Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 1 (2024)

Some Results on Odd-Even Congruence Labeling of Graphs

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Article History:

Received: *05-10-2023*

Revised: *15-11-2023*

Accepted: *02-12-2023*

Abstract:

Introduction: Labeling of graphs has been introduced in 1966. Assignment of natural numbers to vertices and/or edges is referred as graph labeling. Inspired by the ample application of graph labeling technique in real life problems, multifarious labeling strategy was adopted and investigated by many researchers.

Objectives: Graph labeling plays a vital role in various fields and can be implemented in multitudinous discipline including coding theory, X-ray, Psychology, crystallography, circuit design, communication networks, astronomy, radar, data security, secret sharing, data base management and so on. Apart from these labeling techniques serve as a model to understand discrete mathematical domains.

Methodology: In this paper, an attempt has been made to introduce new labeling such as o dd-even congruence labeling. Congruence Graph Labeling is an allocation of natural numbers as labels for the edges and vertices of a graph based on modular arithmetic property. Odd-even congruence labeling is an allocation of odd integers to vertices and even integers to edges in addition to congruence graph labeling.

Result: The suggested labeling has been identified on complete bipartite graph, comb graph and spliting graph of a star graph. Further, it is proved that graph acquired by connecting two copies of even cycle C_r by a path P_t, $K_{1,t} \otimes P_2$ and $D_2(P_t)$ are odd-even congurence graph.

Keywords: Labeling, Congruence labeling, Odd-even congruence labeling, Tensor graph.

1. Introduction

The relation between any of the objects in the real world can be represented as graphs, whose structures, properties, interrelations and correlations are interpreted in Graph Theory. The graph comprises of dots associated by lines which are referred as vertices and edges respectively. It aids the researchers to frame the hypothesis and establish the solution certainly.

Graph labeling was procured as the most prominent one among the multifarious conception of graph theory to design the graphical model of the real life situations. In the middle of 19th century, A.Rosa introduced graph labeling[8]. The vertices and edges of the graphs are labeled with natural numbers with certain constraints in order to discriminate individually. Inspired by its comprehensive application in modeling all the circumstances, numerous labeling technique has been proposed and overworked by several researchers.

In this paper, a new labeling procedure named odd-even congruence labeling is established based on modular division. A graph G is identified as odd-even congruence graph, if its vertex set and edge set are tagged with distinct odd and even integers respectively, further

$f(s_p) \equiv f(s_q)(mod \ \ g(w))$, s_p and s_q are adjacent vertices in G. This paper is devoted for investigating the existence of odd-even congruence graphs of complete bipartite graph, Comb graph and spliting graph of a star graph. In addition, graph acquired by connecting two copies of even cycle C_r by a path P_t, the tensor product of $K_{1,t}$ & P_2 and the shadow graph of the path P_t are proved as odd-even congruence graph.

2. Preliminaries

Definition 2.1[2] Bipartite graph $G[X,Y]$ is recognized as *complete bipartite graph* $K_{r,t}$, the edges occur in between every distinct pair of $s_p f_q$ such that $s_p \in X$ and $f_q \in Y$.

Definition 2.2[4] C*omb graph* $P_t \odot K_1$ is constructed by introducing single pendent edge to link all the vertices in P_t.

Definition 2.3[7] In a graph $G(V,E)$, if the edge set is $E = \{s_p f \,/\, s_p f \in V \ \ \& \ \ s_p \neq f\}$ and f is a fixed vertex then G is stated as a *star graph*, it is denoted by S_t.

Definition 2.4[5] The *spliting graph*, $Spl(G)$ is arrived by introducing a new vertex f_p for each existing vertex s_p moreover $N(s_p) = N(f_p)$, where $N(s_p)$ is the neighborhood of s_p.

Definition 2.5[11] *Tensor product* $G_1 \otimes G_2$ of G_1 and G_2 is a graph whose vertices and edges are $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \otimes G_2) = \{(s_1,s_2)(s_3,s_4) | s_1 s_3 \in E(G_1)$ and $s_2 s_4 \in E(G_2)\}$.

