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Oscillatory and Property Conditions for Third Order Difference Equation with Negative Middle-Term

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Abstract:
This paper presents a detailed analysis of the oscillation and Property A conditions for third-order advanced difference equations with negative middle terms. The authors employ the generalized Riccati transformation technique and summation by parts to establish sufficient criteria for the solutions to be oscillatory or have Property A. The main results and their proofs are presented clearly and concisely. The difference equations have been a topic of interest in various scientific fields such as economics, physics, and mathematical biology. The results presented in this paper can be used to inform the development of more accurate models for real-world problems in various scientific fields. Additionally, this paper explores the economic theory of business cycles and its mathematical foundations, specifically Samuelson's model, which outlines five potential trajectories or trends that economic activity can follow based on different combinations of the marginal propensity to consume multiplier parameter(α) and the accelerator coefficient (β).

Keywords: Advanced equation, property A, oscillation criteria, Riccati transformation, semi-canonical.

1. Introduction

In this article, we will discuss the economic theory of business cycles and its mathematical foundations. Specifically, we will explore Samuelson's model, which outlines five potential trajectories or trends that economic activity can follow. These paths are determined by different combinations of the marginal propensity to consume (α) and the accelerator coefficient (β). Each combination results in a unique pattern of economic movement. We will examine the model in detail and highlight its key features, including the dynamic of economic cycles and their relationship to changes in consumption and investment.

The article will also delve into the mathematical foundations of the model, including second-order difference equations in the context of business cycles. We will explore the oscillatory behavior and Property A conditions for third-order difference equations, which have applications in economics, physics, and mathematical biology. The paper will highlight the connection between these equations and Samuelson's model, demonstrating how they can be used to gain insights into the behavior of economic cycles.

Finally, we will present practical examples to demonstrate the main results, providing to apply in real-world scenarios. This chapter aims to provide readers with a comprehensive overview of Samuelson's model and its mathematical foundations, highlighting its relevance and potential implications for various fields.

Third-order advanced difference equation

$$\Delta\left(a(\varphi)\Delta\big(b(\varphi)(\Delta y(\varphi))^{\kappa}\big)\right) - p(\varphi)(\Delta y(\varphi+1))^{\kappa} + q(\varphi)f(y(\sigma(\varphi))) = 0, \varphi \geq \varphi_0 \quad (1)$$

where $\{a(\varphi)\}, \{b(\varphi)\}, \{p(\varphi)\}$ and $\{q(\varphi)\}$ are the positive real sequences for $\varphi \in N$.

Terms considered:

H1) κ is a quotient of odd positive integers;

H2) $\sigma(\varphi)$ is an increasing sequence such that $\sigma(\varphi) \geq \varphi + 1$ for all $\varphi \geq \varphi_0$;

H3) $\sum_{s=\varphi_0}^{\infty} \frac{1}{a(s)} < \infty, \sum_{s=\varphi_0}^{\infty} \frac{1}{b(s)} = \infty$;

H4) function $f: R \to R$ is continuous such that $uf(u) > 0$ for $u \neq 0$ and $f(uv) \geq f(u)f(v)$:

A solution of (1) is a real sequence $\{y(\varphi)\}$ defined for all $\varphi \geq \varphi_0$. A nontrivial solution $\{y(\varphi)\}$ of (1) is said to be oscillatory if it is neither positive nor negative, otherwise it is non-oscillatory. By Property A of (1) it is meant that every solution $y(\varphi)$ of (1) is strictly increasing

$$y(\varphi)) > 0, d(\varphi)\Delta(y(\varphi))^{\kappa} < 0, c(\varphi)\Delta\big(d(\varphi)(\Delta(y(\varphi))^{\kappa})\big) > 0,$$

$$\Delta\left(c(\varphi)\Delta\left(d(\varphi)\big(\Delta(y(\varphi))^{\kappa}\big)\right)\right) < 0$$

Recent research has explored the oscillation criteria of difference equations, primarily focused on [1]-[4] , [15]. However, third-order equations are also essential, as they have applications in various fields, including economics, physics, and mathematical biology ([7],[10], [12]). For instance, Samuelson's business cycles theory, a central economic theory related to the business cycle, can be modelled by linear second and third-order difference equations. This capability allows advanced arguments to depict events dependent on another action. Recent studies have investigated the oscillatory criteria and Property A conditions for difference equations, and their findings are available in the literature ([6],[8],[9],[11]). This paper's results and recent research can inform the development of more accurate models for real-world problems.

