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Convergence Analysis of S-iteration Process of Generalized Nonlinear Variational Inclusion Problem

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Abstract:

To obtain the solution of generalized variational inclusion involving A(.,.) co-coercive operators, a proposal for E-iteration has also been proposed and analyzed. Existence theorems for the solution of generalized variational inclusion are proved by using co-coercive and relaxed co-coercive mappings. Also, certain particular cases, along with their comparison with some methods, have been studied. Finally, we present a numerical example to exemplify and show the convergence of the suggested algorithm in support of our main result, which has been formulated by using MATLAB programming.

Keywords: Algorithm, S-iterative process, A(.,.)-co-coercive operator, Resolvent operator, Sequence analysis

1. Introduction

Variational inclusions represent an extended category of problems beyond variational inequalities, and they hold a significant and elegant position in the fields of optimization and nonlinear analysis. Variational inclusions/inequalities involve applications in different fields like mechanics, physics, non-linear programming, optimization, and control theory. For details, see [1, 4–11, 13–15, 17–19] and the references therein. To solve variational inclusion many iterative techniques have been developed; See for example, [6,8,10,12,15,16].

In 2016, Buong et al. [6] proposed an explicit iterative algorithm to find out the solution for variational inequalities with a uniformly Gâteaux differentiable norm. To make a clear understanding, some examples have been illustrated. In 2017, Sahu et al. [15] proposed a system of generalized variational inequalities. In their research, they introduced two parallel iterative methods, namely the parallel S-iteration process and the parallel Mann iteration process, to address a particular problem. They also examined the convergence of the sequences produced by these parallel iteration methods using a numerical example. Their analysis demonstrated that the recommended parallel S-iteration process outperforms the parallel Mann iteration process. Later Ha et al. [10] suggested a simple parallel iterative method in finding out the solution to variational inequalities. It has been claimed [10] that the parallel iterative method is more straightforward the one proposed by Buong et al. [6].

In addition to this, numerical examples have been [10] to illustrate the effectiveness and superiority of the proposed algorithm. Recently Gursoy et al. [9] proposed and analyzed an S-iteration process for solving a class of variational inclusion H-monotone operator. A comparison of the suggested method has been performed along with some existing methods considered by Fang and Huang [7] and Zeng et al. [19].

Motivated by ongoing research in this direction, we have designed a S-iteration for finding the solution of generalized variational inclusion problem. Also, existence theorems are proved by using cocoercive and relaxed co-coercive mappings. A numerical example has been presented as well to illustrate convergence results.

2. Preliminaries

We represent the sets of nonnegative real numbers and nonnegative integers as R^+ and N_0 respectively. Consider a real Hilbert space denoted as X, where its inner product and norm are symbolized as and $\|.\|$ respectively.

Let $S, T, g: \mathscr{H} \to \mathscr{H}$ be three single-valued functions and $N: \mathscr{H} \to 2^{\mathscr{H}}$ be a multi-valued function. Consider the generalized variational inclusion problem (GVIP): for some real number ρ and find $w \in \mathscr{H}$ such as

$$\rho \in S(x) - T(x) + \tau N(g(x)). \tag{2.1}$$

Some exceptional cases of (2.1) are as follows:

a) If $\rho = 0, \tau = 1, S = 0$ and N is a single-valued function, then (2.1) becomes the problem of finding $w \in \mathscr{H}$ such as

$$0 \in N(g(w)) - T(w). \tag{2.2}$$

Problem (2.2) was proposed by Noor et al. [14].

b) If $\rho = 0, \tau = 1, T = 0$ and $g = I$ (identity function), then (2.1) becomes the problem of finding $w \in \mathscr{H}$ such as

$$0 \in S(w) + N(w). \tag{2.3}$$

Problem (2.3) was considered by Fang and Huang [7].

It's evident that by appropriately selecting the functions used in equation (2.1), one can identify numerous variational inclusion or inequality problems that have been investigated in recent studies, as observed in references such as [5, 11, 13].

Now, we provide certain definitions and outcomes to reach the primary conclusion of this paper.

Definition 2.1 ([2,15]) Consider a mapping $P: \mathscr{H} \to \mathscr{H}$ that takes one value at a time. A mapping $R: \mathscr{H} \to \mathscr{H}$ is termed

a) monotone (in short MT) if $\langle Rw - Ry, w - y\rangle \geq 0, \forall w, y \in \mathscr{H}$,

b) strictly MT if R is MT and

$$\langle Rw - Ry, w - y\rangle = 0, \text{ifand only if } w = y,$$

c) strongly MT if there exists $r > 0$ such as

$$\langle Rw - Ry, w - y\rangle \geq r\|w - y\|^2, \forall w, y \in \mathscr{H},$$

d) strongly MT with respect to P if there exists $\gamma > 0$ such as

$$\langle Rw - Ry, Pw - Py\rangle \geq \gamma\|w - y\|^2, \forall w, y \in \mathscr{H},$$

e) Lipschitz continuous if there exists $\lambda_R > 0$ such as

$$\|Rw - Ry\| \leq \lambda_R\|w - y\|, \forall w, y \in \mathscr{H},$$

f) α-expansive if there exists $\alpha > 0$ such as

$$\|Rw - Ry\| \geq \alpha\|w - y\|, \forall w, y \in \mathscr{H},$$

if $\alpha = 1$, then it is expansive.

g) co-coercive if there exists $\mu' > 0$ such as

$$\langle Rw - Ry, w - y\rangle \geq \mu'\|Rw - Ry\|^2, \forall w, y \in \mathscr{H},$$

h) relaxed co-coercive if there exists a constant $\gamma' > 0$ such as

$$\langle Rw - Ry, w - y\rangle \geq (-\gamma')\|Rw - Ry\|^2, \forall w, y \in \mathscr{H}.$$

Definition 2.2 ([2]) A set-valued function $N: \mathscr{H} \to 2^{\mathscr{H}}$ is termed:

a) MT if

$$\langle w - y, u - v\rangle \geq 0, \forall u, v \in \mathscr{H}, w \in Nu, y \in Nv,$$

b) strongly MT if there exists $\eta > 0$ such as

$$\langle w - y, u - v\rangle \geq \eta\|u - v\|^2, \forall u, v \in \mathscr{H}, w \in Nu, y \in Nv,$$

c) maximal MT if N is MT and $(I + \lambda N)(\mathscr{H}) = \mathscr{H}$ hold for all $\lambda > 0$, where I stands the identity function on $\mathscr{H}$;

d) maximal strongly MT if N is strongly MT and $(I + \lambda N)(\mathscr{H}) = \mathscr{H}$ hold for all $\lambda > 0$;

e) cocoercive if there exists μ'' such as

$$\langle w - y, u - v\rangle \geq \mu''\|u - v\|^2, \forall u, v \in \mathcal{H}, w \in Nu, y \in Ny.$$

Definition 2.3 ([**2**, **3**]) Let $A: \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ and $P, R: \mathcal{H} \to \mathcal{H}$ are the the functions.

a) A(P,.) is termed co-coercive with respect to P if there exists $\mu > 0$ such as

$$\langle A(Pw, u) - A(Py, u), w - y\rangle \geq \mu^2\|Pw - Py\|^2, \forall w, y \in \mathcal{H}.$$

b) $A(., R)$ is termed relaxed co-coercive with respect to R if there exists $\mu > 0$ such as

$$\langle A(u, Rw) - A(u, Ry), w - y\rangle \geq \mu^2\|Rw - Ry\|^2, \forall w, y \in \mathcal{H}.$$

c) A(P,.) is termed r_1-Lipschitz continuous with respect to P if there exists $r_1 > 0$ such as

$$\|A(Pw, .) - A(Py, .)\| \leq t_1\|w - y\|, \forall w, y \in \mathcal{H}.$$

d) $A(.,R)$ is termed r_2-Lipschitz continuous with respect to R if there exists $r_2 > 0$ such as

$$\|A(.,Rw) - A(.,Ry)\| \leq t_2\|w - y\|, \forall w, y \in \mathcal{H}.$$

Definition 2.4 ([2]) Let function $A: \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ and $P, R: \mathcal{H} \to \mathcal{H}$ are the singlevalued functions. Let $N: \mathcal{H} \to 2^{\mathcal{H}}$ be a multi-valued function. N is termed $A(.,.) -$ coocercive with respect to the functions P and R (or simply $A(.,.) -$ cocoerciveinthesequel) if N is cocoercive with respect to P and R and $[A(P,R) + \lambda N](\mathcal{H}) = \mathcal{H}$, for every $\lambda > 0$.