Definition 2.6[12] S*hadow graph* $D_2(G)$ is constituted by picking G' and G'' alike G and introduce edges in between $s' \in G'$ and $s'' \in G''$ where s'' is the neighbors of parallel vertex of s'.

Definition 2.7[13] A bijection $h: V \to \{1,2,....d\}$ and $k: E \to \{1,2,\ldots.d-1\}$ of G is claimed as *congruence graph*, if $h(s_p) \equiv h(s_q)(mod \ \ k(w_p))$, where $d = min\{2|V|, 2|E|\}$.

Definition 2.8[14] A bijection $h: V \to \{1,2,....2d+1\}$ and $k: E \to \{2,4,\ldots.2d\}$ of G is referred as *odd-even congruence graph*, if $h(s_p) \equiv h(s_q)(mod \ \ k(w_p))$, where $d = min\{2|V|, 2|E|\}$.

3. Main Results

In this section, simple finite connected graph $G = (V,E)$ with $|V| = r$ and $|E| = t$ were considered and proved that it admits odd-even congruence labeling.

Theorem 3.1 Every $K_{r,t}$ is odd-even congruence graph for $r \geq 1, t \geq 1$.

Proof:

Consider, $K_{m,n}$ with $|V| = r + t$ and $|E| = rt$.

$d = min\{2|V|, 2|E|\}$

For $G = K_{r,t}$ we have,

$d = min\{2(r+t), 2rt\}$

$= 2(r+t)$

there exist two independent and disjoint vertex sets such as

$V_1 = \{s_1, s_2, \ldots s_r\}$ and $V_2 = \{f_1, f_2, \ldots. f_t\}$. The edge set be $E = \{w_1, w_2, \ldots w_{rt}\}$.

where $w_1 = s_1 f_1$, $w_2 = s_2 f_1, \ldots\ldots., w_m = s_r f_1, \ldots, e_{r+1} = s_1 f_2, \ldots., e_{rt} = s_r f_t$.

Bijection $h: V(G) \to \{1,3,\ldots.,4r+4t+1\}$

and $k: E(G) \to \{2,4,\ldots.,4m+4n\}$ is defined as

$V_1: \ h(s_p) = 2p - 1$, for all $p = 1$ to r

$V_2: \ h(f_q) = 2r(t+1) - 2rq + 1$, for all $q = 1$ to t and

$k(w_l) = 2rt - 2(l-1)$, for all $l = 1$ to rt

To prove the existence of odd-even congruence labeling, consider

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 1 (2024)

$h(s_p) - h(f_q) \equiv (mod\ \ k(w_p))$
$(2p-1)-(2rt+2r-2rq+1) \equiv (mod\ \ (2rt-2l+2))$
$(2rt-2l+2)$ divides $(2p-1-2rt-2r+2rq-1)$ for all p and q.
Hence, every complete bipartite graph is odd-even congruence graph.

Example 3.2

Consider the graph $G = K_{3,2}$ with $|V_1| = 3$ and $|V_2| = 2$.

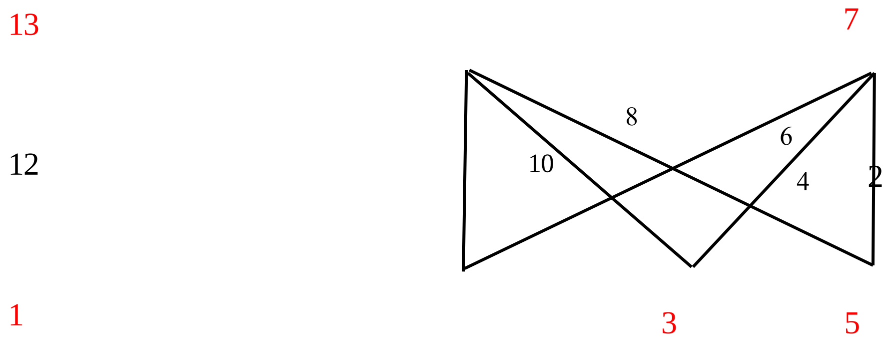

Figure - 1 - $K_{3,2}$

The odd-even congruence labeling of Complete bipartite graph is revealed in Figure - 1.

