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Results on Fixed Points in Vector Valued G-Metric Spaces

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Abstract:

Introduction: Fixed point theory furnishes a robust framework in mathematics for examining the function and system behavior by centering on fixed points. Kirk et al. brought forth the concept of cyclic transformation in 2003. Cyclic contraction broadens the standard contraction transformations in metric space. By relaxing the requirement of the triangle inequality, G-metric space extends the scope of metric spaces. Vector G-metric space is GMS in which the metric is lattice valued.

Objectives: We give the concept of a vector G metric space and define some related concepts. We establish some fixed point results on vector G-metric space by using cyclic contraction. To underscore the practical implications and importance of our findings, we offer a range of examples and corollaries.

Methods: This research paper outlines fixed point theorems for self-transformations in vector G-metric space, with the help of cyclic contraction.

Conclusions: Within this manuscript, we illustrate fixed point theorems for self-transformations in V-complete vector G-metric space by utilizing cyclic contraction. These outcomes are expected to inspire researchers to explore problem-solving opportunities in diverse areas such as differential equations and functional analysis.

Keywords: Cyclic contraction, Vector G-metric space, Vector lattice.

Subject Classification:(2010) 47H10, 47H07

1. Introduction.

Fixed point theory furnishes a robust framework in mathematics for examining the function and system behavior by centering on fixed points. Kirk et al. **Error! Reference source not found.** brought forth the concept of cyclic transformation in 2003. Cyclic contraction(CC) broadens the standard contraction transformations in metric space. It is utilized for exploring transformations and fixed points.The exploration of fixed point theorems(FPT) for cyclic transformations have been widely studied (defined by **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**). By relaxing the requirement of the triangle inequality, G-metric space(GMS) extends the scope of metric

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spaces. Vector G-metric space(VGMS) is GMS in which the metric is lattice valued. This research paper outlines FPT for self-transformations in VGMS, with the help of CC.

We lay out a series of definitions and examples that will be applicable in the forthcoming portion. For an additional comprehensive analysis of the results regarding the vector lattice and GMS, we may refer to (**Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**, **Error! Reference source not found.**).

Definition 1.1[5] Let $\mathfrak{R}$ be a non-null set and consider $\{\chi_\hbar\}_{\mu=1}^{m}$ as a collection of non-void subsets of $\mathfrak{R}$ with $\mu = \cup_{\hbar=1}^{m} \chi_j$. A transformation $\complement: \mu \to \mu$ is termed a cyclic transformation if

$$\complement(\chi_\hbar) \subseteq \chi_{\hbar+1} \quad \hbar = 1,2,\dots,m, \quad where \quad \chi_{m+1} = \chi_1.$$

Definition1.2 Error! Reference source not found. let $\mathfrak{R}$ be a non-null set and V be a vector lattice, consider a function $G: \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R} \to V$ that obeys the properties mentioned below:

(i) $G(\flat_1, \flat_2, \flat_3) = 0$ if $\flat_1 = \flat_2 = \flat_3$,

(ii) $0 < G(\flat_1, \flat_1, \flat_2) \quad \forall \ \flat_1, \flat_2 \in \mathfrak{R}$ with $\flat_1 \neq \flat_2$,

(iii) $G(\flat_1, \flat_1, \flat_2) \preceq G(\flat_1, \flat_2, \flat_3) \quad \forall \ \flat_1, \flat_2, \flat_3 \in \mathfrak{R}$ with $\flat_3 \neq \flat_2$,

(iv) $G(\flat_1, \flat_2, \flat_3) = G(\flat_1, \flat_3, \flat_2) = G(\flat_2, \flat_3, \flat_1) = \cdots$ (tri-variable symmetry)

$(v) G(\flat_1, \flat_2, \flat_3) \preceq G(\flat_1, \sigma, \sigma) + G(\sigma, \flat_2, \flat_3) \quad \forall \ \flat_1, \flat_2, \flat_3, \sigma \in \mathfrak{R}$.

The triplet $(\mathfrak{R}, G, V)$ is defined as vector G-metric space(VGMS).

Definition 1.3 Error! Reference source not found. A VGMS $(\mathfrak{R}, G, V)$ is called symmetric VGMS if $G(\zeta, \flat, \flat) = G(\flat, \zeta, \zeta) \quad \forall \zeta, \flat \in \mathfrak{R}$.

Example 1.4 Let $\mathfrak{R}$ be a non-null set and take V as a vector lattice. Then the function $G: \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R} \to V$ is determined by

$$G(\mu, \flat, \zeta) = \frac{1}{2^n}(|\mu - \flat|, |\flat - \zeta|, |\zeta - \mu|)$$

$\forall \mu, \flat, \zeta \in \mathfrak{R}$ and $\forall n$. Then $(\mathfrak{R}, G, V)$ is a VGMS on $\mathfrak{R}$.

Example 1.5 Let $\mathfrak{R} = [1, \infty)$, V be a vector lattice, G_1 and G_2 be two vector G-metrics on $\mathfrak{R}$. The transformation $G: \mathfrak{R} \times \mathfrak{R} \times \mathfrak{R} \to V$ is determined by

$$G(\mu, \flat, \sigma) = (G_1(\mu, \flat, \sigma) + G_2(\mu, \flat, \sigma))^{\frac{1}{2}}$$

$\forall \quad \mu, \flat, \sigma \in \mathfrak{R}$. Then $(\mathfrak{R}, G, V)$ is a VGMS on $\mathfrak{R}$.