In [13], the authors used advanced arguments to demonstrate the oscillatory and Property B conditions for difference equation,

$$\Delta\left(a_n(\Delta(b_n\Delta x_n))^{\alpha}\right) - p_n f\big(x_{\sigma(n)}\big) = 0, n \geq n_0$$

where a_n, b_n, and p_n are the positive real sequences.

In [14], the oscillatory behavior of difference equation was investigated

$$\Delta\left(c(l)\Delta\big(d(l)(\Delta x(l))^{\alpha}\big)\right) - a(l)(\Delta x(l+1))^{\alpha} - b(l)g(x(\sigma(l))) = 0, l \geq l_0$$

Proceeding the manner, the aim of this paper is to commence sufficient condition to ensure that solution of (1) are oscillatory and also it is closely relevant to the auxiliary second order equation

$$\Delta(a(\varphi)\Delta(z(\varphi))) - \frac{p(\varphi)}{b(\varphi+1)}x(\varphi+1) \tag{2}$$

We also derive a solution for equation (1) that possesses Property A, and its corresponding second-order equation aligns with Samuelson's business cycle model.

2. Main Results

Let us define:

$$c(\varphi) = a(\varphi)z(\varphi)z(\varphi+1),\ d(\varphi) = \frac{b(\varphi)}{z(\varphi)},\ Q(\varphi) = q(\varphi)z(\varphi+1),$$

$$C(\varphi) = \sum_{s=\varphi_1}^{\varphi-1}\frac{1}{c(s)},\ D(\varphi) = \sum_{s=\varphi_1}^{\varphi-1}\frac{1}{d^{\frac{1}{\kappa}}(s)},\ E(\varphi) = \sum_{s=\varphi_1}^{\varphi-1}\frac{1}{d^{\frac{1}{\kappa}}(s)}\left(\sum_{s=\varphi_1}^{\varphi-1}\frac{1}{c(s)}\right)^{\frac{1}{\kappa}}.$$

Theorem 2.1. Assume that

$$\sum_{s=\varphi_0}^{\infty}\frac{1}{d^{\frac{1}{\kappa}}(s)} = \infty, \tag{3}$$

and

$$Ly\ (\varphi) = \Delta\big(a(\varphi)\Delta(b(\varphi)(\Delta y(\varphi))^{\kappa})\big) - p(\varphi)(\Delta y(\varphi+1))^{\kappa}$$

Then $Ly\ (\varphi)$ can be written as

$$Ly\ (\varphi) = \frac{1}{z(\varphi+1)}\Delta\left(a(\varphi)z(\varphi)z(\varphi+1)\Delta\left(\frac{b(\varphi)}{z(\varphi)}(\Delta y(\varphi))^{\kappa}\right)\right).$$

Proof.

$$\frac{1}{z(\varphi+1)}\ \Delta\left(a(\varphi)z(\varphi)z(\varphi+1)\Delta\left(\frac{b(\varphi)}{z(\varphi)}(\Delta y(\varphi))^{\kappa}\right)\right)$$

$$= \frac{1}{z(\varphi+1)}\left[\Delta\big(a(\varphi)(\Delta(b(\varphi)(\Delta y(\varphi))^{\kappa}z(\varphi+1)) - b(\varphi+1)(\Delta y(\varphi+1))^{\kappa}\Delta z(\varphi))\big)\right]$$

$$= \frac{1}{z(\varphi+1)}\left[z(\varphi+1)\Delta\big(a(\varphi)\Delta(b(\varphi)(\Delta y(\varphi+1))^{\kappa})\big) - \frac{b(\varphi+1)}{z(\varphi+1)}(\Delta y(\varphi+1))^{\kappa}\Delta(a(\varphi\Delta z(\varphi)))\right]$$

$$= \Delta\big(a(\varphi)\Delta(b(\varphi)(\Delta y(\varphi+1))^{\kappa})\big) - p(\varphi)(\Delta y(\varphi+1))^{\kappa}.$$

Hence the proof.

Corollary 2.2. Assume that the solution $\{z(\varphi)\}$ of (2) is positive. Then the semi-canonical difference equation (1) is in equivalent canonical form

$$\Delta\left(c(\varphi)\Delta\big(d(\varphi)(\Delta y(\varphi))^{\kappa}\big)\right) + Q(\varphi)f(y(\sigma(\varphi))) = 0. \tag{4}$$

Lemma 2.3. (see [2]) If

$$a(\varphi) \geq 1, \sum_{\varphi=\varphi_0}^{\infty}\frac{p(\varphi)}{b(\varphi+1)} < \infty,\ lim\,sup_{\varphi\to\infty}\, n\sum_{s=\varphi}^{\infty}\frac{p(s)}{b(s+1)} < \frac{1}{4}, \tag{5}$$

then the solution of equation (2) is positive.

