Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 1 (2024)

Fixed Point Results Under Hausdorff Distance in the Fractal Spaces

Rita Pal[1], J. Leo Amalraj[2], V. Venkata Kumar[3], G. Venkat Narayanan[4]

[1]Department of Applied Mathematics
Bhilai Institute of Technology,Bhilai,India
Email ID: ritapal001@gmail.com
[2]Department of Science and Humanities
RMK College of Engineering and Technology
Puduvoyal, Thiruvallur District, Tamilnadu, India
Email ID: leoamalraj@rmkcet.ac.in
[3] Department of Mathematics,
Aditya Engineering College, Surampalem, India,
Email ID: venkatakumar.v@aec.edu.in
[4]Department of Mathematics
St. Joseph's College of Engineering, OMR, Chennai
District Kanchipuram,Tamil Nadu, India
Email ID: gvenkatnarayanan@gmail.com

Article History:

Received: *14-10-2023*

Revised: *28-11-2023*

Accepted: *18-12-2023*

Abstract:

We establish a novel notion of Hausdorff distance and explore some of its topological characteristics by using an extended modular metric. We prove a fixed point theorem on generalized modular fractal space from the concept of iterated function system (IFS) and contraction.

Keywords: Fixed points; Hausdorff ; fractal space; contraction ; IFS ; Modular metric spaces.
2000 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

The Metric modular space structure was modified by Chistyakov (Chistyakov, 2008; Cho, Saadati and Sadeghi, 2012), in an insightful way and proposed the Hausdorff topology on it, is extremely well-liked in modern study. Now, using nonempty compact subsets, we investigate the Hausdorff distance for a certain (GMMS). In order to demonstrate an intriguing (FPT) fixed point theorem, on a generalised metric modular space, we apply the iterated function system (IFS) and idea of contraction together (Abdou, 2016; Abdou, 2020; Chistyakov, 2008; Chistyakov, 2010; Cho et al., 2012; Ege, Park and Ansari, 2020). Hutchinson studied iterated function system (IFS) and thought about the idea of fractal theory (Hutchinson, 1981). By Ri (Ri, 2016),Barnsley (Good, 1990), Bisht (Bisht, 2018), and Imdad (Imdad, Alfaqih and Khan, 2018), this topic was generalized. A singular nonempty compact set F and $F = \bigcup_{i=1}^{m} Q$i (F) of the complete GMM space (L,T) for a GMMIFS , then a fractal set F is known as the attractor of the relevant generalized modular metric iterated function system. The associated attractor generalized modular metric iterated function system in this context is referred to as generalized modular metric fractal space.

2 Preliminiers

Now let's review some ideas and fundamental principles. Here, we let P = [0, 1], P^0 = (0, 1), Q = [0, ∞), Q^0 = (0, ∞) and a set L ≠ ϕ.

Definition 2.1 (Azadifa, Maramaei and Sadeghi, 2013) A function T: L × L × L× Q^0 → Q is referred to as generalized metric modular (GMM) on L a no -empty set , if it follows the axioms listed below:

(GMM-1) $T_\rho(l, l, n) \in Q^0$ *for all* $l, n \in L$ *and* $\rho \in Q^0$ *with* $l \neq n$.

(GMM-2) $T_\rho(l, m, n) = 0$, $\forall \rho \in Q^0$ *if* $l = m = n$.

(GMM-3) $T_\rho(l, l, n) \leq T_\rho(l, m, n)$, $\forall \rho \in Q^0$ *if* $m \neq n$.

(GMM-4) $T_\rho(l, m, n) = T_\rho(l, n, m) = T_\rho(n, l, m)$ *and so on.*

(GMM-5) $T_{\rho+\delta}(l, m, n) \leq T_\rho(l, v, v) + T_\delta(v, m, v)$ *for all* $\rho, \delta \in Q^0$.

Then, (L, T) is referred to as a generalized modular metric on L .

Definition 2.2 (Azadifa et al., 2013) Let us set $l_0 \in L$ and $L_T = \{m \in L; \lim_{\rho\to 0} T_\rho(l_0, m, n) = 0$ for some $n \in L\}$. The set L_T is known as a modular set.

Definition 2.3 (Azadifa et al., 2013) Assume that (L, T) be a generalised modular metric (GMM) space. Then, for $l_0 \in L_T$ and $c > 0$, the T-ball with radius c and center l_0 is $B_T(l_0, c) = \{m \in L_T : T_\rho(l_0, m, m) < c\}$, $\forall \rho > 0$.

