Some Results on Odd-Even Congruence Labeling of Graphs

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\textbf{Abstract:}

\textbf{Introduction:} Labeling of graphs has been introduced in 1966. Assignment of natural numbers to vertices and/or edges is referred as graph labeling. Inspired by the ample application of graph labeling technique in real life problems, multifarious labeling strategy was adopted and investigated by many researchers.

\textbf{Objectives:} Graph labeling plays a vital role in various fields and can be implemented in multitudinous discipline including coding theory, X-ray, Psychology, crystallography, circuit design, communication networks, astronomy, radar, data security, secret sharing, database management and so on. Apart from these labeling techniques serve as a model to understand discrete mathematical domains.

\textbf{Methodology:} In this paper, an attempt has been made to introduce new labeling such as odd-even congruence labeling. Congruence Graph Labeling is an allocation of natural numbers as labels for the edges and vertices of a graph based on modular arithmetic property. Odd-even congruence labeling is an allocation of odd integers to vertices and even integers to edges in addition to congruence graph labeling.

\textbf{Result:} The suggested labeling has been identified on complete bipartite graph, comb graph and splitting graph of a star graph. Further, it is proved that graph acquired by connecting two copies of even cycle $C_r$ by a path $P_t, K_1,t \otimes P_2$ and $D_2 (P_t)$ are odd-even congruence graph.

\textbf{Keywords:} Labeling, Congruence labeling, Odd-even congruence labeling, Tensor graph.

1. Introduction

The relation between any of the objects in the real world can be represented as graphs, whose structures, properties, interrelations and correlations are interpreted in Graph Theory. The graph comprises of dots associated by lines which are referred as vertices and edges respectively. It aids the researchers to frame the hypothesis and establish the solution certainly.

Graph labeling was procured as the most prominent one among the multifarious conception of graph theory to design the graphical model of the real life situations. In the middle of 19\textsuperscript{th} century, A.Rosa introduced graph labeling\cite{8}. The vertices and edges of the graphs are labeled with natural numbers with certain constraints in order to discriminate individually. Inspired by its comprehensive application in modeling all the circumstances, numerous labeling technique has been proposed and overworked by several researchers.

In this paper, a new labeling procedure named odd-even congruence labeling is established based on modular division. A graph $G$ is identified as odd-even congruence graph, if its vertex set and edge set are tagged with distinct odd and even integers respectively, further
Consider the bijection \( h \) that for each vertex \( s_p \) of \( G \) maps it to \( h(s_p) = s_q \mod g(w) \), where \( s_p \) and \( s_q \) are adjacent vertices in \( G \). This paper is devoted for investigating the existence of odd-even congruence graphs of complete bipartite graph, Comb graph and splitting graph of a star graph. In addition, graph acquired by connecting two copies of even cycle \( C_r \) by a path \( P_t \), the tensor product of \( K_{1,t} \& P_2 \) and the shadow graph of the path \( P_t \) are proved as odd-even congruence graph.

2. Preliminaries

**Definition 2.1** [2] Bipartite graph \( G [X,Y] \) is recognized as complete bipartite graph \( K_{r,t} \), the edges occur in between every distinct pair of \( s_pf_q \) such that \( s_p \in X \) and \( f_q \in Y \).

**Definition 2.2** [4] Comb graph \( P_t \odot K_1 \) is constructed by introducing single pendent edge to link all the vertices in \( P_t \).

**Definition 2.3** [7] In a graph \( G(V,E) \), if the edge set is \( E = \{s_pf_/ s_p,h \in V \& s_p \neq s_q \} \) and \( f \) is a fixed vertex then \( G \) is stated as a star graph, it is denoted by \( S_t \).

**Definition 2.4** [5] The splitting graph, \( Sp(G) \) is arrived by introducing a new vertex \( f_p \) for each existing vertex \( s_p \), moreover \( N(s_p) = N(f_p) \), where \( N(s_p) \) is the neighborhood of \( s_p \).

**Definition 2.5** [11] Tensor product \( G_1 \otimes G_2 \) of \( G_1 \) and \( G_2 \) is a graph whose vertices and edges are \( V (G_1 \otimes G_2) = V (G_1) \times V (G_2) \) and \( E(G_1 \otimes G_2) = \{(s_1,s_2)\vert s_1s_2 \in E(G_1) \text{ and } s_2s_4 \in E(G_2) \} \).

**Definition 2.6** [12] Shadow graph \( D_2(G) \) is constituted by picking \( G' \) and \( G'' \) alike \( G \) and introduce edges in between \( s' \in G' \) and \( s'' \in G'' \) where \( s'' \) is the neighbors of parallel vertex of \( s' \).