Definition 1.6 Error! Reference source not found. In a VGMS $(\mathfrak{R}, G, V)$, $\langle \varsigma_n \rangle \in \mathfrak{R}$ is called V convergent to some $\varsigma \in V$ if $\exists \langle \mu_n \rangle \in V$ satisfying $\mu_n \downarrow 0$ and $G(\varsigma, \varsigma_n, \varsigma_m) \leq \mu_n$ $\forall n, m \in \mathbb{Z}^+$.

Definition 1.7 Error! Reference source not found. In a VGMS $(\mathfrak{R}, G, V)$, $\langle \varsigma_n \rangle \in \mathfrak{R}$ is called V-Cauchy sequence(VCS) if $\exists \langle \mu_n \rangle \in V$ satisfying $\mu_n \downarrow 0$ and $G(\varsigma_n, \varsigma_m, \varsigma_\mu) \leq \mu_n$ holds for all $n, m, \mu \in \mathbb{Z}^+$.

Definition 1.8 Error! Reference source not found. In a VGMS $(\mathfrak{R}, G, V)$, the term V-complete implies that every VCS in $\mathfrak{R}$ is V-convergence to a limit in $\mathfrak{R}$.

Definition 1.9[5] A cyclic transformation $\complement$ on μ is said to be cyclic contraction if $\exists$ $0 \leq \rho < 1$ such that

$$G(\complement(\mu), \complement(\flat), \complement(\flat)) \leq \rho G(\mu, \flat, \flat)$$

$\forall \mu \in \chi_q$ and $\flat \in \chi_{q+1}$, $q = 1,2, \dots m$.

2 Main Results

In this section, we establish some FPT and corollaries for self-transformations in VGMS $(\mathfrak{R}, G, V)$ by employing the methodology of CC.

Theorem 2.1 Let $(\mathfrak{R}, G, V)$ be symmetric and G- complete VGMS and $\{\chi_q\}_{q=1}^m$ as a collection of non-empty subset of $\mathfrak{R}$ with $\mu = \cup_{q=1}^m \chi_q$.Let $\complement: \mu \to \mu$ be cyclic transformation satisfying

$$\complement(\chi_q) \subseteq \chi_{q+1} \quad q = 1,2, \dots, m, \quad where \quad \chi_{m+1} = \chi_1.$$

Suppose that $\exists$ constant $\partial_1, \partial_2, \partial_3, \partial_4, \partial_5, \partial_6$ and ∂_7 with $0 \leq \partial_1 + \partial_2 + \partial_3 + \partial_4 + \partial_5 + \partial_6 + \partial_7 < 1$ such that the transformation $\complement$ satisfies

$$\begin{aligned} G(\complement(\mu), \complement(\flat), \complement(r)) \leq{} & \partial_1 G(\mu, \flat, r) + \partial_2 G(\mu, \complement(\mu), \complement(\mu)) + \partial_3 G(\flat, \complement(\flat), \complement(\flat)) \\ & + \partial_4 G(r, \complement(r), \complement(r)) + \partial_5 G(\mu, \complement(\flat), \complement(\flat)) + \partial_6 G(\flat, \complement(r), \complement(r)) \\ & + \partial_7 G(r, \complement(\mu), \complement(\mu)) \end{aligned} \tag{1}$$

$\forall \mu \in \chi_q$ and $\flat, r \in \chi_{q+1}$, $q = 1,2, \dots m$. Then χ has FP in $\mu = \cap_{q=1}^m \chi_q$ which is unique.

Proof: We can write eq (1) as

$$G(\complement(\mu), \complement(\flat), \complement(\flat)) \leq \partial_1 G(\mu, \flat, \flat) + \partial_2 G(\mu, \complement(\mu), \complement(\mu)) + \partial_3 G(\flat, \complement(\flat), \complement(\flat))$$

$$+\partial_4 G(\flat, \mathsf{C}(\flat), \mathsf{C}(\flat)) + \partial_5 G(\mu, \mathsf{C}(\flat), \mathsf{C}(\flat)) + \partial_6 G(\flat, \mathsf{C}(\flat), \mathsf{C}(\flat)) + \partial_7 G(\flat, \mathsf{C}(\mu), \mathsf{C}(\mu))$$

$$G(\mathsf{C}(\mu), \mathsf{C}(\flat), \mathsf{C}(\flat)) \preceq \partial_1 G(\mu, \flat, \flat) + \partial_2 G(\mu, \mathsf{C}(\mu), \mathsf{C}(\mu)) + (\partial_3 + \partial_4 + \partial_6) G(\flat, \mathsf{C}(\flat), \mathsf{C}(\flat)) + \partial_5 G(\mu, \mathsf{C}(\flat), \mathsf{C}(\flat)) + \partial_7 G(\flat, \mathsf{C}(\mu), \mathsf{C}(\mu)) \tag{2}$$