Suppose $G = K_{4,4}$ with $|V_1| = 4$ and $|V_2| = 4$.

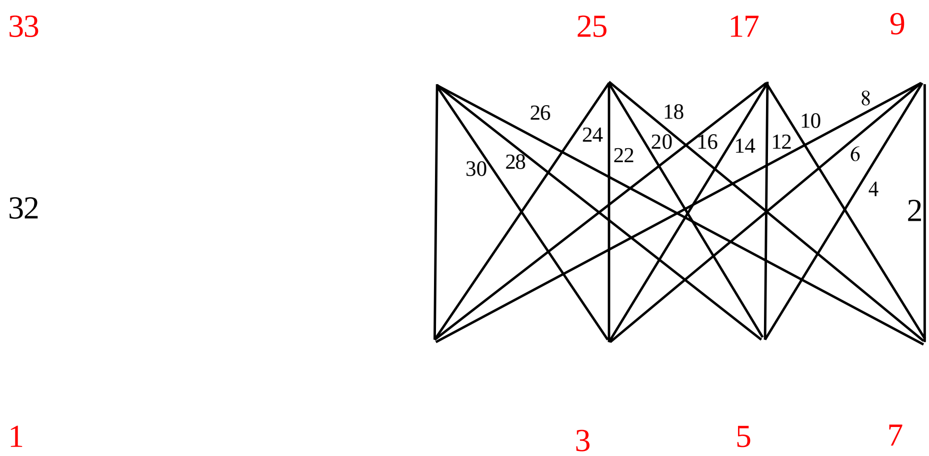

Figure - 2 - $K_{4,4}$

Figure - 2, shows that $K_{4,4}$ receives odd-even congruence labeling

Theorem 3.3

Comb graph $P_t \odot K_1$ is odd-even congruence graph, $t \geq 1$.

Proof:

Let $G = P_t \odot K_1$ with $|V| = 2t$ and $|E| = (2t-1)$.
Then $d = min\,(2\,(2t), 2\,(2t-1))$
$= 4t-2$
Let $\{s_p / 1 \leq p \leq t\}$ and $\{f_p / 1 \leq p \leq t\}$ are vertex sets, here f_p represents pendent vertices. Further, $w_p = \{s_p s_{p+1} / 1 \leq p \leq t-1\}$ are edges of P_t and edges adjacent to f_p are denoted as $\{e_p = s_p f_q / 1 \leq p \leq t\}$.
Define the bijection $h : V \rightarrow \{1, 3, \ldots, 8t-3\}$ as

$h(s_{2p-1}) = 4(t-p)+3$
$h(s_{2p}) = 4p-1$
$h(f_{2p-1}) = 4p-3$
$h(f_{2p}) = 4(t-p)+1$
where $p = 1$ to $(t/2)$ if t is even
$p = 1$ to $(t+1)/2$ if t is odd
The edge labeling $k : E(G) \rightarrow \{2,4,...,8t-4\}$ is defined as
$k(w_p) = 4(t-p), p = 1$ to $t-1$
$k(e_p) = 4(t-p)+2,\ p = 1$ to t
Clearly, $k(w_i)$ divides $(h(s_{2p-1}) - h(s_{2p}))$ and $k(e_p)$ divides $(h(s_p) - h(f_p))$
Hence, the comb graph $P_t \odot K_1$ is odd-even congruence graph.

Example 3.4

Consider a graph $G = P_t \odot K_1$ with $t = 10$.