Definition 2.5 ([2]) Let $A(P,R)$ be μ-cocoercive with respect to P and γ-relaxed cocoercive with respect to R, P is α-expansive, R-Lipschitz continuous, and $\mu > \gamma, \alpha > \beta$. Let N be an $A(.,.) -$ cocoerciveoperatorwithrespectP and R. The resolvent operator $J_{\lambda,N}^{A(\ldots)}: \mathcal{H} \to \mathcal{H}$ is defined by

$$J_{\lambda,N}^{A(\ldots)}(w) = [A(P,R) + \lambda N]^{-1}(w), \forall w \in \mathcal{H}, \lambda > 0. \quad (2.4)$$

Lemma 2.1 ([2]). Let $A(P,Q)$ be μ-cocoercive with respect to P, γ-relaxed coocoercive with respect to R, P is α-expansive, R is β-Lipschitz continuous, and $\mu > \gamma, \alpha > \beta$. Let N be an $A(.,.) -$ cocoerciveoperatorwithtoP and R. Then the resolvent operator $J_{\lambda,N}^{A(\ldots)}: \mathcal{H} \to \mathcal{H}$ is $\frac{1}{\mu\alpha^2 - \gamma\beta^2}$-Lipschitz continuous, that is

$$\left\|J_{\lambda,N}^{A(\ldots)}(w) - J_{\lambda,N}^{A(\ldots)}(y)\right\| \leq \frac{1}{\mu\alpha^2 - \gamma\beta^2}\|w - y\|, \forall w, y \in \mathcal{H} \quad (2.5)$$

3. *S*-iteration Algorithms and Convergence Analysis

The under mentioned lemma ensures the equivalence between fixed point problem and (2.1). This serves as the inspiration for the upcoming outcome we will present.

Lemma 3.1. Let $A: \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ and $P, R, S, T, g: \mathscr{H} \to \mathscr{H}$ are single-valued functions with $g(\mathscr{H}) \cap dom(P) \neq \emptyset$ and $g(\mathscr{H}) \cap dom(R) \neq \emptyset$, and $N: \mathscr{H} \to 2^{\mathscr{H}}$ be a multi-valued function such as $A(.,.)$ co-coercive with respect to and P, R and g . Then $w \in \mathscr{H}$ is a solution of (2.1) if and only if

$$g(w) = J_{\lambda,N}^{A(.,.)}\left[A\big(Pog(w), Rog(w)\big) - \lambda\big(S(w) - T(w)\big) + \lambda\rho\right] \quad (3.1)$$

where $\lambda > 0$.

Algorithm 3.1. The iterative sequence $\{w_n\}$ for all $n \in N_0$ is s stated as

$$\begin{cases} w_0 \in \mathscr{H} \\ w_{n+1} = (1-\alpha_n)w_n + \alpha_n\Big[w_n - g(w_n) + J_{\lambda,N}^{A(\ldots)}\big[A\big(Pog(w_n), Rog(w_n)\big) \\ \qquad -\lambda\big(S(w_n) - T(w_n)\big) + \lambda\rho\big]\Big] \end{cases} \quad (3.2)$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ satisfying the condition$\sum_{n=0}^{\infty}\alpha_n = \infty$.

Algorithm 3.2. The iterative sequence $\{q_n\}$ for all $n \in N_0$ is s stated as

$$\begin{cases} q_0 \in \mathscr{H} \\ q_{n+1} = (1-\xi_n)q_n + \xi_n\Big[r_n - g(r_n) + J_{\lambda,N}^{A(\ldots)}\big[A\big(Pog(r_n), Rog(r_n)\big) \\ \qquad -\lambda\big(S(r_n) - T(r_n)\big) + \lambda\rho\big]\Big] \\ r_{n+1} = (1-\mu_n)q_n + \mu_n\Big[q_n - g(q_n) + J_{\lambda,N}^{A(\ldots)}\big[A\big(Pog(q_n), Rog(q_n)\big) \\ \qquad -\lambda\big(S(q_n) - T(q_n)\big) + \lambda\rho\big]\Big] \end{cases} \tag{3.3}$$

where $\{\xi_n\}$ and $\{\mu_n\}$ are sequences in [0,1] satisfying the condition $\sum_{n=0}^{\infty}\xi_n = \infty$.

Theorem 3.1. Let $\mathscr{H}$ be a real Hilbert space and $A: \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ and $P, R, S, T, g: \mathscr{H} \to \mathscr{H}$ are single-valued functions and $N: \mathscr{H} \to 2^{\mathscr{H}}$ be a multi-valued function such as $A(.,.)$ co- MT with respect to P, R, g operator. Assume that $A(.,.)$ is Lipschitz continuous with constant $t > 0$, mixed strongly MT with respect to P and R with constant $\delta > 0$, g is strongly MT with constant $\delta_g > 0$ and g, P, R, S, T are Lipschitz continuous with constants $\lambda_g, \lambda_P, \lambda_R, \lambda_S$ and λ_T respectively. Let $\{w_n\}$ be a iterative sequences generated by (3.1) with the sequence $\{\alpha_n\} \subset [0,1]$ and satisfying the condition $\sum_{n=0}^{\infty}\alpha_n = \infty$, and there exists a constant $\lambda > 0$ such as

$$\begin{cases} (\mu\alpha^2 - \gamma\beta^2)^2\big(1 - 2\delta_g + \lambda_g^2\big) < \big[\mu\alpha^2 - \gamma\beta^2 - t_1\lambda_P\lambda_g - t_2\lambda_R\lambda_g - \lambda(\lambda_S + \lambda_T)\big]^2, \\ \mu > \gamma \text{ and } \alpha > \beta. \end{cases} \tag{3.4}$$

Then, the following statements hold:

a) There exists $\lambda > 0$ such as

$$\kappa = \sqrt{1 - 2\delta_g + \lambda_g^2} + \frac{t_1\lambda_P\lambda_g + t_2\lambda_R\lambda_g + \lambda\lambda_S + \lambda\lambda_T}{\mu\alpha^2 - \gamma\beta^2} < 1. \tag{3.5}$$

b) The operator $F: \mathscr{H} \to \mathscr{H}$ defined by

$$F(w) = w - g(w) + J_{\lambda,N}^{A(\ldots)}\big[A\big(Pog(w), Rog(w)\big) - \lambda\big(S(w) - T(w)\big) + \lambda\rho\big], \forall w \in X \tag{3.6}$$

is κ-contraction, that is

$$\|F(w) - F(y)\| \le \kappa\|w - y\|, \forall w, y \in \mathscr{H}. \tag{3.7}$$

where κ satisfies (3.5).

c) The iterative sequence $\{w_n\}$ stated as (3.1) converges strongly to a unique solution $w^* \in \mathscr{H}$ of (2.1).

$$\begin{cases} y_0 \in \mathscr{H} \\ y_{n+1} = (1-\xi_n)y_n + \xi_n\Big[y_n - g(y_n) + J_{\lambda,N}^{A(\ldots)}\big[A\big(Pog(y_n), Rog(y_n)\big) \\ \qquad -\lambda\big(S(y_n) - T(y_n)\big) + \lambda\rho\big]\Big], \end{cases} \tag{3.8}$$

converges strongly to w^*.