**Definition 2.7** [13] A bijection \( h : V \rightarrow \{1,2,\ldots,d\} \) and \( k : E \rightarrow \{1,2,\ldots,d-1\} \) of \( G \) is claimed as congruence graph, if \( h(s_p) = h(s_q) \mod k(w_p) \), where \( d = \min \{2|V|,2|E|\} \).

**Definition 2.8** [14] A bijection \( h : V \rightarrow \{1,2,\ldots,2d+1\} \) and \( k : E \rightarrow \{2,4,\ldots,2d\} \) of \( G \) is referred as odd-even congruence graph, if \( h(s_p) = h(s_q) \mod k(w_p) \), where \( d = \min \{2|V|,2|E|\} \).

3. Main Results

In this section, simple finite connected graph \( G = (V,E) \) with \( |V| = r \) and \( |E| = t \) were considered and proved that it admits odd-even congruence labeling.

**Theorem 3.1** Every \( K_{r,t} \) is odd-even congruence graph for \( r \geq 1, t \geq 1 \).

**Proof:**

Consider, \( K_{m,n} \) with \( |V| = r + t \) and \( |E| = rt \).

\[ d = \min \{2|V|,2|E|\} \]

For \( G = K_{r,t} \) we have,

\[ d = \min \{2(r+t),2rt\} \]

\[ = 2(r+t) \]

there exist two independent and disjoint vertex sets such as

\[ V_1 = \{s_1,s_2,\ldots,s_r\} \] and \( V_2 = \{f_1,f_2,\ldots,f_t\} \). The edge set be \( E = \{w_1,w_2,\ldots,w_{rt}\} \),

where \( w_1 = s_1f_1, w_2 = s_2f_1,\ldots,w_m = s_df_1,\ldots, e_{r+1} = s_df_2,\ldots,e_{rt} = s_df_t \).

Bijection \( h : V(G) \rightarrow \{1,3,\ldots,4r+4t+1\} \) and \( k : E(G) \rightarrow \{2,4,\ldots,4m+4n\} \) is defined as

\[ V_1 : h(s_p) = 2p-1, \text{ for all } p = 1 \text{ to } r \]

\[ V_2 : h(f_q) = 2r(t+1)-2r q+1, \text{ for all } q = 1 \text{ to } t \]

\[ k(w_l) = 2rt-2(l-1), \text{ for all } l = 1 \text{ to } rt \]

To prove the existence of odd-even congruence labeling, consider...
\( h(s_p) - h(f_q) \equiv (mod \ k(w_p)) \)

\((2p - 1) - (2rt + 2r - 2rq + 1) \equiv (mod \ (2rt - 2l + 2))\)

\((2rt - 2l + 2)\) divides \((2p - 1 - 2rt - 2r + 2rq - 1)\) for all \( p \) and \( q \).

Hence, every complete bipartite graph is odd-even congruence graph.

**Example 3.2**

Consider the graph \( G = K_{3,2} \) with \( |V_1| = 3 \) and \( |V_2| = 2 \).

![Figure 1 - K3,2](https://www.example.com/figure1)

The odd-even congruence labeling of Complete bipartite graph is revealed in Figure - 1.

Suppose \( G = K_{4,4} \) with \( |V_1| = 4 \) and \( |V_2| = 4 \).

![Figure 2 - K4,4](https://www.example.com/figure2)

Figure - 2, shows that \( K_{4,4} \) receives odd-even congruence labeling

**Theorem 3.3**

Comb graph \( P_t \odot K_1 \) is odd-even congruence graph, \( t \geq 1 \).

**Proof:**

Let \( G = P_t \odot K_1 \) with \( |V| = 2t \) and \( |E| = (2t - 1) \).

Then \( d = \min (2(2t), 2(2t - 1)) \)

\( = 4t - 2 \)

Let \( \{s_p/1 \leq p \leq t\} \) and \( \{f_p/1 \leq p \leq t\} \) are vertex sets, here \( f_p \) represents pendent vertices. Further, \( w_p = \{s_p s_{p+1}/1 \leq p \leq t-1\} \) are edges of \( P_t \) and edges adjacent to \( f_p \) are denoted as \( \{e_p = s_p f_p/1 \leq p \leq t\} \).

Define the bijection \( h : V \rightarrow \{1,3,\ldots,8t-3\} \) as
\[ h(s_{2p-1}) = 4(t-p) + 3 \]
\[ h(s_{2p}) = 4p - 1 \]
\[ h(f_{2p-1}) = 4p - 3 \]
\[ h(f_{2p}) = 4(t-p) + 1 \]

where \( p = 1 \) to \( (t/2) \) if \( t \) is even
\[ p = 1 \) to \( (t+1)/2 \) if \( t \) is odd.

