Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 1 (2024)

Group Actions on Manifolds

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Article History:
Received: *28-09-2023*
Revised: *07-11-2023*
Accepted: *29-11-2023*

Abstract:
In this paper some Riemann Geometry aspects will be gathered with classical time, the study naturally concentrate to PDE's from the relation outside geometry. The classical frame work of differential geometry (intending smooth manifolds) and later it endured with Riemannian metric is briefly described in one of the sections that were formed to be essential for our understanding of the inner structure of the space.

Keywords: Diffeomorphism, Locally integrable structures and Integral curves.

1. Introduction

In this paper we discuss the inner structure of the space, we mean a smooth manifold of dimension$n$, where $n \in N$ .Naturally, it sets a pace with geometry ever since the human civilization started confronting the enchanting beauty of the nature in which geometry manifests. If one were to specifically mention its historical anecdotes, then it is certainly of Greek times. Euclid's axiomatic approach to mathematics and in particular his flat geometry model. Space is curved is altogether another revelation and that guides us to move forward with the agenda. But this picture underwent radical changes during 19th century, we notice these changes in the significant work of Gauss and Riemann, he was a student of Gauss. Infact, Riemann revolutionized our notions of space and liberating mathematics from its Euclidean sholders. That laed to believe formally that objects no longer lead to be confined to the flat, linear space of Euclidean geometry. Riemann proposed a much more abstract conception of space , of dimension n for any $n \in N$ in which we could describe distance and curvature and a form of calculus that intend to this idea of abstract space, Riemann geometry base the stump, as non- Euclidean geometry, and get in a standard Euclidean spirit. Any investigation /research in geometry in the middle of 20th century lead to forever, Einstein realized that this kind of geometry , which involved curved spaces with exactly what was needed by him to unity geometry (Newton's) with special relativity and that later lead to the famous theory of general relativity.

Under classification theme we distinguish all 2-dimensional oriented closed manifolds. Indeed this has been a classical problem of topology and was successfully done by the people in 1940's later, this study lead to higher analogies. Here, we shall give a simple treatment of Riemann's idea of presenting a surface, from topological view point, to begin with a sphere S^2 and torus T^2 both are oriented closed 2-dimensional manifolds and are least homoeomorphic to each other. Other part

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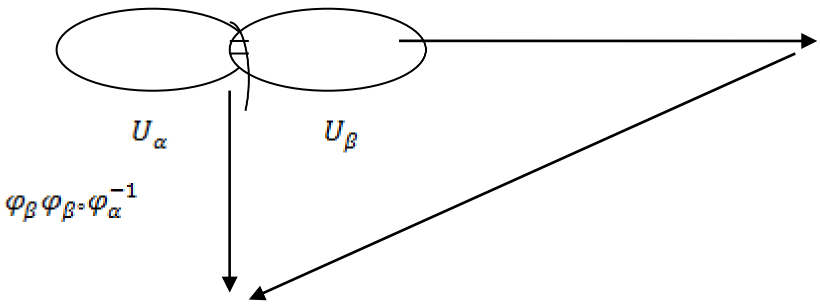
of discussion is vector fields and some inerrability related issues; this is an important and deeply studied theme of geometry under Riemannian metric.
The ambient space being either R^nor C^n over Ror C (as the case may be) a word about low – dimensional topology and finite space (infinite dimensional space) is not out of place, rather it is relevant we come across linear groups, to address the stability issues relating to their dynamics (Dynamical systems, perturbations , etc seen from classical mechanics).

2.Differentiable manifolds and some examples

Let M be a smooth manifold of dimension n , $n \geq 1$ .Then locally we have , coordinate chatrs , which are smooth and φ_α on $U_\alpha \rightarrow R^n$ thus ($\varphi_\alpha , U_\alpha$) pairs as compertable , when it comes to $U_\alpha \cap U_\beta \neq \emptyset$ for different α , β and the transformation maps being smooth the following diagram clarifies these issues.

$\varphi_\alpha R^n \supseteq \varphi_\alpha(U_\alpha \cap U_\beta)$



$R^n \supseteq \varphi_\beta(U_\alpha \cap U_\beta)$

$\varphi_\beta \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ is smooth.Similarly, we can take the other one as well i.e $\varphi_\alpha \varphi_\beta^{-1}$ being smooth.To make things simple we shall use notation (U, x) instead $(\varphi_\alpha . U_\alpha)'$, $\alpha \in \wedge$ and $\cup U = M$, $x: M \rightarrow R^n$, $x(U)$ is open in R^n some times we keep switching from one notion to the other , without any compassion .