Proposition 2.3.1 (Azadifa et al., 2013) Assume that (L, T) *be a* generalised modular metric (GMM) space. Then for $l_0 \in L_T$ and $c > 0$,

(i) *if* $T_\rho(l_0, l, m) < c$, $\forall \rho > 0$, *then* $l, m \in B_T(l_0, c)$.

(ii) *if* $m \in B_T(l_0, c) : B_T(m, \delta) \subseteq B_T(l_0, c)$ and $\delta > 0$.

Definition 2.4 (Azadifa et al., 2013) Assume that (L, T) is a generalized (GMMS) modular metric space. Sequence $\{l_n\} \subseteq$ L and T_l is *T*- convergent to *l* if it converges to *l* of $\tau(T_\rho)$, ∀ n∈ N .

Proposition 2.4.1 (Azadifa et al., 2013) Assume that (L, T) is a generalized (GMMS) modular metric space and sequence $\{l_n\} \subseteq L_T$, ∀ n ∈ N. Then the followings are satisfied :

(1) Sequence$\{l_n\}$ *is T-convergent to l.*

(2) $\sigma_\rho^T(l_n, l) \to 0$ *when* $n \to \infty$,

(3) $T_\rho(l_n, l_n, l) \to 0$ *when* $n \to \infty$ *for all* $\rho > 0$;

(4) $T_\rho(l_n, l, l) \to 0$ *as* $n \to \infty$ *for all* $\rho > 0$;

(5) $T_\rho(l_m, l_n, l) \to 0$ *when* $m, n \to \infty$, $\forall \rho > 0$.

Definition 2.5 (Azadifa et al., 2013) Assume that (L,T) is a generalized (GMMS) modular metric space. Then sequence $\{l_n\} \subseteq L_T$, is called T- Cauchy sequence if, $N_\varepsilon \in$ N : $T_\rho(l_n, l_m, l_q) < \varepsilon$, ∀

, $n, m, q \geq N_\varepsilon$ and for every $\varepsilon, \rho > 0$. If every T- Cauchy sequence in a GMM-space L is a T- convergent sequence in that space, the space is said to be " T-complete."

Proposition 2.5.1 (Azadifa et al., 2013) Assume that (L,T) is a generalized(GMMS) modular metric space and sequence $\{l_n\} \subseteq L_T$ for all n∈N. Then the followings are equivalent:

(1) Sequence $\{l_n\}$ *is a T-Cauchy* sequence.

(2) We can locate $N_\varepsilon \in$ N: $T_\rho(l_n, l_m, l_m) < \varepsilon$, *for each* $\varepsilon > 0$, $\rho > 0$, *for every* $n, m \geq N_\varepsilon$.

(3) Sequence$\{l_n\}$ *is a Cauchy sequence.*

Proposition 2.5.2 (Azadifa et al., 2013) Assume that (L, T) is a (GMM) space. Then for $L \times L \times L \times$ Q^0 , T is a continuous function. Let us assume that a GMM-space (L, T) has two (nonempty) subsets, T and W. $T_\rho(l, T, W) = \inf\{T_\rho(l, t, w)$: $t \in T, w \in W\}$ for $l \in L$ *and* $\rho > 0$,

Proposition 2.5.3 (Azadifa et al., 2013) Assume that (L, T) is a generalized modular metric space . For each M , N, P ∈ H_0(L), the function δ |→ $\sup_{m\in M} T_\rho$(m, N, P) is continuous on Q^0

Proposition 2.5.4 (Alihajimohammad and Saadati, 2021) Assume that (L,T) is a generalized (GMMS) modular metric space . Suppose sequence $\{l_n\} \subseteq L$: $T_\varphi n(\rho)(l_n, l_{n+1}, l_{n+1}) \leq T_\rho(l_0, l_1, l_1)$ for all $\rho \in$ Q^0. Then$\{l_n\}$ is a T-Cauchy sequence.

Proposition 2.5.5 (Alihajimohammad and Saadati, 2021) Let (L,T) is a generalized (GMM) modular metric space. If $T_\rho(l, m, n) = C$ for all l,m,n ∈ L and $\rho \in$ Q^0, then C = 0.