Proof. Using Algorithm 3.1 and the Lipschitz continuity of the resolvent operator, we have

$$\|w_{n+1}-w_n\| = \|(1-\alpha_n)w_n+\alpha_n\Big[w_n-g(w_n)+J_{\lambda,N}^{A(....)}\big[A\big(Pog(w_n),Rog(w_n)\big)-\lambda(S(w_n)$$

$$-T(w_n))+\lambda\rho\big]\Big]-\big[(1-\alpha_n)w_{n-1}+\alpha_n[w_{n-1}-g(w_{n-1})$$

$$+J_{\lambda,N}^{A(....)}\big[A\big(Pog(w_{n-1}),Rog(w_{n-1})\big)-\lambda\big(S(w_{n-1})-T(w_{n-1})\big)+\lambda\rho\big]\big]\|$$

$$\leq (1-\alpha_n)\|w_n-w_{n-1}\|+\alpha_n\big\|w_n-w_{n-1}-\big(g(w_n)-g(w_{n-1})\big)\big\|$$

$$+\alpha_n\|J_{\lambda,N}^{A(....)}\big[A\big(Pog(w_n),Rog(w_n)\big)-\lambda\big(S(w_n)-T(w_n)\big)+\lambda\rho\big]$$

$$-J_{\lambda,N}^{A(....)}\big[A\big(Pog(w_{n-1}),Rog(w_{n-1})\big)-\lambda\big(S(w_{n-1})-T(w_{n-1})\big)+\lambda\rho\big]\|$$

$$\leq (1-\alpha_n)\|w_n-w_{n-1}\|+\alpha_n\big\|w_n-w_{n-1}-\big(g(w_n)-g(w_{n-1})\big)\big\|$$

$$+\frac{\alpha_n}{\mu\alpha^2-\gamma\beta^2}\|A\big(Pog(w_n),Rog(w_n)\big)-\lambda\big(S(w_n)-T(w_n)\big)$$

$$-A\big(Pog(w_{n-1}),Rog(w_{n-1})\big)-\lambda\big(S(w_{n-1})-T(w_{n-1})\big)\| \qquad (3.9)$$

Since g is strongly MT with δ_g and Lipschitz continuous with λ_g, we have

$$\| w_n-w_{n-1}-(g(w_n)-g(w_{n-1}))\|^2 \leq \| w_n-w_{n-1}\|^2-2\langle g(w_n)-g(w_{n-1}), w_n-w_{n-1}$$

$$+\| g(w_n)-g(w_{n-1})\|^2$$

$$\leq (1-2\delta_g+\lambda_g^2)\| w_n-w_{n-1}\|^2$$

which implies that

$$\| w_n-w_{n-1}-\big(g(w_n)-g(w_{n-1})\big)\| \leq \sqrt{1-2\delta_g+\lambda_g^2}\,\| w_n-w_{n-1}\|. \qquad (3.10)$$

Since $A(.,.)$ is Lipschitz continuous P and R, and Lipschitz continuous of P and g, we have

$$\|A\big(Pog(w_n),Rog(w_n)\big)-\lambda\big(S(w_n)-T(w_n)\big)-\big(A\big(Pog(w_{n-1}),Rog(w_{n-1})\big)-\lambda\big(S(w_{n-1})-T(w_{n-1})\big)\|$$

$$=\|A\big(Pog(w_n),Rog(w_n)\big)-A\big(Pog(w_{n-1}),Rog(w_{n-1})\big)-\lambda\big(S(w_n)-S(w_{n-1})\big)-\lambda\big(T(w_n)-T(w_{n-1})\big)\|$$

$$\leq \big\|A\big(Pog(w_n),Rog(w_n)\big)-A\big(Pog(w_{n-1}),Rog(w_{n-1})\big)\big\|+\lambda\|S(w_n)-S(w_{n-1})\|+\lambda\|T(w_n)-T(w_{n-1})\|$$

$$\leq \|A\big(Pog(w_n),Rog(w_n)\big)-A\big(Pog(w_{n-1}),Rog(w_n)\big)+A\big(Pog(w_{n-1}),Rog(w_n)\big)$$

$$-A\big(Pog(w_{n-1}),Rog(w_{n-1})\big)\|+\lambda\|S(w_n)-S(w_{n-1})\|+\lambda\|T(w_n)-T(w_{n-1})\|$$

$$\leq \big\|A\big(Pog(w_n),Rog(w_n)\big)-A\big(Pog(w_{n-1}),Rog(w_n)\big)\big\|+\|A\big(Pog(w_{n-1}),Rog(w_n)\big)$$

$$-A\big(Pog(w_{n-1}),Rog(w_{n-1})\big)\|+\lambda\|S(w_n)-S(w_{n-1})\|+\lambda\|T(w_n)-T(w_{n-1})\|$$

$$\leq t_1\lambda_P\lambda_g\|w_n-w_{n-1}\|+t_2\lambda_R\lambda_g\|w_n-w_{n-1}\|+\lambda\lambda_S\|w_n-w_{n-1}\|+\lambda\lambda_T\|w_n-w_{n-1}\|$$

$\leq (t_1\lambda_P\lambda_g + t_2\lambda_R\lambda_g + \lambda\lambda_S + \lambda\lambda_T)\|w_n - w_{n-1}\|$
(3.11)

On using Equations (3.10) and (3.11), Equation (3.9) becomes

$$\| w_{n+1} - w_n \| \leq (1-\alpha_n) \| w_n - w_{n-1} \| + \alpha_n\sqrt{1-2\delta_g+\lambda_g^2} \| w_n - w_{n-1} \|$$
$$+ \frac{\alpha_n}{\mu\alpha^2-\gamma\beta^2}\sqrt{1-2\lambda(\delta_S+\delta_T)+\lambda\lambda_S^2+\lambda\lambda_T^2} \| w_n - w_{n-1} \|$$
$$\leq [1-\alpha_n+\alpha_n\kappa] \| w_n - w_{n-1} \|$$
$$= [1-\alpha_n(1-\kappa) \| w_n - w_{n-1} \|, \quad (3.12)$$

where

$$\kappa = \sqrt{1-2\delta_g+\lambda_g^2} + \frac{t_1\lambda_P\lambda_g + t_2\lambda_R\lambda_g + \lambda\lambda_S + \lambda\lambda_T}{\mu\alpha^2-\gamma\beta^2}.$$

By condition (3.4), we have $0 \leq \kappa < 1$, thus the sequence $\{w_n\}$ is a Cauchy sequence in $\mathscr{H}$ and as $\mathscr{H}$ is complete, there exists $w^* \in \mathscr{H}$ such as $w_n \to w^*$, as $n \to \infty$. By using the continuity of the functions $g, P, R, S, T, A, J_{\lambda,N}^{A(...,)}$, and Algorithm 3.1, we have

$$g(w) = J_{\lambda,N}^{A(.,.)}[A(Pog(w), Rog(w)) - \lambda(S(w) - T(w)) + \lambda\rho].$$

From Lemma 3.1, we conclude that w^* is a solution of (2.1).

Theorem 3.2. Let $P, R, S, T, A, N, g, \kappa$ and w^* be the same as in Theorem 3.1, and let $\{w_n\}, \{q_n\}, \{r_n\}$ be the sequences defined by (3.2), (3.3) and (3.8), respectively with the sequences $\xi_n \subset [0,1]$ and $\{\mu_n\} \subset [0,1]$ satisfying the conditions $\lim_{n\to\infty} \xi_n = 0$ and $\sum_{n=0}^{\infty} \xi_n = \infty$. Then the following assertions are identical

a) $\{w_n\}$ converges to $w^* \in \mathscr{H}$;

b) $\{q_n\}$ converges to $w^* \in \mathscr{H}$;

c) $\{r_n\}$ converges to $w^* \in \mathscr{H}$.