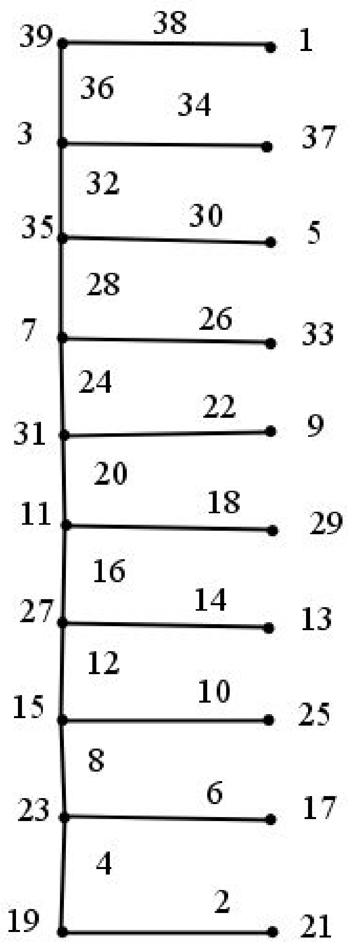

Figure - 3 - $P_{10} \odot K_1$

Figure - 3 exhibits the odd-even congruence labeling of the comb graph $P_{10} \odot K_1$.

Theorem 3.5

Spliting graph of a star graph S_t is odd-even congruence graph.

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Proof:

Suppose $G = Spl\,(S_t)$ is splitting graph with $|V| = 2t+2$ and $|E| = 3t$.

The vertex set be $V(G) = \{s, s_1, s_2, \ldots s_t\} \cup \{f, f_1, f_2, \ldots f_t\}$

where, $s, s_1, s_2, \ldots s_t$ is vertex set of star graph and s is apex vertex

$f, f_1, f_2, \ldots f_t$ are the vertices added to form G.

Also the edge set be

$E(G) = \{w_1, w_2, \ldots, w_t, w_{t+1}, \ldots., w_{2t}, w_{2t+1}, \ldots, w_{3t}\}$

here, $w_1, w_2, \ldots, w_t$ are the edges of S_n, $w_{t+1}, \ldots., w_{2t}$ are edges adjacent to f and s_p and $w_{2t+1}, \ldots.., w_{3t}$ are edges adjacent to s and f_p.

Now, $d = min\,(4t+4, 6t)$

$$= 4t+4$$

Then $h : V(G) \to \{1, 3, \ldots., 8t+9\}$ and

$k : E(G) \to \{2, 4, \ldots., 8t+8\}$ are assigned as

$h(s_p) = 6t - 4p + 7, p = 1$ to t

$h(s) = 1$

$h(f) = 3$

$h(f_p) = 2p + 3,\ p = 1$ to t

$$k\left(w_q\right) = \begin{cases} 7t - 4q + 2; & for\ q = 1\ to\ t \\ 10\mathrm{t} - 4\mathrm{q} + 4\,; & for\ \mathrm{q} = \mathrm{t} + 1\ \ to\ 2\mathrm{t} \\ 2\mathrm{q} - 4\mathrm{t} + 2\,; & for\ \mathrm{q} = 2\mathrm{t} + 1\ to\ 3\mathrm{t} \end{cases}$$

The above labeling construction satisfies $h(f_p) \equiv h(s)(mod\ \ k(w_q))$ for every edge of G.

Hence, spliting graph of a star graph is odd-even congruence graph.