Interchanging the role of μ and $\flat$, we get

$$G(\mathsf{C}(\flat), \mathsf{C}(\mu), \mathsf{C}(\mu)) \preceq \partial_1 G(\flat, \mu, \mu) + \partial_2 G(\flat, \mathsf{C}(\flat), \mathsf{C}(\flat)) + (\partial_3 + \partial_4 + \partial_6) G(\mu, \mathsf{C}(\mu), \mathsf{C}(\mu)) + \partial_5 G(\flat, \mathsf{C}(\mu), \mathsf{C}(\mu)) + \partial_7 G(\mu, \mathsf{C}(\flat), \mathsf{C}(\flat))$$

$$G(\mathsf{C}(\mu), \mathsf{C}(\flat), \mathsf{C}(\flat)) \preceq \partial_1 G(\mu, \flat, \flat) + \partial_2 G(\flat, \mathsf{C}(\flat), \mathsf{C}(\flat)) + (\partial_3 + \partial_4 + \partial_6) G(\mu, \mathsf{C}(\mu), \mathsf{C}(\mu)) + \partial_5 G(\flat, \mathsf{C}(\mu), \mathsf{C}(\mu)) + \partial_7 G(\mu, \mathsf{C}(\flat), \mathsf{C}(\flat)) \tag{3}$$

Adding eq(2) and eq (3), we get

$$2G(\mathsf{C}(\mu), \mathsf{C}(\flat), \mathsf{C}(\flat)) \preceq 2\partial_1 G(\mu, \flat, \flat) + (\partial_2 + \partial_3 + \partial_4 + \partial_6) G(\mu, \mathsf{C}(\mu), \mathsf{C}(\mu)) + (\partial_2 + \partial_3 + \partial_4 + \partial_6) G(\flat, \mathsf{C}(\flat), \mathsf{C}(\flat)) + (\partial_5 + \partial_7) G(\mu, \mathsf{C}(\flat), \mathsf{C}(\flat)) + (\partial_5 + \partial_7) G(\flat, \mathsf{C}(\mu), \mathsf{C}(\mu))$$

$$G(\mathsf{C}(\mu), \mathsf{C}(\flat), \mathsf{C}(\flat)) \preceq \partial_1 G(\mu, \flat, \flat) + \frac{\partial_2 + \partial_3 + \partial_4 + \partial_6}{2} [G(\mu, \mathsf{C}(\mu), \mathsf{C}(\mu)) + G(\flat, \mathsf{C}(\flat), \mathsf{C}(\flat))] + \frac{\partial_5 + \partial_7}{2} [G(\mu, \mathsf{C}(\flat), \mathsf{C}(\flat)) + G(\flat, \mathsf{C}(\mu), \mathsf{C}(\mu))] \tag{4}$$

Putting $\flat = \mathsf{C}(\mu)$ in eq (4)

$$G(\mathsf{C}(\mu), \mathsf{C}^2(\mu), \mathsf{C}^2(\mu)) \preceq \partial_1 G(\mu, \mathsf{C}(\mu), \mathsf{C}(\mu)) + \frac{\partial_2 + \partial_3 + \partial_4 + \partial_6}{2} [G(\mu, \mathsf{C}(\mu), \mathsf{C}(\mu)) + G(\mathsf{C}(\mu), \mathsf{C}^2(\mu), \mathsf{C}^2(\mu))] + \frac{\partial_5 + \partial_7}{2} [G(\mu, \mathsf{C}^2(\mu), \mathsf{C}^2(\mu)) + G(\mathsf{C}(\mu), \mathsf{C}(\mu), \mathsf{C}(\mu))] \tag{5}$$

By using definition of VGMS, we get

$$G(\mu, \mathsf{C}^2(\mu), \mathsf{C}^2(\mu)) \preceq G(\mu, \mathsf{C}(\mu), \mathsf{C}(\mu)) + G(\mathsf{C}(\mu), \mathsf{C}^2(\mu), \mathsf{C}^2(\mu)) \tag{6}$$

Then eq(5) becomes

$$G(\mathsf{C}(\mu), \mathsf{C}^2(\mu), \mathsf{C}^2(\mu)) \preceq \partial_1 G(\mu, \mathsf{C}(\mu), \mathsf{C}(\mu)) + \frac{\partial_2 + \partial_3 + \partial_4 + \partial_6}{2} [G(\mu, \mathsf{C}(\mu), \mathsf{C}(\mu)) + G(\mathsf{C}(\mu), \mathsf{C}^2(\mu), \mathsf{C}^2(\mu))] + \frac{\partial_5 + \partial_7}{2} [G(\mu, \mathsf{C}(\mu), \mathsf{C}(\mu)) +$$

$$G(\text{C}(\mu), \text{C}^2(\mu), \text{C}^2(\mu))]$$

$$\left(1 - \frac{\sum_{i=2}^{7} \partial_i}{2}\right) G\big(\text{C}(\mu), \text{C}^2(\mu), \text{C}^2(\mu)\big) \preceq \left(\partial_1 + \frac{\sum_{i=2}^{7} \partial_i}{2}\right) G\big(\mu, \text{C}(\mu), \text{C}(\mu)\big)$$

$$G\big(\text{C}(\mu), \text{C}^2(\mu), \text{C}^2(\mu)\big) \preceq \frac{\partial_1 + \frac{\sum_{i=2}^{7} \partial_i}{2}}{1 - \frac{\sum_{i=2}^{7} \partial_i}{2}} G(\mu, \text{C}(\mu), \text{C}(\mu)) \tag{7}$$