Algorithm 3.3. The iterative sequence $\{s_n\}$ for all $n \in N_0$ is stated as

$$\begin{cases} s_0 \in \mathscr{H} \\ s_{n+1} = t_n - g(t_n) + J_{\lambda,N}^{A(...,)}[A(Pog(t_n), Rog(t_n)) - \lambda(S(t_n) - T(t_n)) + \lambda\rho] \\ t_n = (1-\mu_n)s_n + \mu_n\Big[s_n - g(s_n) + J_{\lambda,N}^{A(...,)}[A(Pog(s_n), Rog(s_n)) \\ \qquad -\lambda(S(s_n) - T(s_n)) + \lambda\rho]\Big], \end{cases} \quad (3.13)$$

where $\{\mu_n\}$ is a sequence in (0,1) satisfying certain control conditions.

Definition 3.1([4]). Consider two real sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\theta_n\}_{n=0}^{\infty}$ with limits α and θ, respectively. Assume there exists

$$\lim_{n\to\infty} \frac{|\alpha_n-\alpha|}{|\theta_n-\theta|} = l.$$

a) If $l = 0$, in such instances, it can be expressed that $\{\alpha_n\}_{n=0}^{\infty}$ converges faster to α than $\{\theta_n\}_{n=0}^{\infty}$ to θ.

b) If $l \in (0,\infty)$, in such scenarios, we can affirm $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\theta_n\}_{n=0}^{\infty}$ have the same rate of convergence.

Remark 3.1.

a) When $l = \infty$, it indicates that the sequence $\{\theta_n\}_{n=0}^{\infty}$ converges more rapidly than $\{\alpha_n\}_{n=0}^{\infty}$.

b) In situations where both sequences $\{w_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ within the space $\mathscr{H}$ converge to the same point p, the ensuing error predictions apply:

$\|w_n - p\| \leq \alpha_n, \forall n \in N_0,$ (3.14)

$\|y_n - p\| \leq \theta_n, \forall n \in N_0,$ (3.15)

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\theta_n\}_{n=0}^{\infty}$ are sequences consisting of positive numbers (converges to zero).

Definition 3.2 ([4]). Suppose we have two sequences, $\{w_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ both within the space $\mathscr{H}$, converging to the same point p and satisfying conditions (3.14) and (3.15), respectively. When the sequence $\{\alpha_n\}_{n=0}^{\infty}$ converges more rapidly than $\{\theta_n\}_{n=0}^{\infty}$, we characterise $\{w_n\}_{n=0}^{\infty}$ as converging faster than $\{y_n\}_{n=0}^{\infty}$ to p.

Lemma 3.1 ([18]). Consider two sequences, $\{\sigma_n\}_{n=0}^{\infty}$ and $\{\rho_n\}_{n=0}^{\infty}$, consisting of positive real numbers, which satisfy the inequality given as:

$\sigma_{n+1} \leq (1-\epsilon_n)\sigma_n + \rho_n,$ (3.16)

where $\epsilon_n \in (0,1)$ for all $n \geq n_0, \Sigma_{n=1}^{\infty}\epsilon_n = \infty$, and $\rho_n = o(\epsilon_n)$. Then, $\lim_{n\to\infty}\sigma_n = 0$. Now, we are ready to establish the strong convergence of S-iteration process (3.13) to a unique solution w^* of (2.1).

Theorem 3.3. Let $\mathscr{H}$ be a real Hilbert space and $A: \mathscr{H}\times\mathscr{H} \to \mathscr{H}$ and $P,R,S,T,g: \mathscr{H} \to \mathscr{H}$ are single-valued functions and $N: \mathscr{H} \to 2^{\mathscr{H}}$ be a multi-valued function such as $A(.,.)co-$MT with respect to P,R,g operator. Assume that $A(.,.)$ is Lipschitz continuous constant $t > 0$, mixed strongly MT with respect to P and R with constant $\delta > 0$, g is strongly MT with constant $\delta_g > 0$ and g,P,R,S,T are Lipschitz continuous with $\lambda_g, \lambda_P, \lambda_R, \lambda_S$ and λ_T respectively such as

$$\begin{cases}(\mu\alpha^2-\gamma\beta^2)^2(1-2\delta_g+\lambda_g^2) < [\mu\alpha^2-\gamma\beta^2-t_1\lambda_P\lambda_g-t_2\lambda_R\lambda_g-\lambda(\lambda_S+\lambda_T)]^2 \\ \mu > \gamma \text{ and } \alpha > \beta\end{cases} \quad (3.17)$$

Let $\{s_n\}$ be an iterative sequence in $\mathscr{H}$ defined by (3.1) with the sequence $\{\mu_n\} \subset (0,1)$ satisfying $\sum_{n=0}^{\infty}\mu_n = \infty$. Then, the sequence $\{s_n\}$ demonstrates converges strongly towards a unique solution w^* of equation (2.1), and this convergence is associated with the following estimate:

$$\| s_n - w^* \| \leq \kappa^n \prod_{i=0}^{n-1}[1-\mu_i(1-\kappa)] \| s_0 - w^* \|, \text{ for } n \in N$$

Proof. Utilizing (2.3), (3.6) and (3.13), we obtain

$$\| s_{n+1} - w^* \| = \| t_n - g(t_n) + J_{\lambda,N}A(\ldots)[A(Pog(t_n),Rog(t_n)) - \lambda(S(t_n) - T(t_n)) + \lambda\rho]$$

$$-\left(w^* - g(w^*) + J_{\lambda,N}A(\ldots)[A(Pog(w^*),Rog(w^*)) - \lambda(S(w^*) - T(w^*)) + \lambda\rho]\right) \|$$

$$= \| t_n - w^* - g(t_n) - g(w^*) + J_{\lambda,N}A(\ldots)[A(Pog(t_n),Rog(t_n)) - A(Pog(w^*),Rog(w^*))$$

$$-\lambda(S(t_n) - S(w^*)) + \lambda(T(t_n) - T(w^*))] + \lambda\rho \|. \quad (3.18)$$

We have

$$\| J_{\lambda,N}^{A(.,.)}\left[A\left(Pog(t_n),Rog(t_n)\right) - \lambda\left(S(t_n) - T(t_n)\right) + \lambda\rho\right]$$

$$-J_{\lambda,N}^{A(....)}\left[A\left(Pog(w^*),Rog(w^*)\right) - \lambda\left(S(w^*) - T(w^*)\right) + \lambda\rho\right]\|$$

$$\leq \frac{1}{\mu\alpha^2 - \gamma\beta^2}\|A\left(Pog(t_n),Rog(t_n)\right) - \lambda\left(S(t_n) - T(t_n)\right)$$

$$-\left(A\left(Pog(w^*),Rog(w^*)\right) - \lambda\left(S(w^*) - T(w^*)\right)\right)\|$$

$$\leq \frac{1}{\mu\alpha^2 - \gamma\beta^2}\|A\left(Pog(t_n),Rog(t_n)\right) - A\left(Pog(w^*),Rog(w^*)\right)$$

$$-\lambda(S(t_n) - S(w^*) + T(t_n) - T(w^*))\|. \quad (3.19)$$

Since $A(.,.)$ is Lipschitz continuous with respect to P and R, and Lipschitz continuous of P and g, we have

$$\|A\left(Pog(t_n),Rog(t_n)\right) - \lambda\left(S(t_n) - T(t_n)\right) - \left(A\left(Pog(w^*),Rog(w^*)\right) - \lambda\left(S(w^*) - T(w^*)\right)\right\|$$

$$= \|A\left(Pog(t_n),Rog(t_n)\right) - A\left(Pog(w^*),Rog(w^*)\right) - \lambda\left(S(t_n) - S(w^*)\right)$$

$$-\lambda\left(T(t_n) - T(w^*)\right)\|$$

$$\leq \left\|A\left(Pog(t_n),Rog(t_n)\right) - A\left(Pog(w^*),Rog(w^*)\right)\right\| + \lambda\|S(t_n) - S(w^*)\|$$

$$+\lambda\|T(t_n) - T(w^*)\|$$

$$\leq \|A\left(Pog(t_n),Rog(t_n)\right) - A\left(Pog(w^*),Rog(t_n)\right) + A\left(Pog(w^*),Rog(w^*)\right)$$

$$-A\left(Pog(w^*),Rog(w^*)\right)\| + \lambda\|S(t_n) - S(w^*)\| + \lambda\|T(t_n) - T(w^*)\|$$

$$\leq \left\|A\left(Pog(t_n),Rog(t_n)\right) - A\left(Pog(w^*),Rog(w^*)\right)\right\| + \|A\left(Pog(w^*),Rog(t_n)\right)$$

$$-A\left(Pog(w^*),Rog(w^*)\right)\| + \lambda\|S(t_n) - S(w^*)\| + \lambda\|T(t_n) - T(w^*)\|$$