Lemma 2.4. (see [2]) Assume that (5) holds. Then ∃ a non-oscillatory solution to the equation (2) fulfilling

$$\sum_{\varphi=n_0}^{\infty}\frac{1}{a(\varphi)z(\varphi)z(\varphi+1)}=\infty. \tag{6}$$

Remark:

From (3)

$$\sum_{s=\varphi_0}^{\infty}\frac{z(s)}{b(s)}=\infty, \tag{7}$$

where $\{y(\varphi)\}$ is a positive solution.

In the continuation, we examine the positive solution of (4) without losing generality. So, we introduce the following classes:

$y(\varphi)\in N_0: y(\varphi))>0, d(\varphi)\Delta(y(\varphi))^{\kappa}<0, c(\varphi)\Delta\big(d(\varphi)(\Delta(y(\varphi))^{\kappa})\big)>0,$
$\Delta\left(c(\varphi)\Delta\left(d(\varphi)\left(\Delta(y(\varphi))^{\kappa}\right)\right)\right)<0$

$y(\varphi)\in N_1: y(\varphi))>0, d(\varphi)\Delta(y(\varphi))^{\kappa}>0, c(\varphi)\Delta\big(d(\varphi)(\Delta(y(\varphi))^{\kappa})\big)>0,$

$\Delta\left(c(\varphi)\Delta\left(d(\varphi)\left(\Delta(y(\varphi))^{\kappa}\right)\right)\right)<0.$

for all $\varphi\geq\varphi_2\geq\varphi_1$.

Lemma 2.5. Suppose that $\{y(\varphi)\}$ is a positive solution of (4) which satisfies Case N_1 and

$$\left(\sum_{s=\varphi_3}^{\varphi-1}\frac{1}{c(s)}\sum_{t=s}^{\infty}Q(t)f(D(\sigma(t)))\right)^{\frac{1}{\kappa}}=\infty. \tag{8}$$

Then

(i) $\left\{\frac{y(\varphi)}{D(\varphi)}\right\}$ is increasing $\forall\ \varphi\geq N$,

(ii) $\left\{\frac{y(\varphi)}{E(\varphi)}\right\}$ is decreasing $\forall\ \varphi\geq N$,

(iii) $\left\{\left(\frac{d(\varphi)(\Delta y(\varphi))^{\kappa}}{C(\varphi)}\right)\right\}$ is decreasing $\forall\ \varphi\geq N$.

Proof. Let $\{y(\varphi)\}$ be a positive solution to (4) that fulfills $\{y(\varphi)\}\in N_1$ for all $\varphi\in N$. Since $c(\varphi)\Delta(d(\varphi)(\Delta y(\varphi))^{\kappa})$ is decreasing, we get

$$d(\varphi)(\Delta y(\varphi))^{\kappa}\ =d(\varphi_1)\big(\Delta y(\varphi_1)\big)^{\kappa}+\sum_{i=\varphi_1}^{\varphi-1}\frac{c(i)\Delta(d(i)(\Delta y(i))^{\kappa})}{c(i)}\ \geq C(\varphi)c(\varphi)\Delta(d(\varphi)(\Delta y(\varphi))^{\kappa}).$$

This gives

$$\Delta\left(\frac{d(\varphi)(\Delta y(\varphi))^{\kappa}}{C(\varphi)}\right)=\frac{C(\varphi)c(\varphi)\Delta\big(d\big(\varphi(\Delta y(\varphi))^{\kappa}\big)\big)-d(\varphi)(\Delta y(\varphi))^{\kappa}}{c(\varphi)C(\varphi)C(\varphi+1)}\leq 0.$$

So $\left(\frac{d(\varphi)(\Delta y(\varphi))^{\kappa}}{C(\varphi)}\right)$ is decreasing and further, this fact yields

$$y(\varphi) \ = y(\varphi_1) + \left(\sum_{s=\varphi_1}^{\varphi-1} \frac{C^{\frac{1}{\kappa}}(s) d^{\frac{1}{\kappa}}(s)(\Delta y(s))}{C^{\frac{1}{\kappa}}(s) d(s)} \right) \ \geq E(\varphi) \left(\frac{d^{\frac{1}{\kappa}}(\varphi)(\Delta y(\varphi))}{C^{\frac{1}{\kappa}}(\varphi)} \right). \tag{9}$$

Hence

$$\Delta\left(\frac{y(\varphi)}{E(\varphi)}\right) = \frac{E(\varphi) d^{\frac{1}{\kappa}}(\varphi)\Delta y(\varphi) - y(\varphi) C^{\frac{1}{\kappa}}(\varphi)}{d^{\frac{1}{\kappa}}(\varphi) E(\varphi) E(\varphi+1)} \leq 0,$$

which gives $\frac{y(\varphi)}{E(\varphi)}$ is decreasing.