2.1 Definition:Diffeomorphisms: Diff(M) is a group under composition of maps and if G is any group then $G \rightarrow$ Diff(M) is an important group of homomorphisms. i.e $g \in G$, then $g \rightarrow \mathcal{A}_g(m)$, where the map A_g is an automorphism (group action naturally arise in this fashion). Observe that $g \in G,$ and $m \in M$ things go inthat way .i.e $G \times M \rightarrow M$ is at the map is at the map i.e$(g, m) \xrightarrow{\mathcal{A}} M$ as $\mathcal{A}(g, m) \in M,$ which we denote it by $\mathcal{A}_g(m)$. i.e $\mathcal{A}_g: M \rightarrow M$,

2.2 Proposition: Diffeomorphism group (M) is smooth in fact it is a lie group.

Proof:A lie group is a group and at the same times a topological space. For our study we confine to $GL(n, R)$ and its subgroups, which are smooth manifolds at the same time. S. T. Yaninitiated and came up without studying results on PDE's and their intense connections with the geometry of the underlying manifold. Recently,there are results from the study on weak and strong unique continuationfor systems of linear and non-linear PDE's which arise as sections of a vector sub-bundle of the complex tangent bundle which we take it as ϑ of $\mathbb{C}TM$ and $\mathbb{C}T^*M$, its cotangent versions, orthogonal to ϑ and is locally generated as exact forms (intigrebilityissues,incorporated).

Firstly, we shall deal the local integrability for the sub-bundle ϑ of $\mathbb{C}TM$ and provide explicit expressions for the basis$\{L_1, \dots \dots, L_n\}$ a local basis of ϑ over a neighbourhood$U$ for each point p in M.

3. Formulation of local integrable structure

Since, PDE's arise as sections of vector bundles rather sub-bundles of vector bundles $\mathbb{C}TM$ where M is a connected smooth manifold. The bundle ϑ satisfies the involutionly condition

$[\vartheta, \vartheta] \subseteq \vartheta$ ………… (1)

The usual [,] lie bracket of sections in the sub-bundle ϑ of the complexified tangent bundle of M.If X, Y are C' sections of ϑ, the lie bracket $[X, Y]$ is also section of ϑ. We always assume that ϑ is locally integrable.Then there exists mC' sections $z_1, \dots \dots .. z_m$ which are solutions of

$L_j h = 0$, $1 \leq j \leq n$, in U ………(2)

And $\{dz_1, \dots .., dz_m\}$ are linearly independent over $\mathbb{C}$ at each part of U areterm $\{ z_1, \dots \dots ., z_n\}$ a complete set of first integrals on U. Since ϑ satisfies local integrability condition hence it satisfies the involutionly conditionalso. Thus we refer to the pair (M, ϑ)as a locally integrable structure. In such a structure, given any point $p \in M$ these are local co-ordinates $x_1, \dots \dots .. x_m$, $t_1, \dots \dots . t_n$

(i.e $m = dimM - n$, for the sections $z_j^{'s}, j = 1, \dots .., m)$ which are satisfies of (2), vanishing at p such that ϑ is generated locally by basis of the form (m and n).

$L_j = \frac{\partial}{\partial t_j} + \sum_{k=1}^{m} a_{jk}(x,t) \frac{\partial}{\partial x_k}$, $1 \leq j \leq n$ …………(3).

3.1 Examples: For the non-linear systems $F_j(x, u, u_x) = 0$,$1 \leq j \leq n$,$x = (x_1, \dots \dots . x_N)$, there exists local co-ordinates$(x, t) \in R^m \times R^n$ $(m + n = N)$. In which the equations take the form

$u_{t_j} = f_j(x, t, u, u_x)$, $1 \leq j \leq n$ ……..(4)

Following are some examples of locally integrable structures, first one is for R^N and the later one$\mathbb{C}^N$.

(a) Let $L_1 \dots \dots . L_n$ be n smoothlinearly independent vector fields on a domain $\Omega \subseteq \mathrm{R}^\mathrm{N}$ such that the lie bracket $[L_i, L_j]$ is in the linear span of $L_j^{'}s$. Let ϑ denote the sub bundles of $\mathbb{C}T\Omega$ generated by these vector fields. By Frobenius theorem, each $p \in \Omega$ is centre of local coordinates $x_1, \dots \dots . x_N$ in which the bundles is locally generatedby$\frac{\partial}{\partial x_j}$, $1 \leq j \leq n$. Hence $\vartheta$is locally integrable and in these coordinates any solution $\mathcal{U}$ has the form, $\mathcal{U}(x_1, \dots \dots . x_N) = \mathcal{U}(x_{n+1} \dots \dots . x_N)$. For the case,$\Omega \subseteq \mathbb{C}^\mathrm{N}$, we have the following description.