Because g exhibits strongly MT with parameter δ_g and is also Lipschitz continuous with constant λ_g, it follows that

$$\| t_n - w^* - g(t_n) + g(w^*) \|^2 \leq \| t_n - w^* \|^2 - 2\langle g(t_n) - g(w^*), t_n - w^* \rangle$$

$$+\| g(t_n) - g(w^*) \|^2$$

$$\leq \left(1 - 2\delta_g + \lambda_g^2\right) \| t_n - w^* \|^2. \quad (3.20)$$

which implies that

$$\| t_n - w^* - (g(t_n) - g(w^*)) \| \leq \sqrt{1 - 2\delta_g + \lambda_g^2} \| t_n - w^* \|. \quad (3.21)$$

Using (3.19) and (3.21), (3.18) becomes

$$\| t_n - w^* - (g(t_n) - g(w^*)) + J_{\lambda,N}^{A(.,.)}[A(Pog(t_n), Rog(t_n)) - \lambda(S(t_n) - T(t_n)) + \lambda\rho]$$

$$-J_{\lambda,N}^{A(.,.)}[A(Pog(w^*), Rog(w^*)) - \lambda(S(w^*) - T(w^*)) + \lambda\rho] \|$$

$$\leq \sqrt{1 - 2\delta_g + \lambda_g^2} + \frac{t_1\lambda_P\lambda_g + t_2\lambda_R\lambda_g + \lambda\lambda_S + \lambda\lambda_T}{\mu\alpha^2 - \gamma\beta^2} \| t_n - w^* \|$$

$$= \kappa \| t_n - w^* \|, \quad (3.22)$$

Where,

$$\kappa = \sqrt{1 - 2\delta_g + \lambda_g^2} + \frac{t_1\lambda_P\lambda_g + t_2\lambda_R\lambda_g + \lambda\lambda_S + \lambda\lambda_T}{\mu\alpha^2 - \gamma\beta^2}.$$

Using (2.3), (3.6) and (3.13), we have

$$\| t_n - w^* \| = \| (1 - \mu_n)(s_n - w^*) + \mu_n(s_n - g(s_n) - (w^* - g(w^*))$$

$$+ J_{\lambda,N}^{A(.,.)}[A(Pog(s_n), Rog(s_n)) - \lambda(S(s_n) - T(s_n)) + \lambda\rho]$$

$$-J_{\lambda,N}^{A(.,.)}[A(Pog(w^*), Rog(w^*)) - \lambda(S(w^*) - T(w^*)) + \lambda\rho] \|$$

$$\leq (1 - \mu_n) \| s_n - w^* \| + \mu_n \| s_n - w^* - (g(s_n) - g(w^*))$$

$$+ \mu_n \| J_{\lambda,N}^{A(.,.)}[A(Pog(s_n), Rog(s_n)) - \lambda(S(s_n) - T(s_n)) + \lambda\rho]$$

$$-J_{\lambda,N}^{A(.,.)}[A(Pog(w^*), Rog(w^*)) - \lambda(S(w^*) - T(w^*)) + \lambda\rho] \|$$

$$\leq (1 - \mu_n) \| s_n - w^* \| + \mu_n \sqrt{1 - 2\delta_g + \lambda_g^2} \| s_n - w^* \|$$

$$+ \frac{\mu_n}{\mu\alpha^2 - \gamma\beta^2}(t_1\lambda_P\lambda_g + t_2\lambda_R\lambda_g + \lambda\lambda_S + \lambda\lambda_T) \| s_n - w^* \|$$

$$\leq [1 - \mu_n(1 - \kappa)] \| s_n - w^* \|, \quad (3.23)$$

where

$$\kappa = \sqrt{1 - 2\delta_g + \lambda_g^2} + \frac{t_1\lambda_P\lambda_g + t_2\lambda_R\lambda_g + \lambda\lambda_S + \lambda\lambda_T}{\mu\alpha^2 - \gamma\beta^2}.$$

By (3.22), (3.2) becomes

$$\| s_{n+1} - w^* \| \leq \kappa[1 - \mu_n(1 - \kappa)] \| s_n - w^* \|$$

inductively, we have

$$\| s_{n+1} - w^* \| \leq \kappa^{n+1} \prod_{i=0}^{n} [1 - \mu_i(1-\kappa) \| s_0 - w^* \|. \tag{3.24}$$

As per classical analysis, it's a widely recognized fact that for any value of $a \in [0,1]$, the inequality $1 - a \leq e^{-a}$ holds true. Hence, from (3.24), we have

$$\| s_{n+1} - w^* \| \leq \| s_0 - w^* \| \kappa^{n+1} e^{-(1-\kappa)\Sigma_{i=1}^{n}\mu_i}. \tag{3.25}$$

It follows that from the assumption $\sum_{i=0}^{\infty} \mu_i = \infty$ that $e^{-(1-\kappa)\sum_{i=1}^{n}\mu_i} \to 0$ as $n \to \infty$, which implies that $\lim_{n\to\infty} \|s_n - w^*\| = 0$.

The underneath results shows that convergence rate of the sequences generated by (3.13) is faster than (3.2). Therefore, this result has a great importance both from numerical and theoretical aspects.

Theorem 3.4. Let $\mathscr{H}, S, T, N, A, g, \kappa$ and w^* be defined as Theorem 3.3 and suppose $\{\mu_n\}$ be a sequence in (0,1) such as $\mu \leq \mu_n$ for all $n \in N_0$ and for some $\mu > 0$. For given $w_0 = s_0 \in \mathscr{H}$, let $\{w_n\}$ and $\{s_n\}$ be the iterative sequences generated by (3.2) and (3.12), respectively. Then, the sequence $\{s_n\}$ converges to w^* at a rate faster than $\{w_n\}$ does.

Proof. From Theorem 3.1, we have

$$\|w_n - w^*\| \leq \kappa^n \|w_0 - w^*\|.$$

From (3.25), we have

$$\|s_{n+1} - w^*\| \leq \kappa^{n+1} \prod_{i=0}^{n} [1 - (1-\kappa)\mu_i] \|s_0 - w^*\|,$$

or equivalently

$$\|s_n - w^*\| \leq \kappa^n \prod_{i=0}^{n-1} [1 - (1-\kappa)\mu_i] \|s_0 - w^*\|.$$

It follows from the assumption that

$$\|s_n - w^*\| \leq \kappa^n \prod_{i=1}^{n} [1 - (1-\kappa)\mu_i] \|s_0 - w^*\| = \kappa^n [1 - (1-\kappa)\mu]^n \|s_0 - w^*\|.$$

Set

$$\alpha_n = \kappa^n [1 - (1-\kappa)\mu]^n \|s_0 - w^*\|,$$
$$\theta_n = \kappa^n \|u_0 - w^*\|.$$

Given that $\lim_{n\to\infty} \Theta^n = 0, \lim_{n\to\infty} \alpha_n = 0$ and $lim \theta_n = 0$, which means that both sequences $\{\alpha_n\}$ and $\{\theta_n\}$ converge to zero, as stipulated in Definition 3.2. Define

$$\pi_n = \frac{\alpha_n - 0}{\theta_n - 0} = \frac{\kappa^n [1 - (1-\kappa)\mu]^n \|s_0 - w^*\|}{\kappa^n \|u_0 - w^*\|}$$

$$= [1-(1-\kappa)\mu]^{n}.$$

Note that $1-(1-\kappa)\mu \in (0,1)$. This allows us to conclude that

$$\lim_{n\to\infty} \pi_n = \lim \frac{\alpha_n - 0}{\theta_n - 0} = 0$$

Thus, according to Definition 3.1(a), we can infer that the convergence of $\{\alpha_n\}$ is faster than that of $\{\theta_n\}$, and as a consequence, $\{s_n\}$ converges faster than $\{w_n\}$.