Next, since $\left\{ d^{\frac{1}{\kappa}}(\varphi)\Delta y(\varphi) \right\}$ is increasing for all $\varphi \geq N$, it is apparent $\forall\ \varphi \geq \varphi_2 \geq \varphi_1$

$$\begin{aligned} y(\varphi) \ &= y(\varphi_2) + \sum_{s=\varphi_2}^{\varphi-1} \frac{d^{\frac{1}{\kappa}}(s)}{d^{\frac{1}{\kappa}}(s)} \Delta y(s) \quad \leq y(\varphi_2) + d^{\frac{1}{\kappa}}(\varphi)\Delta y(\varphi) \sum_{s=\varphi_2}^{\varphi-1} \frac{1}{d^{\frac{1}{\kappa}}(s)} \\ &= y(\varphi_2) - d^{\frac{1}{\kappa}}(\varphi)\Delta y(\varphi) \sum_{s=\varphi_2}^{\varphi_1-1} \frac{1}{d^{\frac{1}{\kappa}}(s)} + d^{\frac{1}{\kappa}}(\varphi)\Delta y(\varphi) \sum_{s=\varphi_2}^{\varphi-1} \frac{1}{d^{\frac{1}{\kappa}}(s)} . \end{aligned} \tag{10}$$

We claim that $d^{\frac{1}{\kappa}}(\varphi)\Delta y(\varphi) \to \infty$ as $\varphi \to \infty$. Suppose $d^{\frac{1}{\kappa}}(\varphi)\Delta y(\varphi) \to 2w$ as $\varphi \to \infty$, we have $d^{\frac{1}{\kappa}}(\varphi)\Delta y(\varphi) \geq w$. This gives $y(\varphi) \geq wD(\varphi)$, summing (4) from φ to ∞, we have

$$\Delta\big(d(\varphi)(\Delta y(\varphi))^{\kappa}\big) \geq \frac{1}{c(\varphi)} f(w) \sum_{s=\varphi}^{\infty} Q(s) f\left(D\big(\sigma(s)\big)\right).$$

Again summing from φ_3 to $\varphi - 1$, we obtain

$$d^{\frac{1}{\kappa}}(\varphi)\Delta y(\varphi) \geq f^{\frac{1}{\kappa}}(w) \left(\sum_{s=\varphi_3}^{\varphi-1} \frac{1}{c(s)} \sum_{t=s}^{\infty} Q(t) f(D(\sigma(t))) \right)^{\frac{1}{\kappa}}, \tag{11}$$

for all $\varphi \geq \varphi_3$.

$$2w \geq f^{\frac{1}{\kappa}}(w) \left(\sum_{s=\varphi_3}^{\varphi-1} \frac{1}{a(s)} \sum_{t=s}^{\infty} Q(t) f(B(\sigma(t))) \right)^{\frac{1}{\kappa}},$$

which contradicts (8) so $d^{\frac{1}{\kappa}}(\varphi)\Delta y(\varphi) \to \infty$ as $\varphi \to \infty$. Hence, (10) becomes

$$y(\varphi) \leq d^{\frac{1}{\kappa}}(\varphi)\Delta y(\varphi) D(\varphi).$$

Since

$$\Delta\left(\frac{y(\varphi)}{D(\varphi)}\right) = \frac{d^{\frac{1}{\kappa}(\varphi)} D(\varphi)\Delta y(\varphi) - y(\varphi)}{d^{\frac{1}{\kappa}}(\varphi) D(\varphi) D(\varphi+1))} \geq 0,$$

$\frac{y(\varphi)}{D(\varphi)}$ is increasing. This concludes the proof.

