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Product of Semi – Lattices of Certain Graphs

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Abstract:
Introduction: In this article, author tries to construct a relation between graphs of product of meet-semilattices L = L_1 X L_2, where L_1 and L_2 are two semilattices and obtain some properties of such graphs. Author investigated that for meet-semilattices L_1 and L_2 has a cycle of length n-1 and n.

Objectives: author reveals that if L_1 and L_2 be two meet-semilattices with 0 and L = L_1 X L_2, then it is a star graph. In this paper, we have covered some definitions, examples and theorems on zero devisor graph edge of a 4 - cycles or a 5 - cycles. Γ(L) is a star graph.

Methods: Theorem 1.1 [9] The zero-divisor graph of a finite meet-semilattice with only one atom is the empty graph. The zero-divisor graph of the meet semilattice is the empty graph. However, this does not hold for infinite meet-semilattices with one atom. For consider, the infinite meet-semilattice, where the descending dots represent infinite descending chain. It has only one atom c but its graph Γ(L) is an infinite star graph.

Theorem 1.2 [12] Every disconnected graph cannot be a graph of any meet- semilattice L with 0.

Remark 1.2 A graphs of product of meet-semilattices and obtain some properties of such graphs. In this section, we consider two integral meet-semilattices L_1 and L_2 with L ~=L_1 X L_2 and show that if |L_1| = m+1, |L_2| = n+1, then Γ(L) is the complete bipartite graph K (m, n).

Results: If L does not contain any atom, then any edge in Γ(L) is contained in a cycle of length ≤ 6, and therefore Γ(L) is a union of 4 - cycles and 5 - cycles. Let L be a meet-semilattice with 0. If Γ(L) contains a cycle, then the core K of Γ(L) is a union of 4- cycles and 5 – cycles and any vertex in Γ(L) is either a vertex of the core K of Γ(L) or is a pendant of Γ(L). Let L_1 and L_2 be two meet-semilattices with 0 and L = L_1 X L_2. Then exactly one of the following holds:

1. Γ(L) has a cycle of length n-1 or n (that is gr Γ(L)≤ n).

2. Γ(L) is a star graph.

Conclusions: In this article we have studied the concept of the zero-divisor graph derived from meet-semilattice L with 0 on the lines of Anderson and Livingston [6]. Also, we generalized certain results from Demeyer, Mckenzie and Schneider [18] to meet-semilattice L with 0.

Keywords: Lattices, meet semi lattice, star graph.

1. Introduction

Graph theory is an interesting discipline of human enquiry. It has multiple hundred significant subareas. Many researchers worked on two primary areas of arithmetic which are lattice hypothesis and theory hypothesis.

Graph theory is a thriving discipline containing a group of lovely and strong hypotheses of wide appropriateness. Its touchy development is fundamentally because of its job as a fundamental design supporting current applied math, software engineering, combinatorial enhancement and activity research specifically yet in addition to its rising applications in the more applied sciences.

Likewise, graph theory hypothesis is an awesome jungle gym for the investigation of confirmation procedure in Discrete Mathematics and its outcomes have applications in numerous space of the registering, social and innate sciences.

Numerous issues of functional interest can be addressed by theories. The paper composed by Leonhard Euler on the seven Bridge of Konigsberg and distributed in 1736 is viewed as the principal paper throughout the entire existence of diagram hypothesis. This paper, as well as the one composed by Vandermonde on the Knight issues continued with the examination situs started by Leibnitz. Eulers equation relating the quantity of edges, vertices and appearances of a raised polyhedron was examined and summed up by Cauchy and L'Huillies. The principal course reading on diagram hypothesis was composed by Denes Konig which is distributed in 1936.

Garrett Birkhoff's work during the thirties of the 20th century began the overall improvement of lattice hypothesis. In a progression of papers, he exhibited the significance of lattice hypothesis. Lattice hypothesis assumes a significant part in numerous areas of math for instance - Boolean variable-based math, rationale and different regions like exchanging hypothesis, software engineering, quantum mechanics. An alternate part of lattice hypothesis concerns the underpinning of set hypothesis (counting geotherm and genuine investigation).

Lattice have a few associations with the group of gathering like designs since meet and join both are commutative and affiliated a cross section can be seen as comprising of two commutative semigroups having a similar space. For a limited cross section, these semigroups are truth be told commutative monoids.