In Theorem 3.3, we have discussed that S-iteration algorithm (3.13) is a better algorithm with a more efficient convergence rate. Now, we establish new convergence implications between iterative sequences generated by (3.3) and (3.15).

Theorem 3.5. Let $\mathscr{H}, S, T, N, A, g, \kappa$ and w^* be defined as Theorem 3.3 and suppose $\{q_n\}$ and $\{s_n\}$ be iterative sequences generated by (2.4) and (3.1), respectively, with the sequences $\{\xi_n\}$ and $\{\mu_n\}$ in $\subset (0,1)$. Then the subsequent claims are applicable:

a) If $\left\{\frac{1-\xi_n}{\mu_n}\right\}$ is bounded, $\sum_{n=0}^{\infty}\mu_n = \infty$ and $\{q_n\}$ converges strongly to w^*, then $\{s_n - q_n\}$ converges strongly to 0 with the following estimate :

$\|q_{n+1} - s_{n+1}\| \le \kappa[1-(1-\kappa)\mu_n]\|q_n - s_n\| + (1-\xi_n)\{1+\kappa\}\|q_n - w^*\|, \forall n \in N_0$, and $\{s_n\}$ converges strongly to w^*.

b) If $\left\{\frac{1-\xi_n}{\mu_n\xi_n}\right\}$ is bounded and $\sum_{n=0}^{\infty}\xi_n\mu_n = \infty$, then $\{s_n - q_n\}$ converges strongly to 0 with the following estimate :

$$\|q_{n+1} - s_{n+1}\| \le [1-(1-\kappa)\xi_n\mu_n]\|s_n - q_n\| + (1+\kappa)(1-\xi_n)\|s_n - w^*\|, \forall n \in N_0$$

and $\{q_n\}$ converges strongly to w^*.

Proof.

a) Suppose that $\left\{\frac{1-\xi_n}{\mu_n}\right\|$ is bounded, $\sum_{n=0}^{\infty}\mu_n = \infty$ and $\{q_n\}$ converges strongly to w^*. We show that $\{s_n - q_n\}$ converges strongly to 0. It derives from (3.3), (3.6), (3.7) and (3.15) that

$\| q_{n+1} - s_{n+1} \|$

$=\| (1-\xi_n)q_n + \xi_n[r_n - g(r_n) + J_{\lambda,N}^{A(.,.)}[A(Pog(r_n),Rog(r_n)) - \lambda(S(r_n) - T(r_n))$

$\quad +\lambda\rho]] - t_n - g(t_n) + J_{\lambda,N}^{A(.,.)}[A(Pog(t_n),Rog(t_n)) - \lambda(S(t_n) - T(t_n)) + \lambda\rho] \|$

$=\| (1-\xi_n)q_n + \xi_n[r_n - g(r_n) + J_{\lambda,N}^{A(.,.)}[A(Pog(r_n),Rog(r_n)) - \lambda(S(r_n) - T(r_n))$

$\quad +\lambda\rho]] - [r_n - g(r_n) + J_{\lambda,N}^{A(.,.)}[A(Pog(r_n),Rog(r_n)) - \lambda(S(r_n) - T(r_n))$

$\quad +\lambda\rho]] + [r_n - g(r_n) + J_{\lambda,N}^{A(.,.)}[A(Pog(r_n),Rog(r_n)) - \lambda(S(r_n) - T(r_n))$

$\quad +\lambda\rho]] - [t_n - g(t_n) + J_{\lambda,N}^{A(.,.)}[A(Pog(t_n),Rog(t_n)) - \lambda(S(t_n) - T(t_n)) + \lambda\rho]] \|$

$$=\| (1-\xi_n)q_n-(1-\xi_n)(r_n-g(r_n)+J_{\lambda,N}^{A(.,.)}[A(Pog(r_n),Rog(r_n))-\lambda(S(r_n)$$
$$-T(r_n))+\lambda\rho]]+[r_n-g(r_n)+J_{\lambda,N}^{A(.,.)}[A(Pog(r_n),Rog(r_n))-\lambda(S(r_n)$$
$$-T(r_n))+\lambda\rho]]+[r_n-g(r_n)+J_{\lambda,N}^{A(.,.)}[A(Pog(r_n),Rog(r_n))-\lambda(S(r_n)$$
$$-T(r_n))+\lambda\rho]]-[t_n-g(t_n)+J_{\lambda,N}^{A(.,.)}[A(Pog(t_n),Rog(t_n))-\lambda(S(t_n)-T(t_n))+\lambda\rho]] \|$$
$$=(1-\xi_n)\| q_n-(r_n-g(r_n)+J_{\lambda,N}^{A(.,.)}[A(Pog(r_n),Rog(r_n))-\lambda(S(r_n)-T(r_n))$$
$$+\lambda\rho])\| +\xi_n \| r_n-g(r_n)+J_{\lambda,N}^{A(.,.)}[A(Pog(r_n),Rog(r_n))-\lambda(S(r_n)-T(r_n))$$
$$+\lambda\rho]\| -(t_n-g(t_n)+J_{\lambda,N}^{A(.,.)}\left[A\left(Pog(t_n),Rog(t_n)\right)-\lambda\left(S(t_n)-T(t_n)\right)+\lambda\rho\right])$$
$$\leq (1-\xi_n)\| q_n-F(r_n)\| +\xi_n \| F(r_n)-F(t_n)\|$$
$$\leq (1-\xi_n)\| q_n-w^*+F(w^*)-F(r_n)\| +\xi_n \| F(r_n)-F(t_n)\|$$
$$\leq (1-\xi_n)(\| q_n-w^*\| +\| F(w^*)-F(r_n)\|)+\xi_n \| F(r_n)-F(t_n)\|$$
$$\leq (1-\xi_n)(\| q_n-w^*\| +\kappa \| w^*-r_n\|)+\xi_n\kappa \| r_n-t_n\|$$
$$\leq (1-\xi_n)\| q_n-w^*\| +(1-\xi_n)\kappa \| r_n-w^*\| -(1-\xi_n)\kappa \| r_n-t_n\| +\xi_n\kappa \| r_n-t_n\|$$
$$\leq (1-\xi_n)\| q_n-w^*\| +\kappa \| t_n-r_n\| +(1-\xi_n)\kappa \| r_n-w^*\|. \tag{3.26}$$

We have

$$\| r_n-w^*\| =\| (1-\mu_n)q_n+\mu_n[q_n-g(q_n)+J_{\lambda,N}^{A(.,.)}[A(Pog(q_n),Rog(q_n))-\lambda(S(q_n)-T(q_n))$$
$$+\lambda\rho]]-w^*\|$$
$$=\| (1-\mu_n)q_n+\mu_n[q_n-g(q_n)+J_{\lambda,N}^{A(.,.)}[A(Pog(q_n),Rog(q_n))-\lambda(S(q_n)-T(q_n))$$
$$+\lambda\rho]]-[(1-\mu_n)w^*+\mu_n[w^*-g(w^*)+J_{\lambda,N}^{A(.,.)}[A(Pog(w^*),Rog(w^*))$$
$$-\lambda(S(w^*)-T(w^*))+\lambda\rho]]\|$$
$$\leq\| (1-\mu_n)q_n-w^*+\mu_n(F(q_n)-F(w^*))\|$$
$$\leq (1-\mu_n)\| q_n-w^*\| +\mu_n\kappa \| q_n-w^*\|$$
$$=[1-\mu_n(1-\kappa)]\| q_n-w^*\|, \tag{3.27}$$