Theorem 2.6. Let (8) hold. Assume that (2) has a positive solution $\{z(\varphi)\}$ and

$$lim\, sup_{u\to\infty} \frac{u^{\kappa}}{f(u)} = K < \infty. \tag{12}$$

If

$$\lim\sup_{\varphi\to\infty}\begin{pmatrix}\frac{E^{\kappa}(\sigma(\varphi))}{C^{\kappa}(\sigma(\varphi))}f\left(\frac{1}{E(\sigma(\varphi))}\right)\sum_{s=\varphi_1}^{\sigma(\varphi)-1}C(s+1)Q(s)f\left(E\left(\sigma(s)\right)\right)\\ +E^{\kappa}\left(\sigma(\varphi)\right)f\left(\frac{1}{E(\sigma(\varphi))}\right)\sum_{t=\sigma(\varphi)}^{\varphi-1}Q(t)f\left(E\left(\sigma(t)\right)\right)\\ +E^{\kappa}(\sigma(\varphi))f\left(\frac{1}{D(\sigma(\varphi))}\right)\sum_{t=\varphi}^{\infty}Q(t)f(D(\sigma(t)))\end{pmatrix}>K, \tag{13}$$

then $y(\varphi)\notin N_1$ which implies (1) has Property A.

Proof:

Assume that $\{y(\varphi)\}$ is a positive solution of (1) and so a solution of (4) which belonging to either N_0 or N_1 for all $\varphi\geq N$. Now we take $y(\varphi)\in N_1$, summation of (4) from φ to ∞ gives

$$\Delta\left(d(\varphi)(\Delta y(\varphi))^{\kappa}\right)\geq\frac{1}{c(\varphi)}\sum_{s=\varphi}^{\infty}Q(s)f\left(y\left(\sigma(s)\right)\right).$$

Summing the aforementioned inequality from φ_1 to $\varphi-1$, we get

$$d(\varphi)(\Delta y(\varphi))^{\kappa}\ \geq\sum_{s=\varphi_1}^{\varphi-1}\frac{1}{c(s)}\sum_{t=s}^{\infty}Q(t)f\left(y\left(\sigma(t)\right)\right)$$

$$=\sum_{s=\varphi_1}^{\varphi-1}C(s+1)Q(s)f\left(y\left(\sigma(s)\right)\right)+C(\varphi)\sum_{t=\varphi}^{\infty}Q(t)f(y(\sigma(t)))\ .$$

From (9), we obtain

$$\frac{C(\varphi)y^{\kappa}(\varphi)}{E^{\kappa}(\varphi)}\geq\sum_{s=\varphi_1}^{\varphi-1}C(s+1)Q(s)f(y(\sigma(s)))+C(\varphi)\sum_{t=\varphi}^{\infty}Q(t)f(y(\sigma(t))).$$

or

$$\frac{C(\sigma(\varphi))y^{\kappa}(\sigma(\varphi))}{E^{\kappa}(\sigma(\varphi))}\geq\sum_{s=\varphi_1}^{\sigma(\varphi)-1}C(s+1)Q(s)f\left(y\left(\sigma(s)\right)\right)+C\left(\sigma(\varphi)\right)\ \sum_{t=\sigma(\varphi)}^{\varphi-1}Q(t)f\left(y\left(\sigma(t)\right)\right)$$

$$+C\left(\sigma(\varphi)\right)\sum_{t=\varphi}^{\infty}Q(t)f\left(y\left(\sigma(t)\right)\right).$$

Using the monotonic properties (i)-(ii) of Lemma 2.5 and applying f, we get

$$\frac{C(\sigma(\varphi))y^{\kappa}(\sigma(\varphi))}{E^{\kappa}(\sigma(\varphi))}\geq f\left(\frac{y(\sigma(\varphi))}{E(\sigma(\varphi))}\right)\sum_{s=\varphi_1}^{\sigma(\varphi)-1}C(s+1)Q(s)f\left(E\left(\sigma(s)\right)\right)$$

$$+C\left(\sigma(\varphi)\right)f\left(\frac{y(\sigma(\varphi))}{E(\sigma(\varphi))}\right)\sum_{t=\sigma(\varphi)}^{\varphi-1}Q(t)f\left(E\left(\sigma(t)\right)\right)$$

$$+C(\sigma(\varphi))f\left(\frac{y(\sigma(\varphi))}{D(\sigma(\varphi))}\right)\sum_{t=\varphi}^{\infty}Q(t)f\left(D\left(\sigma(t)\right)\right),$$

or

$$\frac{y^{\kappa}(\sigma(\varphi))}{f\left(y(\sigma(\varphi))\right)}\geq\frac{E^{\kappa}(\sigma(\varphi))}{C(\sigma(\varphi))}f\left(\frac{1}{E(\sigma(\varphi))}\right)\sum_{s=\varphi_1}^{\sigma(\varphi)-1}C(s+1)Q(s)f\left(E\left(\sigma(s)\right)\right)$$

$$+E^{\kappa}\left(\sigma(\varphi)\right)f\left(\frac{1}{E(\sigma(\varphi))}\right)\sum_{t=\sigma(\varphi)}^{\varphi-1}Q(t)f\left(E\left(\sigma(t)\right)\right)$$

$$+E^{\kappa}\left(\sigma(\varphi)\right)f\left(\frac{1}{D(\sigma(\varphi))}\right)\sum_{t=\varphi}^{\infty}Q(t)f\left(D\left(\sigma(t)\right)\right).$$