Putting $q = \frac{\partial_1 + \frac{\sum_{i=2}^{7} \partial_i}{2}}{1 - \frac{\sum_{i=2}^{7} \partial_i}{2}}$. Since $0 \leq \sum_{i=1}^{7} \partial_i < 1$, so $0 \leq q < 1$. Then eq(7) becomes

$$G(\text{C}(\mu), \text{C}^2(\mu), \text{C}^2(\mu)) \preceq qG(\mu, \text{C}\mu, \text{C}\mu) \tag{8}$$

$\forall \mu \in \chi_q$. Further,

$$G(\text{C}^\rho(\mu), \text{C}^{\rho+1}(\mu), \text{C}^{\rho+1}(\mu)) \preceq q^\rho G(\mu, \text{C}\mu, \text{C}\mu) \quad \forall \rho \in \mathbb{N} \quad \mu \in \chi_q.$$

For $\rho, j \in \mathbb{N}$ and $\mu \in \chi_q$

$$\begin{aligned} G(\text{C}^\rho(\mu), \text{C}^{\rho+j}(\mu), \text{C}^{\rho+j}(\mu)) &\preceq G(\text{C}^\rho(\mu), \text{C}^{\rho+1}(\mu), \text{C}^{\rho+1}(\mu)) + \\ &\quad G(\text{C}^{\rho+1}(\mu), \text{C}^{\rho+j}(\mu), \text{C}^{\rho+j}(\mu)) \\ &\preceq G(\text{C}^\rho(\mu), \text{C}^{\rho+1}(\mu), \text{C}^{\rho+1}(\mu)) + \\ &\quad G(\text{C}^{\rho+1}(\mu), \text{C}^{\rho+2}(\mu), \text{C}^{\rho+2}(\mu)) \\ &\quad + G(\text{C}^{\rho+2}(\mu), \text{C}^{\rho+j}(\mu), \text{C}^{\rho+j}(\mu) \\ &\quad \vdots \\ &\preceq \sum_{i=1}^{j} G(\text{C}^{\rho+i-1}(\mu), \text{C}^{\rho+i}(\mu), \text{C}^{\rho+i}(\mu)) \\ &\preceq \sum_{i=1}^{j} q^{\rho+i-1} G(\mu, \text{C}\mu, \text{C}\mu) \end{aligned}$$

$$G(\text{C}^\rho(\mu), \text{C}^{\rho+j}(\mu), \text{C}^{\rho+j}(\mu)) \preceq \frac{q^\rho}{1-q} G(\mu, \text{C}\mu, \text{C}\mu) \tag{9}$$

This implies $\langle \text{C}^\rho \mu \rangle$ is a VCS in G-complete VGMS and it converges to $\mu_1 \in \mathfrak{R}$. So, $\exists$ a sequence $\mu_\rho \in V$ such that $\mu_\rho \downarrow 0$ and $G(\mu_1, \text{C}^\rho(\mu), \text{C}^\rho(\mu)) \preceq \mu_\rho$.

Now

$$\begin{aligned} G(\mu_1, \text{C}((\mu_1)), \text{C}(\mu_1)) &\preceq G(\mu_1, \text{C}^{\rho+1}(\mu), \text{C}^{\rho+1}(\mu)) + G(\text{C}^{\rho+1}(\mu), \text{C}(\mu_1), \text{C}(\mu_1)) \\ &\preceq G(\mu_1, \text{C}^{\rho+1}(\mu), \text{C}^{\rho+1}(\mu)) + G(\text{C}(\mu_1), \text{C}^{\rho+1}(\mu), \text{C}^{\rho+1}(\mu)) \end{aligned}$$

By using (4), we get

$$G(\mu_1, \text{C}(\mu_1), \text{C}(\mu_1)) \preceq G(\mu_1, \text{C}^{\rho+1}(\mu), \text{C}^{\rho+1}(\mu)) + \partial_1 G(\mu_1, \text{C}^\rho(\mu), \text{C}^\rho(\mu)) +$$

$$\frac{\partial_2+\partial_3+\partial_4+\partial_6}{2}[G(\mu_1, \complement\mu_1, \complement\mu_1) + G(\complement^{\rho}(\mu), \complement^{\rho+1}(\mu), \complement^{\rho+1}(\mu))]$$

$$+\frac{\partial_5+\partial_7}{2}[G(\mu_1, \complement^{\rho+1}(\mu), \complement^{\rho+1}(\mu)) + G(\complement^{\rho}(\mu), \complement\mu_1, \complement\mu_1)]$$