Lemma 2.6 Let (L, T) is a (GMM) space. Then, for each l ∈ L, M , $N \in$ H_0(L) *and* $\rho \in$ Q^0, there are $m_0 \in M$, $n_0 \in N$ *such that* $T(l, M, N) = T_\rho(l, m_0, n_0)$.

Proof Let $l \in L$, $M, N \in H_0(L)$ and $\rho > 0$. By Proposition 2.5.2. the functions t, u |→ $T_\rho(l, m, n)$ are continuous. Thus, by compactness of M and N, $\exists\, m_0 \in M, n_0 \in N$: $\inf T_\rho(l, m, n) = T_\rho(l, m_0, n_0)$, for all $m \in M$, $n \in N$

Lemma 2.7 Assume that (L,T) is a generalized (GMMS) modular metric space .*Then, for every* $M \in H_0(L)$, $N, P \in F_0(L)$ *and* $\rho \in$ Q^0 we can find $m_0 \in M$ *such that* $\sup T_\rho(M, N, P) = T_\rho(m_0, N, P)$.

Proof Put $\delta = \sup_{t\in T} T_\rho(m, N, P)$. Then we get a sequence $(m_n)_n$ in M : $\delta - \frac{1}{n} < T_\rho(m_n, N, P)$ in which $n \in$ N. From $M \in H_0(L)$, a subsequence $(t_{n_k})_k$ of $(m_n)_n$ and $m_0 \in M$: $m_{n_k} \to m_0$ in (L, T).

Select $n \in N$, $p \in P$. From the Proposition 2.5.2, we get $\lim_k T_\rho(m_{n_k}, n, p) = T_\rho(m_0, n, p)$.

Since, for each $k \in$ N, $\delta - \frac{1}{n_k} < T_\rho(m_{n_k}, n, p)$, we get $\delta \leq T_\rho(m_0, n, p)$.

We conclude $\delta = T_\rho(m_0, N, P)$.

Definition 2.8 (Alihajimohammad and Saadati, 2021) Let (L,T) is a generalized (GMM) modular metric and $Q : L \to L$ is called a GMM-φ-contractive mapping if $T_{\varphi(\rho)}(Q(l), Q(t), Q(u)) \leq T_\rho(l, m, n)$ for every $l, m, n \in S$ and $\rho \in$ Q^0.

Definition 2.9 (Alihajimohammad and Saadati, 2021) A generalized modular metric -φ-contractions $\{Q_1, Q_2, \ldots, Q_m ,: m \geq 2\}$ is a finite set on a complete GMM-space (L, T) is known as a generalized modular metric iterated function system .

3 Main Results: Mathematical Theorem

Let (L, T) is a generalized (GMM) modular metric space and assume the followings:

$F_0(L)$ = nonempty subsets of L,

$G_0(L)$ = nonempty finite subsets and

$H_0(L)$ = nonempty compact subset of L.

Then a function H_T on $H_0(L) \times H_0(L) \times H_0(L) \times Q^0$ is defined by

$$H_T(M, N, P, \rho) = \max\{\sup_{m\in M} T_\rho(m, N, P), \sup_{n\in N} T_\rho(M, n, P), \sup_{p\in P} T_\rho(M, N, p)\}$$

for every $M, N, P \in H_0(L)$ and $\rho \in Q^0$.

Lemma 3.1 *Let* (L, T) *is a* generalized (GMM) modular metric space

$l \in L$, M, $N \in H_0(L)$, $P \in F_0(L)$, and $\alpha, \beta \in Q^0$. Then

$$T_{\alpha+\beta}(l, M, P) \leq T_\alpha(l, N, N) + T_\beta(u_l, M, P),$$

where $n_l \in N$ satisfies $T_\alpha(l, N, N) = T_\alpha(l, n_l, n_l)$.

Proof: Using Lemma 2.6, $T_\alpha(l, N, N) = T_\alpha(l, n_l, n_l)$.

For each $m \in M, p \in P$, we have

$$T_{\alpha+\beta}(l, M, P) \leq T_{\alpha+\beta}(l, m, p) \leq T_\alpha(l, n_l, n_l) + T_\beta(n_l, m, p).$$

Then $T_{\alpha+\beta}(l, M, P) \leq T_\alpha(l, N, N) + T_\beta(n_l, M, P)$

Theorem 3.2 Let (L,T) be a generalized modular metric (GMM) space. Then $(H_0(L), H_T)$ *is a* generalized (GMMS) modular metric space.