Taking limsup as $\varphi \to \infty$ on both side of above inequality, we get contradiction with (13). Hence $y(\varphi) \notin N_1$, so $y(\varphi)$ satisfies the case N_0. Therefore (1) has Property A.

Corollary 2.7. Let (8) hold, $f(u) = u^\kappa$ and (2) have a positive solution $\{z(\varphi)\}$. If

$$lim\,sup_{\varphi\to\infty}\begin{pmatrix}\frac{1}{C(\sigma(\varphi))}\sum_{s=\varphi_1}^{\sigma(\varphi)-1} C(s+1)Q(s)E^\kappa(\sigma(s)) \\ +\sum_{t=\sigma(\varphi)}^{\varphi-1} Q(t)f\left(E(\sigma(t))\right) \\ +\frac{E^\kappa(\sigma(\varphi))}{D^\kappa(\sigma(\varphi))}\sum_{t=\varphi}^{\infty} Q(t)f(D(\sigma(t)))\end{pmatrix} > 1. \quad (14)$$

then $y(\varphi) \notin N_1$ which implies (1) has Property A

Throughout the proof of the following theorem, we use $H(\varphi, j): \varphi, j \in N, \varphi \geq j \geq 0$ to denote the double sequence satisfying

$$H(\varphi,\varphi) = 0\ for\ \varphi \geq \varphi_0\,;$$

$$H(\varphi,j) > 0\ for\ \varphi > j \geq \varphi_0;$$

$$\Delta_2 H(\varphi,j) = H(\varphi,j+1) - H(\varphi,j) < 0\ for\ \varphi > j \geq \varphi_0\,.$$

Theorem 2.8.(H1)-(H3) hold and (2) has a positive solution $\{z(\varphi)\}$. If

$$lim\,sup_{\varphi\to\infty}\frac{1}{H(\varphi,M)}\sum_{j=M}^{\varphi-1}\left[H(\varphi,j)\frac{\rho(j)Q(j)f(y(\sigma(j)))}{y(\sigma(j))^\kappa} - \frac{h^2(\varphi,j)\rho(j+1)c(j)d(\sigma(j))}{4(\sigma(j)-j)H(\varphi,j)}\right] = \infty, \quad (15)$$

and

$$\sum_{t=\varphi}^{\infty}\left(\frac{1}{d(t)}\sum_{j=t}^{\infty}\frac{1}{c(j)}\sum_{i=j}^{\infty} Q(i)f(y(\sigma(i)))\right)^{\frac{1}{\kappa}} = \infty, \quad (16)$$

then (1) is oscillatory.

Proof. Let $y(\varphi)$ be a nonoscillatory solution of (4). Without loss of generality, $y(\varphi)$ is eventually positive such that $y(\varphi) \in N_0$ or $y(\varphi) \in N_1$. If $y(\varphi) \in N_1$, then we have to define for $\rho(\varphi) > 0$,

$$w(\varphi) = \rho(\varphi)\frac{c(\varphi)\Delta(d(\varphi)(\Delta y(\varphi))^\kappa)}{(y(\sigma(\varphi)))^\kappa},$$

then $w(\varphi) > 0$ for $\varphi \geq \varphi_1$, and

$$\begin{aligned}\Delta w(\varphi) &= \Delta\left(\rho(\varphi)\frac{c(\varphi)\Delta\big(d(\varphi)(\Delta y(\varphi))^\kappa\big)}{(y(\sigma(\varphi)))^\kappa}\right) \\ &= \Delta\rho(\varphi)\frac{c(\varphi+1)\Delta\big(d(\varphi+1)(\Delta y(\varphi+1))^\kappa\big)}{(y(\sigma(\varphi+1)))^\kappa} + \rho(\varphi)\Delta\left(\frac{c(\varphi)\Delta\big(d(\varphi)(\Delta y(\varphi))^\kappa\big)}{(x(\sigma(\varphi)))^\kappa}\right) \\ &= \frac{w(\varphi+1)}{\rho(\varphi+1)}\Delta\rho(\varphi) + \rho(\varphi)\frac{\Delta\left(c(\varphi)\Delta\big(d(\varphi)(\Delta y(\varphi))^\kappa\big)\right)}{(y(\sigma(\varphi)))^\kappa} \\ &- \frac{\rho(\varphi)w(\varphi+1)}{(y(\sigma(\varphi)))^\kappa\rho(\varphi+1)}\Delta\big(y(\sigma(\varphi))\big)^\kappa.\end{aligned}$$