$$\preceq G(\mu_1, \complement^{\rho+1}(\mu), \complement^{\rho+1}(\mu)) + \partial_1 G(\mu_1, \complement^{\rho}(\mu), \complement^{\rho}(\mu)) + \frac{\partial_2+\partial_3+\partial_4+\partial_6}{2}[G(\mu_1, \complement\mu_1, \complement\mu_1) + G(\complement^{\rho}(\mu), \mu_1, \mu_1) + G(\mu_1, \complement^{\rho+1}(\mu), \complement^{\rho+1}(\mu))] + \frac{\partial_5+\partial_7}{2}[G(\mu_1, \complement^{\rho+1}(\mu), \complement^{\rho+1}(\mu)) + G(\complement^{\rho}(\mu), \mu_1, \mu_1) + G(\mu_1, \complement\mu_1, \complement\mu_1)]$$

$$G(\mu_1, \complement(\mu_1), \complement(\mu_1)) \preceq G(\mu_1, \complement^{\rho+1}(\mu), \complement^{\rho+1}(\mu)) + \partial_1 G(\mu_1, \complement^{\rho}(\mu), \complement^{\rho}(\mu)) + \frac{\partial_2+\partial_3+\partial_4+\partial_6}{2}[G(\mu_1, \complement\mu_1, \complement\mu_1) + G(\mu_1, \complement^{\rho}(\mu), \complement^{\rho}(\mu)) + G(\mu_1, \complement^{\rho+1}(\mu), \complement^{\rho+1}(\mu))] + \frac{\partial_5+\partial_7}{2}[G(\mu_1, \complement^{\rho+1}(\mu), \complement^{\rho+1}(\mu)) + G(\mu_1, \complement^{\rho}(\mu), \complement^{\rho}(\mu)) + G(\mu_1, \complement\mu_1, \complement\mu_1)]$$

$$\preceq \mu_{\rho+1} + \partial_1\mu_\rho + \frac{\partial_2+\partial_3+\partial_4+\partial_6}{2}[G(\mu_1, \complement\mu_1, \complement\mu_1) + \mu_\rho + \mu_{\rho+1}]$$

$$+\frac{\partial_5+\partial_7}{2}[\mu_{\rho+1} + \mu_\rho + G(\mu_1, \complement\mu_1, \complement\mu_1)]$$

$$\preceq \mu_\rho + \partial_1\mu_\rho + \frac{\partial_2+\partial_3+\partial_4+\partial_6}{2}[G(\mu_1, \complement\mu_1, \complement\mu_1) + \mu_\rho + \mu_\rho]$$

$$+\frac{\partial_5+\partial_7}{2}[\mu_\rho + \mu_\rho + G(\mu_1, \complement\mu_1, \complement\mu_1)] \tag{10}$$

$$(1 - \frac{\sum_{i=2}^{7}\partial_i}{2})G(\mu_1, \complement(\mu_1), \complement(\mu_1)) \preceq \frac{2\partial_1 + \sum_{i=2}^{7}\partial_i}{2}\mu_\rho$$

$$G(\mu_1, \complement(\mu_1), \complement(\mu_1)) \preceq \frac{2\partial_1 + \sum_{i=2}^{7}\partial_i}{2 - \sum_{i=2}^{7}\partial_i}\mu_\rho \downarrow 0$$

It follows that $\mu_1 = \complement\mu_1$.

Now, we assert that μ_1 is unique FP. If ∂_1 is another FP of $\complement$, then $\forall \mu_1 \in \chi_q$ and $\partial_1 \in \chi_{q+1}$, $j = 1,2, \dots m$, we have

$$G(\mu_1, \partial_1, \partial_1) = G(\complement\mu_1, \complement\partial_1, \complement\partial_1)$$

$$\preceq \partial_1 G(\mu_1, \partial_1, \partial_1) + \partial_2 G(\mu_1, \complement\mu_1, \complement\mu_1) + \partial_3 G(\partial_1, \complement\partial_1, \complement\partial_1)$$

$$+\partial_4 G(\partial_1, \complement\partial_1, \complement\partial_1) + \partial_5 G(\mu_1, \complement\partial_1, \complement\partial_1) + \partial_6 G(\partial_1, \complement\partial_1, \complement\partial_1)$$
$$+\partial_7 G(\partial_1, \complement\mu_1, \complement\mu_1)$$
$$\preceq \partial_1 G(\mu_1, \partial_1, \partial_1) + \partial_5 G(\mu_1, \partial_1, \partial_1) + \partial_7 G(\partial_1, \mu_1, \mu_1)$$
$$\preceq \partial_1 G(\mu_1, \partial_1, \partial_1) + \partial_5 G(\mu_1, \partial_1, \partial_1) + \partial_7 G(\mu_1, \partial_1, \partial_1)$$
$$\preceq (\partial_1 + \partial_5 + \partial_7) G(\mu_1, \partial_1, \partial_1).$$

Since $0 \leq \partial_1 + \partial_5 + \partial_7 < \sum_{i=1}^{7} \partial_i < 1$. So $G(\mu_1, \partial_1, \partial_1) = 0$ this implies $\mu_1 = \partial_1$. Thus $\complement$ has FP in μ which is unique.

Setting $\partial_2 = \partial$, $\partial_3 = \rho$ and $\partial_i = 0$ for $i = 1,3, \dots, 7$ in the previous theorem, we yield the subsequent corollary.