Example 3.6

Consider the graph $G = Spl\,(S_t)$ with $t = 8$

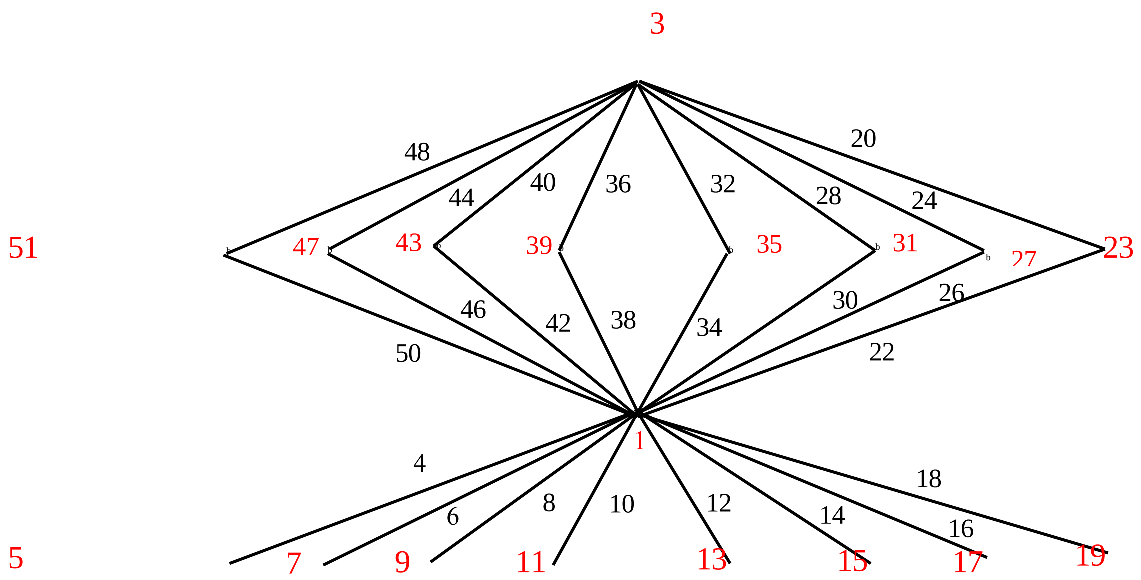

Figure - 4 - $Spl\,(S_8)$

The given $G = Spl\,(S_8)$ admits odd-even congruence labeling and it depicts in figure – 4

Theorem 3.7

The graph acquired by connecting two copies of even cycle C_r by P_n is odd-even congruence graph.

Proof:

Suppose G is acquired by connecting two copies of even cycle C_r by P_t, with $|V| = 2r + t - 2$ and $|E| = 2r + t - 1$ edges.

Let $s_1, s_2, \ldots s_r, s_{r+1}, s_{r+2}, \ldots, s_{2r+t-3}, s_{2r+t-2}$ be the vertices of G.

The path s_1 to s_{2r+t-2} form a spanning path in G.

The vertex s_m and $s_{[r+(t-2])+1}$ are the common vertex of the first and second C_r & P_t respectively.

$d = min(2(2r+t-2), 2(2r+t-1))$

$= 2(2r+t-2)$

Define $h : V(G) \to \{1,3,\ldots,(8r+4t-4)\}$ as following

for $1 \le p \le r-1$

$$h(s_p) = \begin{cases} p & ; p \text{ is odd} \\ d+7-p; & p \text{ is even} \end{cases}$$

for $r \le p \le (3r)/2+t-1$

$$h(s_p) = \begin{cases} d+8-p & ; p \text{ is odd} \\ p+1 & ; p \text{ is even} \end{cases}$$

for $(3r)/2+t \le p \le 2r+t-2$

$$h(s_p) = \begin{cases} d+6-p & ; p \text{ is odd} \\ p+3 & ; p \text{ is even} \end{cases}$$

then the edges of G are labeled as

$k(w_p) = |\, h(s_p) - h(s_q) \,|$

Obviously, $h(s_p)$ satisfies modulo division by $k(w_p)$.

Thus, the graph acquired by connecting two copies of even cycle C_r by a path P_t is odd-even congruence graph.

