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Mixed Stereographic Circular Probability Model: A comparative study using M. L Estimation

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Abstract:

Now a days, circular statistics has a wide range of applications in the fields of Engineering, Geosciences, Environmental studies, Medicine, Psychology, Economics and Finance etc., such as analyzing angular movements in robotics, orientations in structural Engineering, Seismic wave directions, water and air flow patterns, brain waves in EEG signals, eye ball movements, modeling of economic indicators behaviors which cannot be analyzed using linear statistical methods as the data involves periodic nature.

Many researchers have presented discrete and continuous circular probability models; their properties are discussed. By combining two distinct probability distributions rather than utilizing a single probability model, the T-X family of probability distributions is intended to increase the efficiency and reduce errors. We developed a mixed stereographic circular probability model with this interest. Study is done on same attributes. The maximum likelihood estimators, AIC and BIC values were determined in this study. The model's output is compared with those of the current existing single circular probability models by using an eye data set from a glaucoma clinic at the University of Malaya Medical Centre in Malaysia.

Keywords: T-X family, Inverse stereographic projection, Akaike Information Criteria, Maximum likelihood(M L) estimators, Bayesian Information Criteria.

1. Introduction

Mardia and Jupp (2000)[1], Jammalamadaka S. Rao and Sen Gupta (2001)[2] discussed the methodologies like wrapping, stereographic projection, offset, and increasing sun function etc., to develop circular probability models and formulas to evaluate the various characteristics of these models. Minh and Farnum (2003)[3] developed 'inverse stereographic methodology' using bilinear transformation to induce new circular probability models. Motivated by this Toshihiro Abe et al., (2010)[4], A. V. Dattatreya Rao, S. V. S. Girija, and Y. Phani(2006)[8], Phani Yedlapalli et al., (2020)[14] developed various circular probability models and studied their characteristics. Rao AV Dattatreya, and S. V. S. Girija(2019)[12] published a book which describes the research work done in circular statistics in various previously mentioned methodologies.

Transformed-transformer families of linear probability distributions are developed to increase the efficiency and reduce the errors rather than utilizing single probability models. Alzaatreh, Lee et al.,(2013)[5] developed a method for inducing this type of models using quantile functions. Later so many researchers like Rambli, Adzhar et al., (2015)[6], Okasha, Mahmoud K et al., (2015)[7], R. Subba Rao, R. R. L. Kantam and G. Prasad, (2016)[9] and Korkmaz, Mustafa Ç et al.,(2017)[10] worked on mixed linear probability models. New half circular Burr type models were induced by Rambli, Adzhar, et al.,(1029) [13],Abuzaid, Ali H(2018)[11] using inverse stereographic projection and the AIC,BIC and parameter values are estimated. Ayesha Iftikhar et al.,(2022)[15] generated a modified hc burr III distribution and conducted a comparative study with the existing half circular models.

Section 2 describes the fundamental definitions needed for this research work. In section 3 parameters of the Stereographic Weibull Rayleigh Distribution (SWRD) are estimated using M L method of estimation. Section 4 shows the comparative study of SWRD with current existing circular probability models and follows section 5 discusses the conclusion and scope of the work.

2. Fundamental definitions

T-X family:

For a random variable X, the T-X class of distributions' probability distribution function is

$$\mathrm{J(x)} = \int_{a}^{W(L(x))} m(t)dt = M\{W\big(L(x)\big)\} \tag{2.1}$$

and the associated density function is

$$j(x) = \frac{d}{dx} W\big(L(x)\big) m\{W\big(L(x)\big)\} \tag{2.2}$$

When a continuous random variable T>0 specified on $[0,\infty)$ has a cumulative probability distribution M(t) and a probability density function m(t).

L(x) is the cumulative distribution function of a random variable X.

Maximum likelihood estimators:

The most common method for estimating the parameters of the probability model that best fits a given data set is maximum likelihood estimation. Consider a random sample of size n,

$z^T = (z_1, z_2, \ldots, z_n)$, taken from a population with distribution$f\left(\cdot/_{\emptyset}\right)$, with $\emptyset$, a vector of parameters. $\emptyset$ has the likelihood function $L\left(^{\emptyset}/_{z}\right) = \prod_{i=1}^{n} f\left(^{z_i}/_{\emptyset}\right)$. The log-likelihood function $logL\left(^{\emptyset}/_{z}\right)$ can be maximized to estimate the parameters.