From (4), we have

$$\Delta w(\varphi) = \frac{w(\varphi+1)}{\rho(\varphi+1)}\Delta\rho(\varphi) - \frac{\rho(\varphi)Q(\varphi)f(y(\sigma(\varphi)))}{y(\sigma(\varphi))^{\kappa}} - \frac{\rho(\varphi)w(\varphi+1)}{(y(\sigma(\varphi)))^{\kappa}\rho(\varphi+1)}\Delta(y(\sigma(\varphi)))^{\kappa}. \quad (17)$$

Since $\Delta\big(d(\varphi)\Delta(y(\varphi))^{\kappa}\big)$ is decreasing, so $d(\varphi)\Delta(y(\varphi))^{\kappa} = d(j)\Delta(y(j))^{\kappa} + \sum_{s=j}^{\varphi-1}\Delta(d(s)\Delta y(s))$, $\varphi \geq j \geq \varphi_2$, which implies

$$d(\varphi)(\Delta y(\varphi))^{\kappa} \geq (\varphi - j)\Delta(d(\varphi)(\Delta y(\varphi))^{\kappa}).$$

and so

$$(\Delta y(\sigma(\varphi)))^{\kappa} \geq \frac{(\sigma(\varphi)-j)c(\varphi)\Delta(d(\varphi)(\Delta y(\varphi))^{\kappa})}{c(\varphi)d(\sigma(\varphi))}. \quad (18)$$

From (17) and (18), we carry

$$\Delta w(\varphi) \leq \frac{w(\varphi+1)}{\rho(\varphi+1)}\Delta\rho(\varphi) - \frac{\rho(\varphi)Q(\varphi)f\big(y(\sigma(\varphi))\big)}{y(\sigma(\varphi))^{\kappa}} - \frac{\rho(\varphi)w(\varphi+1)}{(y(\sigma(\varphi)))^{\kappa}\rho(\varphi+1)}\frac{(\sigma(\varphi)-j)c(\varphi)\Delta\big(d(\varphi)(\Delta y(\varphi))^{\kappa}\big)}{c(\varphi)d\big(\sigma(\varphi)\big)},$$

$$\Delta w(\varphi) \leq -\frac{\rho(\varphi)Q(\varphi)f(y(\sigma(\varphi)))}{y(\sigma(\varphi))^{\kappa}} + \frac{\Delta\rho(\varphi)}{\rho(\varphi+1)}w(\varphi+1) - \frac{(\sigma(\varphi)-j)}{\rho(\varphi+1)c(\varphi)d(\sigma(\varphi))}w^2(\varphi+1).$$

Multiplying both side by $H(\varphi, j)$ and then summing up from M to $\varphi - 1$ for all $\varphi \geq M$, we obtain

$$\sum_{j=M}^{\varphi-1} H(\varphi,j)\frac{\rho(j)Q(j)f\big(y(\sigma(j))\big)}{y(\sigma(j))^{\kappa}} \leq \sum_{j=M}^{\varphi-1} H(\varphi,j)\frac{\Delta\rho(j)}{\rho(j+1)}w(j+1)$$
$$- \sum_{j=M}^{\varphi-1} H(\varphi,j)\frac{(\sigma(j)-j)}{\rho(j+1)c(j)d(\sigma(j))}w^2(j+1) - \sum_{j=M}^{\varphi-1} H(\varphi,j)w(j). \quad (19)$$

Using summation by parts, we see that

$$-\sum_{j=M}^{\varphi-1} w(j)H(\varphi,j) = H(\varphi,M)w(M) + \sum_{j=M}^{\varphi-1} w(j+1)\Delta_2 H(\varphi,j). \quad (20)$$

