zeta(n\sigma+1-\omega)]\varsigma},$$

which simplifies to

$$\zeta\geq\frac{\varsigma^2\big(\sigma+(1-\omega)\big)\omega(n-1)}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]^2\Theta(n,\vartheta)-\varsigma^2\big(\sigma+(1-\omega)\big)(n\sigma+1-\omega)}.$$

7. Closure Theorems

We shall prove the following closure theorems for the class $TS(\omega,\sigma,\varsigma,\vartheta)$.

Theorem 7.1. Let $u_j\in TS(\omega,\sigma,\varsigma,\vartheta)$.), *j=1,2,…* Then $g(z)=\sum_{j=1}^{s}c_j\,u_j(z)\in TS(\omega,\sigma,\varsigma,\vartheta)$. For

$u_j(z)=z-\sum_{n=2}^{\infty}a_{n,j}\,z^n$, where $\sum_{j=1}^{s}c_j=1$.

Proof.

$$\begin{aligned} g(z)\quad &=\sum_{j=1}^{s}c_j\,u_j(z)\\ &=z-\sum_{n=2}^{\infty}\sum_{j=1}^{s}c_j\,a_{n,j}z^n\\ &=z-\sum_{n=2}^{\infty}e_n\,z^n, \end{aligned}$$

where $e_n=\sum_{j=1}^{s}c_j\,a_{n,j}$. Thus $g(z)\in TS(\omega,\sigma,\varsigma,\vartheta)$ if

$$\sum_{n=2}^{\infty}\frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\vartheta)}{\varsigma\big(\sigma+(1-\omega)\big)}e_n\leq 1,$$

that is, if

$$\begin{aligned}&\sum_{n=2}^{\infty}\sum_{j=1}^{s}\frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\vartheta)}{\varsigma\big(\sigma+(1-\omega)\big)}c_j a_{n,j}\\ =\ &\sum_{j=1}^{s}c_j\sum_{n=2}^{\infty}\frac{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]\Theta(n,\vartheta)}{\varsigma\big(\sigma+(1-\omega)\big)}a_{n,j}\\ \leq\ &\sum_{j=1}^{s}c_j=1.\end{aligned}$$

Theorem 14. *Let* $u,g\in TS(\omega,\sigma,\varsigma,\vartheta)$. *Then* $h(z)=z-\sum_{n=2}^{\infty}(a_n^2+b_n^2)\,z^n\in TS(\omega,\sigma,\varsigma,\vartheta), where$

$$\zeta\geq\frac{2\omega(n-1)\varsigma^2\big(\sigma+(1-\omega)\big)}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]^2\Theta(n,\vartheta)-2\varsigma^2\big(\sigma+(1-\omega)\big)(n\sigma+1-\omega)}.$$

Proof. Since $u,g\in TS(\omega,\sigma,\varsigma,\vartheta)$, so Theorem 2.1, yields

$$\sum_{n=2}^{\infty}\left[\frac{\big(\omega(n-1)+\varsigma(n\sigma+1-\omega)\big)\Theta(n,\vartheta)}{\varsigma\big(\sigma+(1-\omega)\big)}a_n\right]^2\leq 1$$

and

$$\sum_{n=2}^{\infty}\left[\frac{\big(\omega(n-1)+\varsigma(n\sigma+1-\omega)\big)\Theta(n,\vartheta)}{\varsigma\big(\sigma+(1-\omega)\big)}b_n\right]^2\leq 1.$$

We obtain from the last two inequalities

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{\big(\omega(n-1)+\varsigma(n\sigma+1-\omega)\big)\Theta(n,\vartheta)}{\varsigma\big(\sigma+(1-\omega)\big)}\right]^2(a_n^2+b_n^2)\leq 1. \qquad (7.1)$$

But $h(z)\in TS(\omega,\sigma,\zeta,q,m)$, if and only if

$$\sum_{n=2}^{\infty}\frac{[\omega(n-1)+\zeta(n\sigma+1-\omega)]\Theta(n,\vartheta)}{\zeta\big(\sigma+(1-\omega)\big)}(a_n^2+b_n^2)\leq 1, \qquad (7.2)$$

where $0<\zeta<1$, however (7.1) implies (7.2) if

$$\frac{[\omega(n-1)+\zeta(n\sigma+1-\omega)]\Theta(n,\vartheta)}{\zeta\big(\sigma+(1-\omega)\big)}$$
$$\leq \quad \frac{1}{2}\left[\frac{\big(\omega(n-1)+\varsigma(n\sigma+1-\omega)\big)\Theta(n,\vartheta)}{\varsigma\big(\sigma+(1-\omega)\big)}\right]^2.$$

Simplifying, we get

$$\zeta \geq \frac{2\omega(n-1)\varsigma^2\big(\sigma+(1-\omega)\big)}{[\omega(n-1)+\varsigma(n\sigma+1-\omega)]^2\Theta(n,\vartheta)-2\varsigma^2\big(\sigma+(1-\omega)\big)(n\sigma+1-\omega)}.$$

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