The edge labeling \( k: E(G) \rightarrow \{2, 4, ..., 8t - 4\} \) is defined as
\[ k(w_p) = 4(t-p), \ p = 1 \) to \( t-1 \]
\[ k(e_p) = 4(t-p) + 2, \ p = 1 \) to \( t \]

Clearly, \( k(w_i) \) divides \( h(s_{2p-1}) - h(s_{2p}) \) and \( k(e_p) \) divides \( h(s_p) - h(f_p) \)

Hence, the comb graph \( P_t \odot K_1 \) is odd-even congruence graph.

**Example 3.4**
Consider a graph \( G = P_t \odot K_1 \) with \( t = 10 \).

![Diagram](image_url)

Figure - 3 exhibits the odd-even congruence labeling of the comb graph \( P_{10} \odot K_1 \).

**Theorem 3.5**
Spliting graph of a star graph \( S_t \) is odd-even congruence graph.
Proof:
Suppose \( G = Spl(S_t) \) is splitting graph with \(|V| = 2t + 2\) and \(|E| = 3t\).
The vertex set be \( V(G) = \{s, s_1, s_2, \ldots, s_t\} \cup \{f_1, f_2, \ldots, f_t\} \)
where, \( s, s_1, s_2, \ldots, s_t \) is vertex set of star graph and \( s \) is apex vertex 
\( f_1, f_2, \ldots, f_t \) are the vertices added to form \( G \).
Also the edge set be 
\( E(G) = \{w_1, w_2, \ldots, w_t, w_{t+1}, \ldots, w_{2t}, w_{2t+1}, \ldots, w_{3t}\} \)
here, \( w_1, w_2, \ldots, w_t \) are the edges of \( S_n \), \( w_{t+1}, \ldots, w_{2t} \) are edges adjacent to \( f \) and \( s_p \) and 
\( w_{2t+1}, \ldots, w_{3t} \) are edges adjacent to \( s \) and \( f_p \).
Now, \( d = \min(4t + 4, 6t) = 4t + 4 \)

Then \( h : V(G) \to \{1, 3, \ldots, 8t + 9\} \) and 
\( k : E(G) \to \{2, 4, \ldots, 8t + 8\} \) are assigned as 
\( h(s_p) = 6t - 4p + 7, p = 1 \) to \( t \)
\( h(s) = 1 \)
\( h(f) = 3 \)
\( h(f_p) = 2p + 3, p = 1 \) to \( t \)
\( k(w_q) = \begin{cases} 
7t - 4q + 2; & \text{for } q = 1 \text{ to } t \\
10t - 4q + 4; & \text{for } q = t + 1 \text{ to } 2t \\
2q - 4t + 2; & \text{for } q = 2t + 1 \text{ to } 3t
\end{cases} \)

The above labeling construction satisfies \( h(f_p) \equiv h(s)(\text{mod } k(w_q)) \) for every edge of \( G \).
Hence, splitting graph of a star graph is odd-even congruence graph.

Example 3.6
Consider the graph \( G = Spl(S_t) \) with \( t = 8 \)

![Figure - 4 - Spl(S_8)](https://internationalpubls.com)

The given \( G = Spl(S_8) \) admits odd-even congruence labeling and it depicts in figure – 4
Theorem 3.7
The graph acquired by connecting two copies of even cycle $C_r$ by $P_n$ is odd-even congruence graph.
Proof:
Suppose $G$ is acquired by connecting two copies of even cycle $C_r$ by $P_t$, with $|V| = 2r + t - 2$ and $|E| = 2r + t - 1$ edges.
Let $s_1, s_2, \ldots, s_r, s_{r+1}, s_{r+2}, \ldots, s_{2r+t-3}, s_{2r+t-2}$ be the vertices of $G$.
The path $s_1$ to $s_{2r+t-2}$ form a spanning path in $G$.
The vertex $s_m$ and $s_{(r+(t-2))}$ are the common vertex of the first and second $C_r$ & $P_t$ respectively.
$d = \min(2(2r + t - 2), 2(2r + t - 1))$
$= 2(2r + t - 2)$
Define $h: V(G) \to \{1, 3, \ldots, (8r + 4t - 4)\}$ as following
for $1 \leq p \leq r - 1$
$h(s_p) = \begin{cases} 
  p & \text{p is odd} \\
  d + 7 - p & \text{p is even}
\end{cases}$
for $r \leq p \leq (3r)/2 + t - 1$
$h(s_p) = \begin{cases} 
  d + 8 - p & \text{p is odd} \\
  p + 1 & \text{p is even}
\end{cases}$
for $(3r)/2 + t \leq p \leq 2r + t - 2$
$h(s_p) = \begin{cases} 
  d + 6 - p & \text{p is odd} \\
  p + 3 & \text{p is even}
\end{cases}$
then the edges of $G$ are labeled as $k(w_p) = |h(s_p) - h(s_q)|$
Obviously, $h(s_p)$ satisfies modulo division by $k(w_p)$.
Thus, the graph acquired by connecting two copies of even cycle $C_r$ by a path $P_t$ is odd-even congruence graph.
Example 3.8
Let $G$ be a graph acquired by connecting two copies of even cycle $C_{10}$ by a path $P_5$

Figure 5 reveals that the graph acquired by connecting two copies of even cycle $C_{10}$ by a path $P_5$ is an odd-even congruence graph
Theorem 3.9

$K_{1,t} \otimes P_2$ is odd-even congruence graph.