Example 3.8

Let G be a graph acquired by connecting two copies of even cycle C_{10} by a path P_5

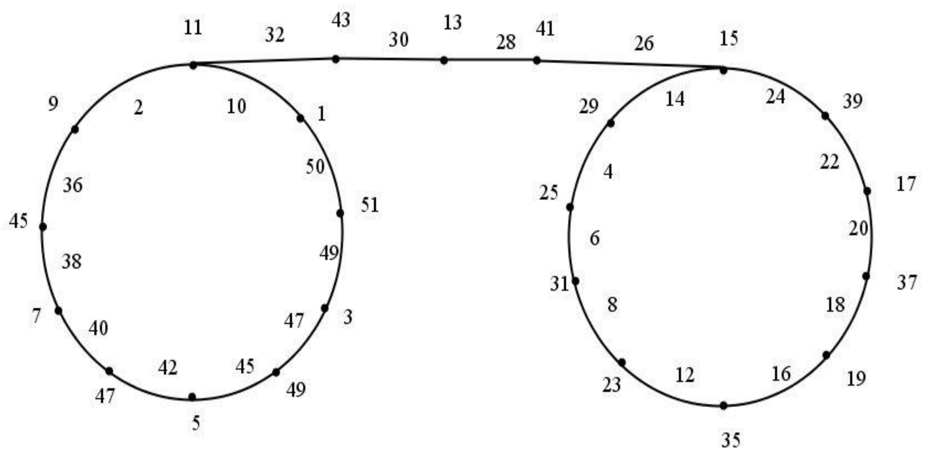

Figure - 5

Figure - 5 reveals that the graph acquired by connecting two copies of even cycle C_{10} by a path P_5 is an odd-even congruence graph

Theorem 3.9

$K_{1,t} \otimes P_2$ is odd-even congruence graph.

Proof:

The tensor product of star graph $K_{1,t}$ and path P_2 is denoted as $G = K_{1,t} \otimes P_2$ with $|V| = 2t+2$ and $|E| = 2t$.

Let $s_1, s_2, \ldots s_{t+1}$ are vertex set of $K_{1,t}$, s_1 is apex vertex and f_1, f_2 is vertex set of P_2.

$V(G) = (s_1, f_1),(s_2, f_1), \ldots, (s_{t+1}, f_1),(s_1, f_2),(s_2, f_2), \ldots, (s_{t+1}, f_2)$

$E(G) = e_1, e_2, \ldots, e_t, e_{t+1}, \ldots, e_{2t}$

where,

$w_1, w_2, \ldots, w_n$ are the edges adjacent with the vertex (s_p, f_1) and

$w_{t+1}, w_{t+2}, \ldots, w_{2t}$ are the edges adjacent with the vertex (s_q, f_2)

$d = min(2(2t+2), 2(2t))$

$= 4t$

Label the vertices $h : V(G) \rightarrow \{1,3,\ldots 8t+1\}$ and edges $k : E(G) \rightarrow \{2,4,\ldots,8t\}$ are labeled in the following way

$h(s_p, f_1) = 2p-1$ for $1 \le p \le t+1$

$h(s_q, f_2) = 2(t+q)+1$ for $1 \le q \le t+1$

$$k(w_p) = \begin{cases} 3t+2(p-1) & ; for\ p = 1\ to\ t \\ 4t-2p+2 & ; for\ p = t+1\ to\ 2t \end{cases}$$

Evidently, $2(t+q)+1-2p+1 \equiv (mod(3t+2p-2))$

$i.e, (3t+2p-2)$ divides $(2(t+q)-2p+2)$

Hence, $K_{1,t} \otimes P_2$ was an odd-even congruence graph.

Example 3.10

Let $G = K_{1,4} \otimes P_2$, $t = 4$

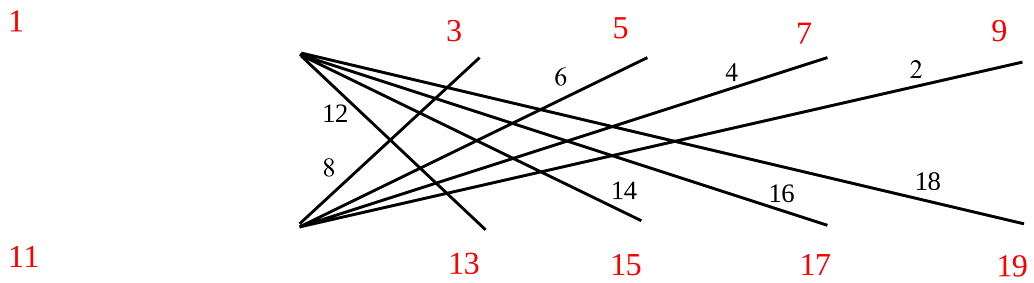

Figure - 6 $K_{1,4} \otimes P_2$

Figure - 6 represents the odd-even congruence labeling of $K_{1,4} \otimes P_2$

Theorem 3.11

The graph $D_2(P_t)$ is odd-even congruence graph.