Akaike Information Criteria:

A statistical measure called the Akaike Information Criterion (AIC) is used to assess and contrast various models according to their complexity and goodness of fit. A better trade-off between model fit and complexity is indicated by lower AIC values. In most cases, the model with the lowest AIC is chosen when evaluation different models.

The formula for calculating AIC is

$$AIC = 2k - 2ln\,\hat{L}$$

Bayesian Information Criteria:

A statistical measure called the Bayesian Information Criterion (BIC) is used to assess a model's goodness of fit while penalizing for model complexity to prevent over fitting.

$$BIC = kln(n) - 2ln\,\hat{L}$$

Where k is the parameters count,

n is the number of data points, and

$\hat{L}$ is the model's M L function value

Inverse Stereographic Projection:

Inverse Stereographic Projection is defined as a one-to-one mapping that is given by,

$T(\Phi) = y = k + l\,\frac{sin\Phi}{1+cos\Phi} = k + ltan\left(\frac{\Phi}{2}\right)$

where y∈(-∞, ∞), $\boldsymbol{\Phi}$∈[-π, π), k∈R, and l>0. (2.3)

Then by Minh and Farnum(2003)[4]

$T^{-1}(y) = \Phi = 2tan^{-1}\left(\frac{y-k}{l}\right)$ (2.4)

is a random point on the unit circle.

Stereographic Weibull Rayleigh Distribution:

If the probability density function and distribution function of a random variable θ on a unit circle are, respectively, defined by,

$h(\theta;\alpha,\lambda) = \frac{l^2\alpha}{2\lambda p^2} tan\left(\frac{\theta}{2}\right) sec^2\frac{\theta}{2}\left(\frac{l^2}{2\lambda p^2} tan^2\left(\frac{\theta}{2}\right)\right)^{\alpha-1} \boldsymbol{e}^{-\left(\frac{l^2}{2\lambda p^2}\boldsymbol{tan}^2\left(\frac{\theta}{2}\right)\right)^{\alpha}}$, (2.5)

$$\alpha > 0, \lambda > 0, p > 0, 0 \leq \theta \leq \pi$$

$\mathrm{H}(\theta;\alpha,\lambda) = 1 - e^{-\left(\frac{l^2}{2\lambda p^2}\boldsymbol{tan}^2\left(\frac{\theta}{2}\right)\right)^{\alpha}}$

then the variable is said to have a Stereographic Weibull Rayleigh Distribution with shape parameter $\alpha > 0$ and scale parameter $\lambda > 0$.

3. Parameter Estimation of SWRD:

The likelihood function of SWRD is given by

$$L = \prod_{i=1}^{n} h(\theta)$$

$$L = \prod_{i=1}^{n}\left(\frac{l^2\alpha}{2\lambda p^2} tan\left(\frac{\theta}{2}\right) sec^2\frac{\theta}{2}\left(\frac{l^2}{2\lambda p^2} tan^2\left(\frac{\theta}{2}\right)\right)^{\alpha-1} e^{-\left(\left(\frac{l^2}{2\lambda p^2} tan^2\left(\frac{\theta}{2}\right)\right)^{\alpha}\right)}\right)$$

$$logL = \sum_{i=1}^{n} log\frac{l^2}{2p^2} + \sum_{i=1}^{n} log\alpha + \sum_{i=1}^{n} log\frac{1}{\lambda} + \sum_{i=1}^{n} log\left(tan\frac{\theta_i}{2}sec^2\frac{\theta_i}{2}\right) +$$

$$(\alpha-1)\sum_{i=1}^{n} log\frac{l^2}{2\lambda p^2}tan^2\frac{\theta_i}{2} - \sum_{i=1}^{n}\left(\frac{l^2}{2\lambda p^2}tan^2\frac{\theta_i}{2}\right)^{\alpha} \quad (3.1)$$