Proof : Let M, N, P, W $\in H_0(L)$ and $\alpha, \beta \in Q^0$. By Lemma 2.7, there exist

$m_0 \in M$, $n_0 \in N$, and $p_0 \in P$ such that: $\sup_{m\in M} T(m, N, P) = T(m_0, N, P)$, $\sup_{n\in N} T(M, n, P) = T(M, n_0, P)$, and $\sup_{p\in P} T(M, P, p) = T(M, P, p_0)$.

Then $H_T(M, N, P, \alpha) \geq 0$.

Moreover,it is clear that

$M = N = P \Leftrightarrow H_T(M, N, P, \alpha) = 0$

Then from Lemma 3.1 we have

$$\text{Sup}_{m\in M} T_{\alpha+\beta}(m, N, W) \leq \text{Sup}_{m\in M} T_\alpha(m, P, P) + \text{Sup}_{m\in M} T_\beta(pm, N, W)$$

since $\{p_m : m \in M\} \subseteq P$, $\text{Sup}_{m\in M} T_\beta(Pm, N, W) \leq \text{Sup}_{p\in P} T_\beta(p, N, W)$

$\text{Sup}_{m\in M}\ T_{\alpha+\beta}$ (m, N, W) $\leq \text{Sup}_{m\in M}\ T_{\alpha}$ (m, P, P) + Sup $_{p\in P}\ T_{\beta}$ (p, N, W)

In the same way, we obtain

$$\sup_{n\in N} T_{\alpha+\beta}(M, n, W) \leq \sup_{n\in N} T_{\alpha}(n, P, P) + \sup_{p\in P} T_{\beta}(p, M, W),$$

$$\sup_{w\in W} T_{\alpha+\beta}(M, N, w) \leq \sup_{w\in W} T_{\alpha}(w, P, P) + \sup_{p\in P} T_{\beta}(p, M, W).$$

Therefore, it is obvious to conclude that

$$H_T(M, N, W, \alpha+\beta) \leq H_T(M, P, P, \alpha) + H_T(P, N, W, \beta).$$

$\alpha \mapsto H_T(M, N, P, \alpha)$ is continuous on Q^0, by the Proposition 2.5.3.

Then $(H_0(L), H_T)$ is a Generalized modular metric space.

Theorem 3.3 Let (L,T) is a (GMM) space .A function $Q : L \to L$ is given by

: $T_{\varphi(\rho)}$ $(Q(l), Q(m), Q(n)$ $\leq T_{\rho}(l, m, n)$, *for all* $\rho \in Q^0$ and $l, m, n \in L$. Then the sequence $Q^n(l)_{n=1}^{\infty}$ is generalized modular metric complete space .

Proof Let $\{l_n$: $Q^n(l)\}_{n=1}^{+\infty}$ }, $\{l_n\}$ is a sequence that complies with the requirements of proposition 2.5.4

$T_{\rho}(l, Q(l), Q(l)$ $\leq T_{\rho}(l, Q(l), Q(l))$ (using the induction)

If $T_{\varphi n(\rho)}(Q^n(l), Q^{n+1}(l), Q^{n+1}(l)) \leq T_{\rho}(l, Q(l), Q(l))$

then

$$T_{\varphi^{n+1}(\rho)}(Q^{n+1}(l), Q^{n+2}(l), Q^{n+2}(l)) = T_{\varphi(\varphi n(\rho))}(Q(Q^n(l), Q(Q^{n+1}(l)))$$

Now we have

$$Q\ (Q^{n+1}(l)) \leq T_{\varphi n(\rho)}(Q^n(l), Q^{n+1}(l), Q^{n+1}(l))$$

$$\leq T_{\rho}(l, Q(l), Q(l))$$

Therefore,

$$T_{\varphi n(\rho)}(l_n, l_{n+1}, l_{n+1}) \leq T_{\rho}(l_0, l_1, l_1),$$

Hence $Q^n(l)_{n=1}^{\infty}$ is generalized modular metric complete space (GMMCS).

Theorem 3.4 Suppose (L,T) is a generalized (GMM) space and map Q ,*GMM-φ-* contractive mapping *for all* $\rho \in Q^0$ and $l, m, n \in$ L :

$$T_{\varphi(\rho)}(Q(l), Q(t), Q(u) \leq T_{\rho}(l, m, n)$$

for every $l, m, n \in L$ and $\rho \in Q^0$.Then in L, Q posseses *a* unique fixed point .