Proof:

Suppose $G = D_2(P_t)$ is the shadow graph of the path P_t with $|V| = 2t$ and $|E| = 4(t-1)$.

Let $s_1, s_2, \ldots s_t$ be the vertices of first P_t and $f_1, f_2, \ldots, f_t$ are vertices of second path P_t.

Here, $d = min(2(2t), 2(4(t-1)))$

$= 4t$

The vertices $h : V(G) \rightarrow \{1,3,......8t+1\}$ and edges $k : E(G) \rightarrow \{2,4,\ldots\ldots 8t\}$ are labeled as given below

$$h(s_p) = \begin{cases} 4p-3 & ; p \text{ is odd} \\ 8t-4p+1 & ; p \text{ is even} \end{cases}$$

$$h(f_p) = \begin{cases} 4p-1 & ; p \text{ is odd} \\ 8t-4p-3 & ; p \text{ is even} \end{cases}$$

The edge $w_p = sf$ are labeled as follows

$k(w_p) = |h(s) - h(f)|$

Apparently, the vertex label satisfies modulo division by its corresponding edge label. Hence $D_2 (P_t)$ is odd-even congruence graph.

Example 3.12

Suppose $G = D_2 (P_6)$ with $t = 6$

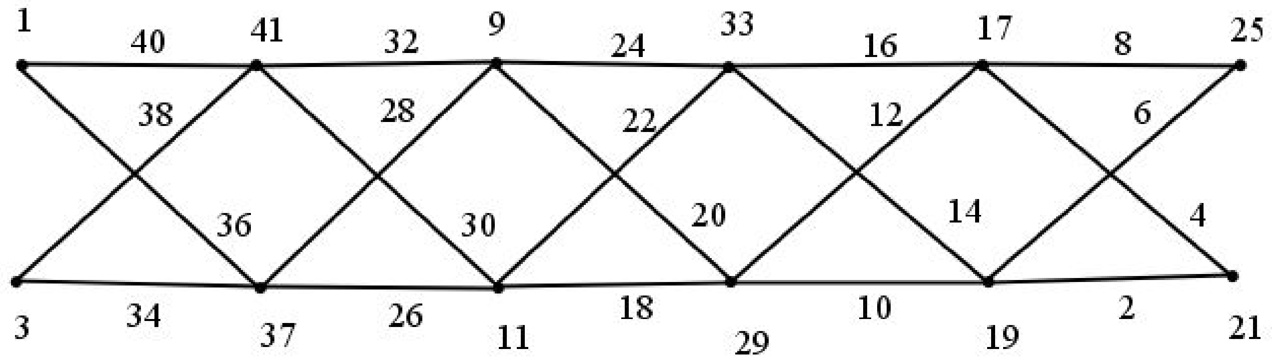

Figure - 7 $D_2 (P_6)$

Odd-even congruence labeling of the shadow graph $D_2 (P_6)$ is exposed in figure – 7

4. Conclusion

Labeling in graph theory has paid more attention for many researchers. New concept of labeling such as odd-even congruence labeling based on modulo division has been defined. This paper examines the existence of odd-even congruence labeling for complete bipartite graph, Comb graph, spliting graph of a star graph and graph acquired by connecting two copies of even cycle C_r by a path P_t were proved. Also, it is proved that $K_{1,t} \otimes P_2$ and $D_2 (P_t)$ admits odd-even congruence labeling. The odd-even congruence labeling is open to investigate for some other family of graphs and can be applied in communication networks to gaurd the informations.

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