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 31 No. 1 (2024)

(b) We know that $\mathbb{C}^n \simeq R^{2n}$hence, the basis in this case will be 2-times the linearly independent set of vector field $L_j^{'}s$(as noticed in the real case) .Then ϑ is the associated bundle generated by

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial y_j}\right), 1 \leq j \leq n.$$

The co-ordinate functions $z_j = x_j + \sqrt{-1}\, y_{j,} 1 \leq j \leq n$ are complete set of first integrals and so (M, ϑ) is locally integrable. Here the solutions are holomorphic functions.

3.2Proposition: Letm= $dimM - n$, as earlier $\Omega \subseteq \mathbb{C}^N$and then$\mathbb{C}T\Omega$ give rise to a locally integrable system (M, ϑ), where the basis is generated by n linearly independent real analytic and complex valued coefficient on Ωwhich is complex valued.

Proof: On the same lines of example (a) and holomorphicversion of Frobeniustheorem, the properties follow.

4. The locally integrable structures and integral curves

Here the setting is $M = GL(n, R)$ and sections are interms of integral curves, generating them. Let $\gamma : [0, T] \to M$ be a smooth curve, such that

(i) $\gamma(0) = p, \gamma(T) = q$, in M

(ii) For $t_0 = 0 < t_1 < \cdots \ldots \ldots < t_k = T$ and $X_i \in X$, X is locally defined smooth vector field for each $i, \gamma : [t_{i-1}, t_i] \to M$ is an integral curve $X_i(-X_i,$ as the case may be we assume that M is orientable).

We shall be interested in $M = GL(n, R)$ thus for $t \in [0, T], t \to \gamma(t)$ in $M = GL(n, R)$ is an $n \times n$ non singular matrix with real entries.
For an open set U of $GL(n, R)$as neighborhood for each p in U , we shall consider its $\mathbb{C}T\Omega$, where $\Omega \subseteq R^N$ and ($(GL(n, R)$, ϑ) a locally integrable structure.
For M connected we shall, simply deal with its analogue i.e $\mathbb{C}T\Omega, \Omega \subseteq \mathbb{C}^N$, i.e $GL(n, C)$ and corresponding locally integrable structure on ($(GL(n, R)$, ϑ).

4.1 Definition: The locally integrable structure (M, ϑ) is said to satisfies the weak unique continuation property if any solution that vanishes on a non-empty open subset vanishes on M.

4.2 Definition: The locally integrable structure (M, ϑ) satisfies the strong unique continuation property if any solution that is flat at a point vanishes on M.
The validity of weak unique continuation property both for linear and non-linear systems is connected with the orbits of the system, which is a very useful geometric objects associated with the given family of real vector fields .The above setting of linear local vector field X_i are the formal setting for this explanation.

The following technical deposits provide the construction of such orbits, for more clarity and clear understanding of the motions .This is one side of the study, the real objective in global theory i.e Globally integrable systems. Before we take up them we will present the following theorem on local integrability structures. From the standard analogues the following theorem in an important result.

4.3 Theorem: Let (M, ϑ) be a locally integrable structure and set,$X = R(\gamma) = \{S: S = RL, L$ is a smooth section of $\vartheta\}$. If $\mathcal{U}$ is a solution on M that vanishes in a neighborhood of p then it vanishes in a neighbourhood of the Sussman orbit X through p .Thus, the support of $\mathcal{U}$ is a union of orbits of X .In particular, if M is an orbit, then(M, ϑ)satisfies the weak unique continuationproperty.

4.4 Example: Let (M, ϑ) be locally integrable and $X = R(\vartheta)$ suppose at each$\in M$, the linearspace of all of the repeated brackets of sections of X equals T_pM. Then M is the only orbit of X and so by the above theorem, the weak unique continuation property holds for (M, ϑ).

These are explains of locally integrable structures, where M is the only orbit of $X = R(\vartheta)$ although the hypothesis in the above example is violated.

5. Conclusion

P.Cohen gives an example for a smooth vector field $L = \frac{\partial}{\partial y} + \alpha(x,y)\frac{\partial}{\partial x}$ in R^2 with a smooth solution n on R^2 , for $L_u = 0$ whose support is $\{(x,y): y \geq 0\}$, for $y > 0$,this is a consequence of certain group acting on the upper local of place H of $\mathbb{C}$.

That is locally integrable structure and maximally real sub manifolds of Mprovide some interesting results characterizing holomorphic maps. And a useful application of locally integrable structure on unique weak continuation.

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