Corollary 2.2: Let $(\Re, G, V)$ be $G-$ complete and symmetric VGMS and $\{\chi_q\}_{q=1}^{m}$ be family of the non-empty subset of μ with $\mu = \cup_{q=1}^{m} \chi_q$. Let $\complement: \mu \to \mu$ be a transformation satisfying

$$\complement(\chi_q) \subseteq \chi_{q+1} \quad q = 1,2, \dots, m, \quad where \quad \chi_{m+1} = \chi_1.$$

Suppose that $\exists$ constant $\partial, \Re$ with $0 < \partial + \rho < 1$ such that the transformation $\complement$ satisfies

$$G(\complement\mu, \complement\flat, \complement\flat) \preceq \partial G(\mu, \complement\mu, \complement\mu) + \rho G(\flat, \complement\flat, \complement\flat)$$

$\forall \mu \in \chi_q$ and $\flat, r \in \chi_{q+1}$, $q = 1,2, \dots m$. Then χ has FP in $\mu = \cap_{q=1}^{m} \chi_q$ which is unique.

Example 2.3: The transformation $\complement: [0,1] \to [0,1]$ defined by

$$\complement\mu = \begin{cases} \frac{\mu}{4} & 0 \preceq \mu < \frac{1}{2} \\ \frac{\mu}{5} & \frac{1}{2} \preceq \mu < 1 \end{cases}$$

is a discontinuous transformation with 0 as unique FP.

Setting $\partial_1 = \partial$ and $\partial_i = 0$ for $i = 2, \dots, 7$ in the theorem 2.1, we yield the subsequent corollary.

Corollary 2.4:(Banach Contraction Principle) Let $(\Re, G, V)$ be $G-$ complete and symmetric VGMS and $\{A_j\}_{j=1}^{m}$ be family of non-empty subset of μ with $\mu = \cup_{q=1}^{m} \chi_q$. Let $\complement: \mu \to \mu$ be a transformation satisfying

$$\complement(\chi_q) \subseteq \chi_{q+1} \quad q = 1,2, \dots, m, \quad where \quad \chi_{m+1} = \chi_1.$$

Suppose that $\exists$ constant ∂ with $0 < \partial < 1$ such that the transformation $\complement$ satisfies

$$G(\complement\mu, \complement\flat, \complement r) \preceq \partial G(\mu, \flat, r)$$

$\forall \mu \in \chi_q$ and $\flat, r \in \chi_{q+1}$, $q = 1,2, \dots m$. Then χ has FP in $\mu = \cap_{q=1}^{m} \chi_q$ which is unique.

Theorem 2.5 Let $(\Re, G, V)$ be $G-$ complete and symmetric VGMS and $\{\chi_q\}_{q=1}^{m}$ be family of non-empty subset of μ with $\mu = \cup_{q=1}^{m} \chi_q$. Let $\complement: \mu \to \mu$ be a transformation satisfying

$$\complement(\chi_q) \subseteq \chi_{q+1} \quad q = 1,2,\dots,m, \quad where \quad \chi_{m+1} = \chi_1.$$

Suppose that there exist constant $\partial_1, \partial_2, \partial_3, \partial_4$ with $0 < \partial_1 + \partial_2 + \partial_3 + \partial_4 < \frac{1}{2}$ such that the transformation $\complement$ satisfies

$$G(\complement\mu, \complement\flat, \complement\flat) \preceq \{\partial_1[G(\mu, \complement\mu, \complement\mu) + G(\flat, \complement\flat, \complement\flat)], \partial_2[G(\mu, \complement\flat, \complement\flat) + G(\flat, \complement\mu, \complement\mu)], \partial_3[G(\mu, \complement\mu, \complement\mu) + G(\mu, \complement\flat, \complement\flat)], \partial_4[G(\flat, \complement\mu, \complement\mu) + G(\flat, \complement\flat, \complement\flat)]\} \quad (11)$$

$\forall \mu \in \chi_q$ and $\flat, r \in \chi_{q+1}$, $q = 1,2, \dots m$. Then χ has FP in $\mu = \cap_{q=1}^{m} \chi_q$ which is unique.

Proof: Interchanging the role of μ and $\flat$ in (11), we get

$$G(\complement\flat, \complement\mu, \complement\mu) \preceq \{\partial_1[G(\flat, \complement\flat, \complement\flat) + G(\mu, \complement\mu, \complement\mu)], \partial_2[G(\flat, \complement\mu, \complement\mu) + G(\mu, \complement\flat, \complement\flat)], \partial_3[G(\flat, \complement\flat, \complement\flat) + G(\flat, \complement\mu, \complement\mu)], \partial_4[G((\mu, \complement\flat, \complement\flat) + G(\mu, \complement\mu, \complement\mu)]\} \quad (12)$$