And

$$\| r_n-t_n\| =\| (1-\mu_n)q_n+\mu_n[q_n-g(q_n)+J_{\lambda,N}^{A(.,.)}[A(Pog(q_n),Rog(q_n))-\lambda(S(q_n)-T(q_n))$$
$$+\lambda\rho]]-(1-\mu_n)s_n+\mu_n[s_n-g(s_n)+J_{\lambda,N}^{A(.,.)}[A(Pog(s_n),Rog(s_n))$$
$$-\lambda(S(s_n)-T(s_n))+\lambda\rho]]\|$$
$$\leq (1-\mu_n)\| q_n-s_n\| +\mu_n \| (F(q_n)-F(s_n))\|$$

$$\leq (1-\mu_n)\parallel q_n - s_n \parallel + \mu_n \kappa \parallel q_n - s_n \parallel$$
$$= [1-\mu_n(1-\kappa)] \parallel q_n - s_n \parallel. \quad (3.28)$$

Combining (3.26), (3.27) and (3.25) and using the fact $\kappa \in (0,1)$, we have

$$\parallel q_{n+1} - s_{n+1} \parallel \leq (1-\xi_n)\parallel q_n - w^* \parallel + (1-\xi_n)\kappa[1-(1-\kappa)\mu_n] \parallel q_n - w^* \parallel$$
$$+\kappa[1-(1-\kappa)\mu_n] \parallel q_n - s_n \parallel$$
$$\leq \kappa[1-(1-\kappa)\mu_n] \parallel q_n - s_n \parallel$$
$$+(1-\xi_n)\{1+\kappa[1-(1-\kappa)\mu_n]\} \parallel q_n - w^* \parallel$$
$$\leq [1-(1-\kappa)\mu_n] \parallel q_n - s_n \parallel + (1+\kappa)(1-\xi_n) \parallel q_n - w^* \parallel. \quad (3.29)$$

Set

$\sigma_n := \|q_n - s_n\|,$

$\epsilon_n := (1-\kappa)\mu_n,$

$\rho_n := (1+\kappa)(1-\xi_n)\|q_n - w^*\|.$

Then, (3.29) becomes

$$\sigma_{n+1} \leq (1-\epsilon_n)\sigma_n + \rho_n, n \geq 0. \quad (3.30)$$

Since $\left\{\frac{1-\xi_n}{\mu_n}\right\}$ is bounded. we have $\rho_n = o(\epsilon_n)$. Therefore, an application of Lemma 3.1 to (3.30) yields $\lim_{n\to\infty}\sigma_n = \lim_{n\to\infty}\|q_n - s_n\| = 0$. Since $\lim_{n\to\infty}\|q_n - w^*\| = 0$, it follows that

$$\|s_n - w^*\| \leq \|q_n - s_n\| + \|q_n - w^*\| \to 0, as \quad n \to \infty.$$

b) Suppose that $\left\{\frac{1-\xi_n}{\xi_n \mu_n}\right\}$ is bounded and $\Sigma_{n=0}^{\infty}\mu_n = \infty$. Then, from Theorem (3.2), $\{s_n\}$ strongly converges to w^*.

We now show that $\{q_n\}$ converges strongly to w^*. Utilizing (3.3), (3.6), (3.7) and (3.13), we have

$$\parallel s_{n+1} - q_{n+1} \parallel = \parallel t_n - g(t_n) + J_{\lambda,N}^{A(.,.)}[A(Pog(t_n), Rog(t_n)) - \lambda(S(t_n) - T(t_n)) + \lambda\rho]$$
$$-(1-\xi_n)q_n + \xi_n[r_n - g(r_n) + J_{\lambda,N}^{A(.,.)}[A(Pog(r_n), Rog(r_n))$$
$$-\lambda(S(r_n) - T(r_n)) + \lambda\rho]] \parallel$$
$$\leq (1-\xi_n) \parallel q_n - F(t_n) \parallel + \xi_n \parallel F(r_n) - F(t_n) \parallel$$
$$\leq (1-\xi_n)\{\parallel F(t_n) - F(w^*) \parallel + \parallel F(w^*) - s_n \parallel + \parallel s_n - q_n \parallel\}$$
$$+\xi_n \parallel F(r_n) - F(t_n) \parallel$$
$$\leq (1-\xi_n)\{\kappa \parallel t_n - w^* \parallel + \parallel s_n - w^* \parallel + \parallel s_n - q_n \parallel\} + \xi_n \kappa \parallel t_n - r_n \parallel, \quad (3.31)$$

Now, we obtain

$$\begin{aligned}\| t_n - w^* \| &= \| (1-\mu_n)s_n + \mu_n[s_n - g(s_n) + J_{\lambda,N}^{A(.,.)}[A(Pog(s_n),Rog(s_n)) \\ &\quad -\lambda(S(s_n) - T(s_n)) + \lambda\rho]] - w^* \| \\ &\le (1-\mu_n)\| s_n - w^* \| + \mu_n \| F(s_n) - F(w^*) \| \\ &\le (1-\mu_n)\| s_n - w^* \| + \mu_n \kappa \| s_n - w^* \| \\ &\le [1-(1-\kappa)\mu_n] \| s_n - w^* \|. \end{aligned} \tag{3.32}$$

And,

$$\begin{aligned}\| t_n - r_n \| &= \| (1-\mu_n)(s_n - q_n) + \mu_n[s_n - g(s_n) + J_{\lambda,N}^{A(.,.)}[A(Pog(s_n),Rog(s_n)) \\ &\quad -\lambda(S(s_n) - T(s_n)) + \lambda\rho]] - [q_n - g(q_n) + J_{\lambda,N}^{A(.,.)}[A(Pog(q_n),Rog(q_n)) \\ &\quad -\lambda(S(q_n) - T(q_n)) + \lambda\rho]] \| \\ &\le (1-\mu_n)\| s_n - q_n \| + \mu_n \| F(s_n) - F(w^*) \| \\ &\le [1-(1-\kappa)\mu_n] \| s_n - q_n \|. \end{aligned} \tag{3.33}$$

Substituting (3.32) and (3.33) into (3.31), we obtain

$$\begin{aligned}\| s_{n+1} - q_{n+1} \| &= \xi_n \kappa([1-(1-\kappa)\mu_n] \| s_n - q_n \| \\ &\quad +(1-\xi_n)\{\kappa[1-(1-\kappa)\mu_n] \| s_n - w^* \| + \| s_n - w^* \| + \| s_n - q_n \|\}) \\ &\le [1-(1-\kappa)\xi_n\mu_n] \| s_n - q_n \| + (1+\kappa)(1-\xi_n) \| s_n - w^* \|. \end{aligned} \tag{3.34}$$

Set $\sigma_n := \| s_n - q_n \|, \epsilon_n := \xi_n\mu_n$, and $\rho_n := (1+\kappa)(1-\xi_n) \| s_n - w^* \|$. Note that $\{\frac{1-\xi_n}{\mu_n \xi_n}\}$ is bounded. Also, $\lim_{n\to\infty} \| s_n - w^* \| = 0, \rho_n = o(\epsilon_n)$ and $\Sigma_{n=0}^{\infty}\mu_n = \infty$. Therefore, an application Lemma 3.1 to (3.34) yield $\lim_{n\to\infty}\sigma_n = \lim_{n\to\infty} \| s_n - q_n \| = 0$. When $\lim_{n\to\infty} \| s_n - w^* \| = 0$ and

$$\| q_n - w^* \| \le \| s_n - q_n \| + \| s_n - w^* \|,$$

we deduce that $\lim_{n\to\infty} \| q_n - w^* \| = 0$.