Proof Now from the above theorem 3.3 , we get $\{l_n$: $\{Q^n(l)\}\}_{n=1}^{+\infty}$ }is generalized(GMMCS) modular metric complete space , for each $l \in L$ and $\lim_{n\to\infty} Q^n(l) = x \in$L .

Letting $l_0 = l$ and $l_n = Q^n(l)$ for each $n \geq 1$, since $\lim_{n\to\infty} Q^n(l) = x$, we have $\lim T_{\rho}(l_n, x, x) = 0$ for each $\rho \in Q^0$.

On the other hand, we recognize

$$T_{\varphi(\rho)}(Q(x), l_{n+1}, l_{n+1}) \leq T_{\rho}(x, l_n, l_n) \text{ for each } n \in N \text{ and each } \rho > 0.$$

$$\text{Then } T_{\varphi(\rho)}(Q(x), x, x) = \lim_{n\to\infty} . T_{\varphi(\rho)}(Q(x), l_{n+1}, l_{n+1})$$

$$\leq \lim_{n\to\infty} . T_{\rho}(x, l_n, l_n) = 0, \text{for each } \rho > 0.$$

$$\Rightarrow x = Q(x),$$

Now , To prove : *U*niqueness

Let $y \in Q$ is another point, and $\rho \in Q^0$

$$T_{\rho}(x, x, y) = T_{\rho}(Q(x), Q(x), Q(y)) \geq T_{\varphi(\rho)}(Q(x), Q(x), Q(y)) .$$

since $T_{\rho}(l, m, m)$ is nonincreasing and $\varphi(\rho) < \rho$,

we have

$$T_{\varphi(\rho)}(Q(x), Q(x), Q(y)) \geq T_{\rho}(Q(x), Q(x), Q(y)) = T_{\rho}(x, x, y).$$

Hence $T_{\rho}(x, x, y) = C$

From proposition 2.5.5, we get $C = 0$.

Therefore, $x = y$.

4. Conclusions

We have studied certain topological aspects of the Hausdorff distance on GMM and defined a (GMFS) in the sense of Chistyakov by iterated function system. as anapplication Some concepts of fixed point have been implemented in generalized modular metric space and generalized modular metric fractal space (GMMF-space).

5. Acknowledgement

The Authors are grateful to the knowledgeable referee for his insightful observations and comments, which substantially assisted us in significantly improving the manuscript.

References

[1] Abdou, A. A. 2016. Some fixed point theorems in modular metric spaces, J. Nonlinear Sci. Appl 9(6): 4381–4387.

[2] Abdou, A. A. 2020. Fixed points of kannan maps in modular metric spaces, AIMS Mathematics 5(6): 6395–6403.

[3] Alihajimohammad, A. and Saadati, R. 2021. Generalized modular fractal spaces and fixed point theorems, Advances in Difference Equations 2021(1): 1–10.

[4] Azadifa, B., Maramaei, M. and Sadeghi, G. 2013. On the modular g-metric spaces and fixed point theorems., Journal of Nonlinear Sciences & Applications (JNSA) 6(4).

[5] Bisht, R. K. 2018. Comment on: A new fixed point theorem in the fractal space, Indagationes Mathematicae 29(2): 819–823.

[6] Chistyakov, V. V. 2008. Modular metric spaces generated by f-modulars, Folia Math 14(3).
[7] Chistyakov, V. V. 2010. Modular metric spaces, i: basic concepts, Nonlinear Analysis: Theory, Methods & Applications 72(1): 1–14.
[8] Cho, Y. J., Saadati, R. and Sadeghi, G. 2012. Quasi-contractive mappings in modular metric spaces, Journal of Applied Mathematics 2012.
[9] Ege, O., Park, C. and Ansari, A. H. 2020. A different approach to complex valued gb g {b}-metric spaces, Advances in Difference Equations 2020(1): 1–13.
[10] Good, I. 1990. Fractals everywhere (michael barnsley).
[11] Hutchinson, J. E. 1981. Fractals and self similarity, Indiana University Mathematics Journal 30(5): 713–747.
[12] Imdad, M., Alfaqih, W. M. and Khan, I. A. 2018. Weak θ-contractions and some fixed point results with applications to fractal theory, Advances in Difference Equations 2018(1): 1– 18.
[13] Ri, S.-i. 2016. A new fixed point theorem in the fractal space, Indagationes Mathematicae 27(1): 85–93.