Adding eq(11) and eq(12), we get

$$2G(\complement\mu, \complement\flat, \complement\flat) \preceq \{2\partial_1[G(\mu, \complement\mu, \complement\mu) + G(\flat, \complement\flat, \complement\flat)], 2\partial_2[G(\mu, \complement\flat, \complement\flat) + G(\flat, \complement\mu, \complement\mu)], (\partial_3 + \partial_4)[G(\mu, \complement\mu, \complement\mu) + G(\mu, \complement\flat, \complement\flat)], \partial_3 + \partial_4)([G(\flat, \complement\mu, \complement\mu) + G(\flat, \complement\flat, \complement\flat)]\}$$

$$G(\complement\mu, \complement\flat, \complement\flat) \preceq \{\partial_1[G(\mu, \complement\mu, \complement\mu) + G(\flat, \complement\flat, \complement\flat)], \partial_2[G(\mu, \complement\flat, \complement\flat) + G(\flat, \complement\mu, \complement\mu)], \frac{\partial_3+\partial_4}{2}[G(\mu, \complement\mu, \complement\mu) + G(\mu, \complement\flat, \complement\flat)], \frac{\partial_3+\partial_4}{2}[G(\flat, \complement\mu, \complement\mu) + G(\flat, \complement\flat, \complement\flat)]\}$$

$$\preceq \{\frac{2\partial_1+\partial_3+\partial_4}{2} G(\mu, \complement\mu, \complement\mu) + \frac{2\partial_1+\partial_3+\partial_4}{2} G(\flat, \complement\flat, \complement\flat) + \frac{2\partial_2+\partial_3+\partial_4}{2} G(\mu, \complement\flat, \complement\flat) + \frac{2\partial_2+\partial_3+\partial_4}{2} G(\flat, \complement\mu, \complement\mu)\}$$

$$\preceq \{\frac{2\partial_1+\partial_3+\partial_4}{2}[G(\mu, \complement\mu, \complement\mu) + G(\flat, \complement\flat, \complement\flat)] + \frac{2\partial_2+\partial_3+\partial_4}{2}[G(\mu, \complement\flat, \complement\flat) + G(\flat, \complement\mu, \complement\mu)]\} \quad (13)$$

Putting $\flat = \complement\mu$ in eq(13), we get

$$G(\complement\mu, \complement^2\mu, \complement^2\mu) \preceq \{\frac{2\partial_1 + \partial_3 + \partial_4}{2}[G(\mu, \complement\mu, \complement\mu) + G(\complement\mu, \complement^2\mu, \complement^2\mu)] + \frac{2\partial_2 + \partial_3 + \partial_4}{2}[G(\mu, \complement^2\mu, \complement^2\mu) + G(\complement\mu, \complement\mu, \complement\mu)]\}$$

$$\preceq \{\frac{2\partial_1 + \partial_3 + \partial_4}{2}[G(\mu, \complement\mu, \complement\mu) + G(\complement\mu, \complement^2\mu, \complement^2\mu)] +$$

$$\frac{2\partial_2+\partial_3+\partial_4}{2} G(\mu, \complement^2\mu, \complement^2\mu)\} \tag{14}$$

By using (VGM5), we get

$$G(\mu, \complement^2\mu, \complement^2\mu) \preceq G(\mu, \complement\mu, \complement\mu) + G(\complement\mu, \complement^2\mu, \complement^2\mu) \tag{15}$$

Then eq(14) becomes

$$G(\complement\mu, \complement^2\mu, \complement^2\mu) \preceq \{\frac{2\partial_1 + \partial_3 + \partial_4}{2}[G(\mu, \complement\mu, \complement\mu) + G(\complement\mu, \complement^2\mu, \complement^2\mu)] +$$

$$\frac{2\partial_2 + \partial_3 + \partial_4}{2} G(\mu, \complement\mu, \complement\mu) + G(\complement\mu, \complement^2\mu, \complement^2\mu)\}$$

$$\preceq \{\sum_{i=1}^{4} \partial_i G(\mu, \complement\mu, \complement\mu) +$$

$$\sum_{i=1}^{4} \partial_i G(\complement\mu, \complement^2\mu, \complement^2\mu)\}$$

$$(1 - \sum_{i=1}^{4} \partial_i) G(\complement\mu, \complement^2\mu, \complement^2\mu) \preceq \sum_{i=1}^{4} \partial_i (G(\mu, \complement\mu, \complement\mu))$$

$$G(\complement\mu, \complement^2\mu, \complement^2\mu) \preceq \frac{\sum_{i=1}^{4} \partial_i}{1-\sum_{i=1}^{4} \partial_i} G(\mu, \complement\mu, \complement\mu) \tag{16}$$

Putting $q = \frac{\sum_{i=1}^{4} \partial_i}{1-\sum_{i=1}^{4} \partial_i}$. Since $0 \le \sum_{i=1}^{4} \partial_i < \frac{1}{2}$, so $0 \le q < 1$. Then eq(16) becomes

$$G(\complement\mu, \complement^2\mu, \complement^2\mu) \preceq qG(\mu, \complement\mu, \complement\mu) \tag{17}$$

$\forall \mu \in \chi_q$. Further

$$G(\complement^\rho\mu, \complement^{\rho+1}\mu, \complement^{\rho+1}\mu) \preceq q^\rho G(\mu, \complement\mu, \complement\mu) \quad \forall \rho \in \mathbb{N} \quad \mu \in \chi_q.$$