Theorem 3.5 (a) establishes the strong convergence of $\{s_n\}$ to w^* under convergence of $\{q_n\}$ and the condition$\Sigma_{n=0}^{\infty}\mu_n = \infty$. Theorem 3.5 (b) establishes a new convergence theorem for $\{q_n\}$ under boundedness of $\{\frac{1-\xi_n}{\mu_n \xi_n}\}$ and divergence of $\Sigma_{n=0}^{\infty}\xi_n\mu_n$.

4. Numerical Results

In this section, we provide an illustrative example and numerical results that serve to exemplify the algorithm's applicability, demonstrating not only the primary outcomes of our paper but also the efficiency and convergence of the sequence generated through the iterative approach.

Example 4.1. Let $\mathscr{H} = \mathscr{R}$ and define $H, M, N, S, T, P, R, g: \mathscr{R} \to \mathscr{R}$ and $A: \mathscr{R} \times \mathscr{R} \to \mathscr{R}$ by $H(w) = \frac{27w}{4}, M(w) = \frac{w}{4} + 1, N(w) = w - 1, S(w) = \frac{9w}{2}, T(w) = \frac{7w}{2},, P(w) = 3w, R(w) = \frac{w}{2}, g(w) = \frac{2w}{3}, A(w) = \frac{3w}{4} - 1, A(P(w), R(w)) = P(w) + R(w)$,for all$w \in \mathscr{H}$.For $\lambda = 1$ and $= -2$, we have $J_{\lambda,N}^{A(.,.)}(w) = \frac{2(w+2)}{9}$ and $F(w) = 3w$, for all $w \in \mathscr{H}$

From (3.2) and (3.13) can be written as, $w_{n+1} = Tw_n$,and $s_{n+1} = T[(1 - \mu_n)s_n + \mu_n Ts_n]$ for $n \in \mathcal{N}_0$,respectively, where

$$T(w) := F(w) = 3w, \text{ for all } w \in \mathscr{H},$$

and $\{\mu_n\}$ is a sequence in (0,1). We consider $\alpha_n = \frac{1}{n}$ and $\mu_n = \frac{1}{n^2+1}$, $n \in \mathcal{N}_0$. Since assumptions of Theorem 3.3 are satisfied, therefore the sequence$\{s_n\}$ converges to a unique solution of (2.1). Similarity, the sequence $\{w_n\}$ convereges to a unique solution of (2.1) by Theorem 3.2. The graphical presentation of the convergence of sequences $\{s_n\}$ generated from $s_0 = 5,8,11$ are given in Figure 1. Numerical values of $\{s_n\}$ are given in Tables 1. From Figure 2 and Table 2, we see that the sequence $\{s_n\}$ converges faster than the sequence $\{w_n\}$.

Table 1: The values of s_n with initial values $s_0 = 5, s_0 = 10$ *and* $s_0 = 15$

No. of Iterations	For $s_0 = 5$ s_n	For $s_0 = 10$ s_n	For $s_0 = 15$ s_n
n=1	5	8	11
n=2	13.8993517972893	22.2389628756629	30.5785739540365
n=3	15.2588266954200	24.4141227126719	33.5694187299239
n=4	9.32628502653346	14.9220560424535	20.5178270583736
n=5	3.70219158018539	5.92350652829662	8.14482147640786
n=6	1.03787208523604	1.66059533637766	2.28331858751928
n=7	0.216827962271176	0.346924739633882	0.477021516996588
n=8	0.0350752760961080	0.0561204417537728	0.0771656074114377
n=9	0.00452286820450332	0.00723658912720530	0.00995031004990729
n=10	0.000475687588453968	0.000761100141526349	0.00104651269459873
n=11	4.15749986892055	6.65199979027288e-05	9.14649971162521e-05
n=12	3.06677733294322e-06	4.90684373270916e-06	6.74691013247509e-06
n=13	1.93448288920915e-07	3.09517262273463e-07	4.25586235626012e-07
n=14	1.05525450594488e-08	1.68840720951181e-08	2.32155991307874e-08
n=17	7.85059315299877e-13	1.25609490447980e-12	1.72713049365973e-12
n=20	2.10651237441896e-17	3.37041979907034e-17	4.63432722372172e-17
n=25	7.70878569921844e-26	1.23340571187495e-25	1.69593285382806e-25
n=27	0	0	0
n=28	0	0	0

Table2: The values of $\{s_n\}$ and $\{w_n\}$ with initial values $s_0 = w_0 = 5$.

Number of Iterations	Proposed S-iteration Algorithm 3.3 ($s_0 = 5$) s_n	Proposed Algorithm 3.1 ($w_0 = 5$) w_n
n=1	5	5
n=2	13.8993517972893	19.4879571810883
n=3	15.2588266954200	27.0958573369905
n=4	9.32628502653346	23.1275375716004
n=5	3.70219158018539	14.3521314352151
n=6	1.03787208523604	7.22777560406802
n=7	0.216827962271176	3.37773785150181
n=8	0.0350752760961080	1.75006635630060
n=9	0.00452286820450332	1.12929969254448
n=10	0.000475687588453968	0.870676048112462
n=13	1.93448288920915e-07	0.572413040490093
n=17	7.85059315299877e-13	0.404900857028524
n=20	2.10651237441896e-17	0.333600636560647
n=25	7.70878569921844e-26	0.258848165970660
n=100	0	0.000606691920380
n=200	0	2.36268998330960e-22

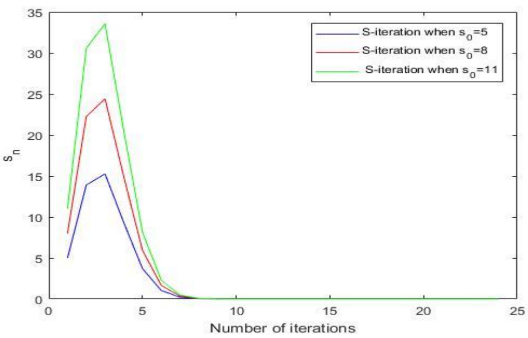

Figure1: The convergence of s_n with initial values $s_0 = 5, s_0 = 10\ and\ s_0 = 15$.

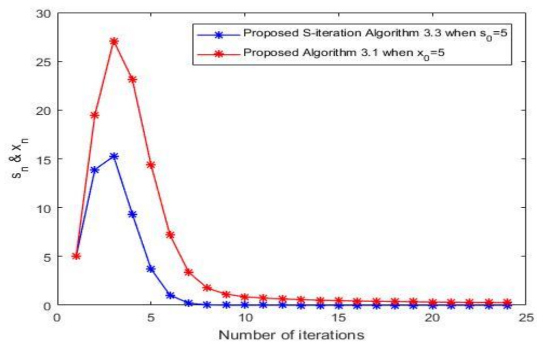

Figure 2: The convergence of s_n and $w_n$with initial values $s_0 = w_0 = 5$.

5. Conclusion

In conclusion, within the scope of this study, we have explored a broader variational inclusion problem that encompasses $A(.,.)$--co-coercive operators within the context of real Hilbert spaces. Through the utilization of the resolvent operator technique, we have established an equivalence between the generalized variational inclusion problem and its associated fixed-point problem. Leveraging this equivalence, we have demonstrated both the existence and uniqueness of a solution for the generalized variational inclusion problem, employing co-coercive and relaxed co-coercive functions. Also, we have proposed the algorithms involving S-iteration and H-MT operators under some suitable conditions. Lastly, we provide a numerical example to substantiate our primary finding. The results we've obtained serve to expand and provide a more comprehensive framework compared to many existing outcomes found in the literature for various systems.

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