The logarithmic translation of Lattice plays and fundamental job in widespread polynomial math. The magnificence of cross section hypothesis gets to some extent from the outrageous straightforwardness of its fundamental ideas for instance poset, least upper bound, most noteworthy lower bound and so forth. The investigation of logarithmic chart hypothesis is an intriguing subject for mathematicians and returns basically to 1973, when N. Biggs see [9] distributed his book on Algebraic diagram hypothesis. As he wrote in the prelude of his book, his point was "to make an interpretation of properties of charts into mathematical properties and afterward utilizing the outcomes and strategies for variable based math, to derive hypotheses about diagrams". Despite the fact that Biggs talked about mathematical techniques and variable based math as a rule, the sort of variable based math he truly utilized was straight variable based math and a few properties of polynomials.

In 1993 Anderson and Naseer [2] addressed this issue adversely and gave a counter model. They further concentrated on the zero divisor graphs of commutative rings by altering the Beck's [7] definition.

They considered just nonzero zero-divisors as vertices of graphs. In 1999 Anderson and Livingston [4] changed the meaning of the zero-divisor graph, characterizing the vertices of the graph to be the nonzero zero-divisors of the commutative ring.

Afterward, Demeyer, Mckenzie and Schneider in [10] concentrated on graphs on commutative semigroup with 0. This review for semigroups was gone on by Demeyer and Demeyer in [9]. Therefore, research has moved in a few headings for instance, Anderson, R. Levy and J. Shapiro in [5] broadened the outcomes by taking a gander at the coterie number what's more, planarity of zero-divisor charts, while R. Akhtar and L. Lee in [1] explored the properties vital for a zero-divisor diagram to be either a planar or a total r - partite. Redmond in [14], took a gander at a portion of the progressions inferred by the zero-divisor diagram on a noncommutative ring. The zero-divisor chart of a commutative ring has additionally been concentrated in ([3], [12], [13], [16]) and the zero-divisor diagram idea has been reached out to noncommutative ring in [15]. Mulay in [13] utilizing Anderson and Livingston's in [4] meaning of the zero-divisor graph examined the cycle construction of Γ(R). M. Axtell et al. in [3] look at the conservation of diagram hypothetical properties of the zero-divisor graph under expansion to polynomial and power series rings.

2. Objectives

Definition 1.1. [8] A meet-semilattice (or lower semilattice) is a partially ordered set which has a meet (or greatest lower bound) for any nonempty finite subset.

Remark 1.1. Every join-semilattice is a meet-semilattice in the inverse order and vice versa.

Definition 1.2. [6] An element $a \in L$ is called a zero-divisor if there exists a nonzero element $b \in L$ such that $a \wedge b = 0$. We denote by Z(L) the set of all zero- divisors of L.

Definition 1.3. [10] A graph Γ(L) to L with vertex set $Z*(L) = Z(L) - \{0\}$, the set of all non-zero zero-divisors of L. Two distinct $x, y \in Z*(L)$ are adjacent if and only if $x \wedge y = 0$ and call this graph as the zero-divisor graph of L.

Remark 1.2 Γ(L) is connected with diam $\Gamma(L) \leq 3$ and if Γ(L) contains a cycle, then $gr\Gamma(L) \leq 4$. We show that if Γ(L) contains a cycle, then the core K of Γ(L) is a union of 3 - cycles and 4 - cycles. Moreover, any vertex in Γ(L) is either a vertex of the core K of Γ(L) or else is a pendant vertex of Γ(L). It is also shown that if L does not contain any atom, then every pair of vertices in Γ(L) is contained in a cycle of length ≤ 6.

Definition 1.4. [1] Let $(L, \leq)$ be a meet-semilattice. For any $a, b \in L$ either $a \leq b$ or $b \leq a$ holds then $(L, \leq)$ is called a chain.

Definition 1.5. [2] In a lattice L with 0, a nonzero element $a \in L$ is called an atom if there is no $x \in$ L such that $0 < x < a$.

Definition 1.6. [4] Let G be a graph. For distinct vertices x and y of G, let d (x, y) be the length of the shortest path from x to y; ($d(x, y) = \infty$ if there is no such path). The diameter of G is diam $G = sup\{d(x, y) \mid x$ and y are distinct vertices of $G\}$.

Definition 1.7. [6] The graph of G, denoted by $gr(G)$, is defined as the length of the shortest cycle in G. ($gr(G) = \infty$ if G contains no cycles).

Definition 1.8. [11] A graph G is called a star graph if it has a vertex adjacent to every other vertex and these are the only adjacency relations.