Proof:

The tensor product of star graph $K_{1,t}$ and path $P_2$ is denoted as $G = K_{1,t} \otimes P_2$ with $|V| = 2t + 2$ and $|E| = 2t$.

Let $s_1, s_2, \ldots, s_{t+1}$ are vertex set of $K_{1,t}$, $s_1$ is apex vertex and $f_1, f_2$ is vertex set of $P_2$.

$V(G) = (s_1, f_1), (s_2, f_1), \ldots, (s_{t+1}, f_1), (s_1, f_2), (s_2, f_2), \ldots, (s_{t+1}, f_2)$

$E(G) = e_1, e_2, \ldots, e_t, e_{t+1}, \ldots, e_2$

where,

$w_1, w_2, \ldots, w_n$ are the edges adjacent with the vertex $(s_p, f_1)$ and $w_{t+1}, w_{t+2}, \ldots, w_{2t}$ are the edges adjacent with the vertex $(s_q, f_2)$

$d = \min (2(2t+2), 2(2t))$

Label the vertices $h : V(G) \rightarrow \{1, 3, \ldots, 8t+1\}$ and edges $k : E(G) \rightarrow \{2, 4, \ldots, 8t\}$ are labeled in the following way

$h(s_p, f_1) = 2p - 1$ for $1 \leq p \leq t + 1$

$h(s_q, f_2) = 2(t + q) + 1$ for $1 \leq q \leq t + 1$

$k(w_p) = \begin{cases} 3t + 2(p - 1) & \text{for } p = 1 \text{ to } t \\ 4t - 2p + 2 & \text{for } p = t + 1 \text{ to } 2t \end{cases}$

Evidently, $2(t+q) + 1 - 2p + 1 \equiv (\text{mod } (3t+2p-2))$

i.e. $(3t+2p-2)$ divides $(2(t+q) - 2p + 2)$

Hence, $K_{1,t} \otimes P_2$ was an odd-even congruence graph.

Example 3.10

Let $G = K_{1,4} \otimes P_2$, $t = 4$

\[ \text{Figure - 6 $K_{1,4} \otimes P_2$} \]

Figure 6 represents the odd-even congruence labeling of $K_{1,4} \otimes P_2$

Theorem 3.11

The graph $D_2(P_t)$ is odd-even congruence graph.

Proof:

Suppose $G = D_2(P_t)$ is the shadow graph of the path $P_t$ with $|V| = 2t$ and $|E| = 4(t-1)$.

Let $s_1, s_2, \ldots, s_t$ be the vertices of first $P_t$ and $f_1, f_2, \ldots, f_t$ are vertices of second path $P_t$.

Here, $d = \min (2(2t), 2(4(t-1)))$
=4t

The vertices \( h : V(G) \rightarrow \{1,3,\ldots,8t+1\} \) and edges \( k : E(G) \rightarrow \{2,4,\ldots,8t\} \) are labeled as given below

\[
h(s_p) = \begin{cases} 4p - 3 & \text{; } p \text{ is odd} \\ 8t - 4p + 1 & \text{; } p \text{ is even} \end{cases}
\]

\[
h(f_p) = \begin{cases} 4p - 1 & \text{; } p \text{ is odd} \\ 8t - 4p - 3 & \text{; } p \text{ is even} \end{cases}
\]

The edge \( w_p = sf \) are labeled as follows

\[ k(w_p) = |h(s) - h(f)| \]

Apparently, the vertex label satisfies modulo division by its corresponding edge label. Hence \( D_2(P_t) \) is odd-even congruence graph.

**Example 3.12**

Suppose \( G = D_2(P_6) \) with \( t = 6 \)

![Graph D_2(P_6)](image)

Odd-even congruence labeling of the shadow graph \( D_2(P_6) \) is exposed in figure – 7

4. **Conclusion**

Labeling in graph theory has paid more attention for many researchers. New concept of labeling such as odd-even congruence labeling based on modulo division has been defined. This paper examines the existence of odd-even congruence labeling for complete bipartite graph, Comb graph, splitting graph of a star graph and graph acquired by connecting two copies of even cycle \( C_r \) by a path \( P_t \) were proved. Also, it is proved that \( K_{1,t} \otimes P_2 \) and \( D_2(P_t) \) admits odd-even congruence labeling. The odd-even congruence labeling is open to investigate for some other family of graphs and can be applied in communication networks to guard the informations.

**References**