Partially differentiate (6) with respect to α and equate this expressions to zero,

$$\frac{\partial logL}{\partial\alpha} = 0$$

$$\frac{n}{\alpha} + \sum_{i=1}^{n} log\left(\frac{l^2}{2\lambda p^2}tan^2\frac{\theta_i}{2}\right) - \sum_{i=1}^{n}\left(\frac{l^2}{2\lambda p^2}tan^2\frac{\theta_i}{2}\right)^{\alpha} log\left(\frac{l^2}{2\lambda p^2}tan^2\frac{\theta_i}{2}\right) = 0$$

Let $\frac{l^2}{2\lambda p^2}tan^2\frac{\theta_i}{2} = A$

$$\frac{n}{\alpha} + \sum_{i=1}^{n} logA - \sum_{i=1}^{n} A^{\alpha} logA = 0 \quad (3.2)$$

Partially differentiate (6) with respect to λ and equate this expressions to zero,

$$\frac{\partial logL}{\partial\lambda} = 0$$

$$n\alpha\,\lambda - \sum_{i=1}^{n} \alpha A^{\alpha-1}\left(\frac{l^2}{2p^2}tan^2\frac{\theta_i}{2}\right) = 0 \quad (3.3)$$

The maximum likelihood estimators (MLEs) of the unknown parameters can be obtained by solving equations (7) and (8) concurrently, although these problems are not analytically solvable.

fitdistrplus, AdequacyModel and pracma packages in R are used to analyze this model at MLE.

4. Comparative Study

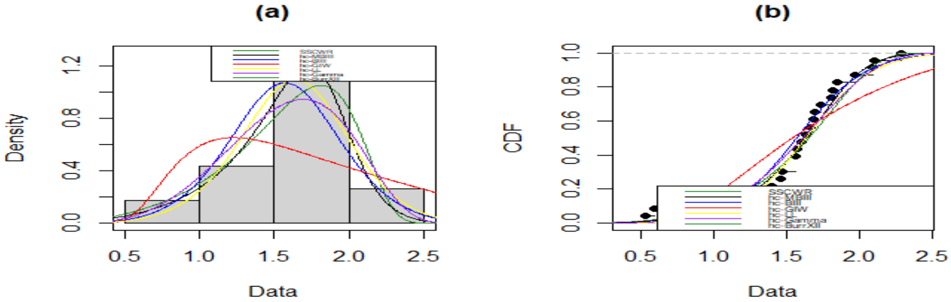
Using the above mentioned R packages the AIC,BIC and parameter values are estimated for

hc-MBIII, hc-Log logistic, hc-Burr III, , hc-Gamma, hc-GIW and hc-Burr-XII distribution(Iftikar Ayesha et al.,[15]) and the results are tabulated below.

Table 1: Comparative study

Distribution	AIC	CAIC	BIC	A	W	K-S (p-value)
SWRD	13.3696	14.03627	15.45864	0.9184	0.15577	0.16388(0.6255)
hc-MBIII	25.23826	26.50142	28.64474	0.44517	0.06778	0.127358 (0.84968)
hc-Burr III	26.74478	7.34478	29.01577	0.815283	0.11913	0.183918(0.41799)
hc-GIW	43.54607	44.80923	6.95255	2.4295	0.41253	0.27321(0.06453)
hc-Log logistic	26.14425	26.74425	28.41524	0.74076	0.10753	0.116541(0.913587)
Hc-Gamma	26.17461	26.77461	28.4456	0.8166877	0.127228	0.16989(0.520301)
Hc-Burr-XII	26.69013	27.29013	28.96112	0.818624	0.119508	0.16551(0.55440)

The proposed stereographic circular distribution, SWRD fits the eye data better than the models previously mentioned since smaller AIC values signify a better fit which are graphically depicted as follows.(In graph SSCWR shows the curve of SWRD)



5. Conclusion

In this research work, we estimated the parameters of the mixed circular probability model 'SWRD' using Maximum Likelihood estimation technique. Also AIC, BIC values for this model are evaluated and compared these results with the existing circular models like hc-MBIII, hc-Log logistic, hc-Burr III, hc-Gamma, hc-GIW and hc-Burr-XII. In comparison to these models, the proposed model is best fit to the data representing the posterior corneal curvature of 23 eye patients. Thus researches and those who work with angular data can considerably benefit from include this distribution in the theoretical framework.

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