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3. Methods

Theorem i1.1 [9] iThe izero-divisor igraph iof ia ifinite imeet-semilattice iwith ionly ione iatom iis ithe iempty igraph. iThe izero-divisor igraph iof ithe imeet semilattice iin iFigure i1.1 iis ithe iempty igraph.

However, ithis idoes inot ihold ifor iinfinite imeet-semilattices iwith ione iatom. iFor iconsider, ithe iinfinite imeet-semilattice igiven iin iFigure i1.2, iwhere ithe idescending idots irepresent iinfinite idescending ichain. iIt ihas ionly ione iatom ic, ibut iits igraph i$\Gamma(L)$ iis ian iinfinite istar igraph.

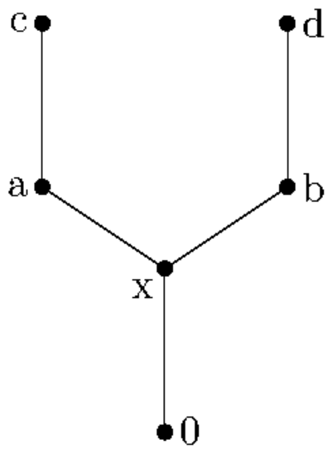

Figure 1.1

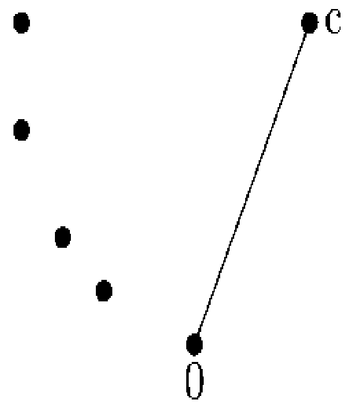

Figure 1.2

Theorem 1.2 [12] Every disconnected graph cannot be a graph of any meet- semilattice L with 0.

Remark 1.2 A graphs of product of imeet-semilattices and obtain some properties of such graphs. In this section, we consider two integral meet-semilattices L_1 and L_2 with L ∼=L_1 X L_2 and show that if |L_1|=m+1, |L_2|= n+1, then $\Gamma(L)$ is the complete bipartite graph K (m, n). Also, it is shown that gr ($\Gamma(L)$) = ∞ if and only if either (i) | $\Gamma(L)$ | ≤ 2 or (ii) | $\Gamma(L)$ | = 3 and $\Gamma(L)$ is not complete or (iii) L ∼ = 〖C〗_2 X L_1, where L_1 an integral meet-semilattice and C_2 is a two-element chain. In this case, $\Gamma(L)$ is a star igraph. Further $\Gamma(L)$ has a cycle of length 3 or 4 (i.e. gr $\Gamma(L) \leq 4$), and $\Gamma(L)$ is a star graph.

Results

Theorem 2.1. If L does not contain any atom, then any edge in $\Gamma(L)$ is contained in a cycle of length ≤ 6, and therefore $\Gamma(L)$ is a union of 4 - cycles and 5 - cycles.

Corollary 2.1. For any meet-semilattice L, let K, the core (a notion that describes behavior of a graph) of $\Gamma(L)$, be the union of cycles in $\Gamma(L)$.

Theorem 2.2. Let L be a meet-semilattice with 0. If $\Gamma(L)$ contains a cycle, then the core K of $\Gamma(L)$ is a union of 4- cycles and 5 – cycles and any vertex in $\Gamma(L)$ is either a vertex of the core K of $\Gamma(L)$ or is a pendant of $\Gamma(L)$.

Remark 2.1. Let L_1 and L_2 be two meet-semilattices with 0 and L = L_1 X L_2, then $\Gamma(L)$ is star graph if and only if one of the L_1 or L_2 is C_2 and the other is an integral meet-semilattice.

Theorem 2.3. Let L_1 and L_2 be two meet-semilattices with 0 and L = L_1 X L_2. Then exactly one of the following holds:

1. Γ(L) has a cycle of length n-1 or n (that is gr Γ(L)≤ n),

2. Γ(L) is a star graph.

4. Discussion

Theorem 2.1. If L does not contain any atom, then any edge in Γ(L) is contained in a cycle of length ≤ 6, and therefore Γ(L) is a union of 4 - cycles and 5 - cycles.

Proof. Let a− x be an edge in Γ(L). Since Γ(L) is connected and | Γ(L)| ≥ 4, there exists a vertex b in Γ(L) with a − x − b or x − a – b is a path in Γ(L).