For $\rho, j \in \mathbb{N}$ and $\mu \in \chi_q$

$$G(\complement^\rho(\mu), \complement^{\rho+j}(\mu), \complement^{\rho+j}(\mu)) \preceq G(\complement^\rho(\mu), \complement^{\rho+1}(\mu), \complement^{\rho+1}(\mu)) +$$

$$G(\complement^{\rho+1}(\mu), \complement^{\rho+j}(\mu), \complement^{\rho+j}(\mu))$$

$$\preceq G(\complement^\rho(\mu), \complement^{\rho+1}(\mu), \complement^{\rho+1}(\mu)) +$$

$$G(\complement^{\rho+1}(\mu), \complement^{\rho+2}(\mu), \complement^{\rho+2}(\mu))$$

$$+G(\complement^{\rho+2}(\mu), \complement^{\rho+j}(\mu), \complement^{\rho+j}(\mu)$$

$$\vdots$$

$$\preceq \sum_{i=1}^{j} G(\complement^{\rho+i-1}(\mu), \complement^{\rho+i}(\mu), \complement^{\rho+i}(\mu))$$

$$\preceq \sum_{i=1}^{j} q^{\rho+i-1} G(\mu, \complement\mu, \complement\mu)$$

$$G(\complement^{\rho}(\mu), \complement^{\rho+j}(\mu), \complement^{\rho+j}(\mu)) \preceq \frac{q^{\rho}}{1-q} G(\mu, \complement\mu, \complement\mu) \tag{18}$$

This implies $\{\complement^{\rho}\mu\}$ is a VCS in G-complete VGMS and so it converges to $\mu_1 \in \partial$.

Now

$$G(\mu_1, \complement\mu_1, \complement\mu_1) \preceq G(\mu_1, \complement^{\rho+1}\mu, \complement^{\rho+1}\mu) + G(\complement^{\rho+1}\mu, \complement\mu_1, \complement\mu_1)$$
$$\preceq G(\mu_1, \complement^{\rho+1}\mu, \complement^{\rho+1}\mu) + \frac{2\partial_1+\partial_3+\partial_4}{2}[G(\complement^{\rho}\mu, \complement^{\rho+1}\mu, \complement^{\rho+1}\mu)$$
$$+G(\mu_1, \complement\mu_1, \complement\mu_1)] + \frac{2\partial_2+\partial_3+\partial_4}{2}[G(\complement^{\rho}\mu, \complement\mu_1, \complement\mu_1)$$
$$+G(\mu_1, \complement^{\rho+1}\mu, \complement^{\rho+1}\mu)] \tag{19}$$

By using

$$\mu_1 = \lim_{\rho\to\infty} \complement^{\rho}\mu = \lim_{\rho\to\infty} \complement^{\rho+1}\mu.$$

Then eq(19) becomes

$$G(\mu_1, \complement\mu_1, \complement\mu_1) \preceq 0 + \frac{2\partial_1 + \partial_3 + \partial_4}{2}[0 + G(\mu_1, \complement\mu_1, \complement\mu_1)]$$
$$+\frac{2\partial_2 + \partial_3 + \partial_4}{2}[G(\mu_1, \complement\mu_1, \complement\mu_1) + 0]$$
$$\preceq \sum_{l=1}^{4} \partial_i G(\mu_1, \complement\mu_1, \complement\mu_1) \tag{20}$$

Since $0 \preceq \sum_{i=1}^{4} \partial_i < 1$, it follows that $\mu_1 = \complement\mu_1$.

Now, we assert that μ_1 is unique FP. If ∂_1 is another FP of $\complement$, then $\forall\mu_1 \in \chi_q$ and $\partial_1 \in \chi_{q+1}$, $j = 1,2, \dots m$, we have

$$G(\mu_1, \partial_1, \partial_1) = G(\complement\mu_1, \complement\partial_1, \complement\partial_1)$$
$$\preceq \{\partial_1[G(\mu_1, \complement\mu_1, \complement\mu_1) + G(\partial_1, \complement\partial_1, \complement\partial_1)], \partial_2[G(\mu_1, \complement\partial_1, \complement\partial_1) +$$
$$G(\partial_1, \complement\mu_1, \complement\mu_1)], \partial_3[G(\mu_1, \complement\mu_1, \complement\mu_1) + G(\mu_1, \complement\partial_1, \complement\partial_1)],$$
$$\partial_4[G(\partial_1, \complement\mu_1, \complement\mu_1) + G(\partial_1, \complement\partial_1, \complement\partial_1)]\}$$
$$\preceq \{\partial_1(0), \partial_2[G(\mu_1, \partial_1, \partial_1) + G(\partial_1, \mu_1, \mu_1)], \partial_3 G(\mu_1, \partial_1, \partial_1),$$
$$\partial_4 G(\mu_1, \partial_1, \partial_1)\}$$
$$\preceq \{2\partial_2 G(\mu_1, \partial_1, \partial_1), \partial_3 G(\mu_1, \partial_1, \partial_1), \partial_4 G(\mu_1, \partial_1, \partial_1)\}$$
$$\preceq (2\partial_2 + \partial_3 + \partial_4) G(\mu_1, \partial_1, \partial_1).$$

Since $0 \preceq 2\partial_2 + \partial_3 + \partial_4 < 1$. So $\complement$ has FP in μ which is unique.

3 Conclusion

Within this manuscript, we illustrate FPT for self-transformations in V-complete VGMS by utilizing cyclic contraction. These outcomes are expected to inspire researchers to explore problem-solving opportunities in diverse areas such as differential equations and functional analysis.

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