In the first case, if b ∧ a = 0 then a − x − b − a is a 4 - cycle. If b ∧ a ≠ 0,

since x is not an atom then there exists a nonzero c < x. Then a ∧ c = 0, b ∧ c = 0. Hence, a − x − b − c – d – a is a cycle of length 4. Thus, x is contained in a cycle of length ≤ 4, so a − x is an edge of either a 4 - cycles or a 5 - cycles. In the second case, if x ∧ b = 0 then a − x − b − c – d – a is a 4 - cycle. If x ∧ b ≠ 0, since a is not an atom then there exists a nonzero d < a. Then d ∧ x = 0, d ∧ b = 0.

Hence d − x − a − b – c – d is a cycle of length 6. Thus, a is contained in a cycle of length ≤ 6, so a−x is an edge of a 4 - cycle. Hence a−x is an edge of a 4 - cycles or a 5 - cycles.

Corollary 2.1. For any meet-semilattice L, let K, the core (a notion that describes behavior of a graph) of Γ(L), be the union of cycles in Γ(L).

Theorem 2.2. Let L be a meet-semilattice with 0. If Γ(L) contains a cycle, then the core K of Γ(L) is a union of 4- cycles and 5 – cycles and any vertex in Γ(L) is either a vertex of the core K of Γ(L) or is a pendant of Γ(L).

Proof. Let a1 ∈ K and suppose that a1 does not belong to any 4 – cycles or a 5 - cycle in Γ(L). Then a_1 is in some in – cycle a_1, a_2, a_3 ⋯a_n, a_1 with in ≥ 6. By Theorem 2.1, a_1 is an atom in L. Then a_1 ≤ a_5 implies that a_1∧a_4 = 0, which is a contradiction.

Hence Γ(L) is a union of 4- cycles and 5 – cycles.

Now suppose that a is any vertex in Γ(L). If a ∉K and a is not a pendant vertex then the following possibility holds.

(i) a is contained in a path of the form x − y − a – b – c with c ∈ K or

(ii) a is contained in a path of the form x − a – b – c with c ∈ K.

Since c ∈ K, b is contained in a 4 - cycles or a 5 - cycles, say b − c − d − e − b

or b − c − d − e – a – b.

In (i), we get d (x, c) = 4, contradicts | Γ(L)| ≤ 4. Hence (i) cannot hold.

In (ii), we get x − a − b − c − d − b or x − a − b − c − d − e − b.

Hence by Theorem 2.1., a must be an atom. Therefore, a ∧ c = a.

This gives a ∧ d = 0, a contradiction as a ∉K. Thus (ii) cannot hold.

Hence either a ∈ K or a is a pendant vertex.

Remark 2.1. Let L_1 and L_2 be two meet-semilattices with 0 and L = L_1 X L_2, then Γ(L) is star graph if and only if one of the L_1 or L_2 is C_2 and the other is an integral meet-semilattice.

Remark 2.2. For any star graph with n elements there corresponds a meet-semilattice as in figure 2.1

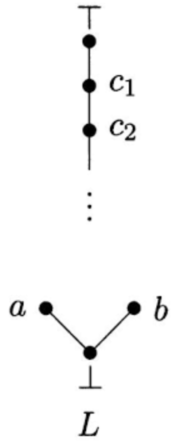

Figure 2.1

Theorem 2.3. Let L_1 and L_2 be two meet-semilattices with 0 and L = L_1 X L_2. Then exactly one of the following holds:

1. Γ(L) has a cycle of length n-1 or n (that is gr Γ(L)≤ n),

2. Γ(L) is a star graph.

Proof. Suppose L = L_1 X L_2, where at least one of L_1 and L_2 is not an integral meet-semilattice, say L_1 is not an integral (join) meet -semilattice.

Then there exists nonzero a, b ∈ L_1, with a ∧ b = 0 and

choose nonzero c ∈ L_2. Then (a, 0) − (b, 0) − (0, c) –…. form a cycle of length n-1 in Γ(L).

Let L = L_1 X L_2, where L_1 and L_2, both are integral meet semi-lattices with

|L_1| > n-2, |L_2| > n-2. Let a, b ∈ L_1, and c, d ∈ L_2 be ia non zero elements. Then

(a, 0) − (0, c) − (b, 0) − (0, d) − … form a cycle of length in Γ(L).

Let L ∼= L_1 X L_2, where either |L_1| = n- 2 or |L_2| = n-2.

Let |L_1| = n-2 and L_2 is an integral meet-semilattice then by Theorem 2.2, Γ(L) is a star graph.

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