Substituting (20) in (19), we carry

$$\sum_{j=M}^{\varphi-1} H(\varphi,j)\frac{\rho(j)Q(j)f\big(y(\sigma(j))\big)}{y(\sigma(j))^{\kappa}} \leq H(\varphi,M)w(M) + \sum_{j=M}^{\varphi-1} h(\varphi,j)w(j+1)$$
$$-\sum_{j=M}^{\varphi-1} H(\varphi,j)\frac{(\sigma(j)-j)}{\rho(j+1)c(j)d(\sigma(j))}w^2(j+1),$$

where

$$h(\varphi,j) = \Delta_2 H(\varphi,j) + H(\varphi,j)\frac{\Delta\rho(j)}{\rho(j+1)}.$$

$$\sum_{j=M}^{\varphi-1} H(\varphi,j)\frac{\rho(j)Q(j)f(y(\sigma(j)))}{y(\sigma(j))^{\kappa}} \leq H(\varphi,M)w(M) + \sum_{j=M}^{\varphi-1}\frac{h^2(\varphi,j)\rho(j+1)c(j)d(\sigma(j))}{4(\sigma(j)-j)H(\varphi,j)},$$

or

$$\sum_{j=M}^{\varphi-1}\left[H(\varphi,j)\frac{\rho(j)Q(j)f(y(\sigma(j)))}{y(\sigma(j))^{\kappa}} - \frac{h^2(\varphi,j)\rho(j+1)c(j)d(\sigma(j))}{4(\sigma(j)-j)H(\varphi,j)}\right] \leq H(\varphi,M)w(M). \quad (21)$$

Taking $limsup$ as $\varphi \to \infty$, we obtain

$$\limsup_{\varphi\to\infty}\frac{1}{H(\varphi,M)}\sum_{j=M}^{\varphi-1}\left[H(\varphi,j)\frac{\rho(j)Q(j)f(y(\sigma(j)))}{y(\sigma(j))^{\kappa}} - \frac{h^2(\varphi,j)\rho(j+1)c(j)d(\sigma(j))}{4(\sigma(j)-j)H(\varphi,j)}\right] \leq w(M).$$

which contradicts (15).

Now we take $y(\varphi) \in N_0$, summing (4) from φ to ∞, we have

$$c(\varphi)\Delta\big(d(\varphi)(\Delta y(\varphi))^{\kappa}\big) \geq \sum_{t=\varphi}^{\infty} Q(t) f\Big(y\big(\sigma(t)\big)\Big) \ ,$$

$$\Delta\big(d(\varphi)(\Delta y(\varphi))^{\kappa}\big) \geq \frac{1}{c(\varphi)} \sum_{t=\varphi}^{\infty} Q(t) f(y(\sigma(t))) \, .$$

Again summing from φ to ∞

$$(\Delta y(\varphi))^{\kappa} \leq -\frac{1}{d(\varphi)} \sum_{t=\varphi}^{\infty} \frac{1}{c(t)} \sum_{j=t}^{\infty} Q(j) f\Big(y\big(\sigma(j)\big)\Big),$$

$$\Delta y(\varphi) \leq -\left(\frac{1}{d(\varphi_1)} \sum_{t=\varphi}^{\infty} \frac{1}{c(t)} \sum_{j=t}^{\infty} Q(j) f(y(\sigma(j)))\right)^{\frac{1}{\kappa}}.$$

Summing once again from φ_1 to ∞, we obtain

$$y(\varphi_1) \geq \sum_{t=\varphi_1}^{\infty} \frac{1}{d(t)} \left(\sum_{j=t}^{\infty} \frac{1}{c(j)} \sum_{i=j}^{\infty} Q(i) f(y(\sigma(i)))\right)^{\frac{1}{\kappa}},$$

which contradicts (16). Therefore, every solution of (1) is oscillatory. Hence the proof.

3 Examples

Example 3.1. Look into the equation

$$\Delta\big(\varphi^2 \Delta(\varphi(\Delta y(\varphi)))\big) - \frac{(2\varphi+1)(\varphi+1)}{\varphi+2}(\Delta(\varphi+1)) + \frac{\varphi+1}{\varphi+2} f(y(3\varphi))) = 0, \varphi \geq 1. \qquad (22)$$

Here $a(\varphi) = \varphi^2,\ b(\varphi) = \varphi,\ p(\varphi) = \frac{(2\varphi+1)(\varphi+1)}{\varphi+2},\ q(\varphi) = \frac{\varphi+1}{\varphi+2},\ \kappa = 1,$

$\sigma(\varphi) = 3\varphi,\ f(y) = y^{\kappa},\ c(\varphi) = \varphi^2(\varphi+1)(\varphi+2),\ d(\varphi) = \frac{\varphi}{\varphi+1},$

$Q(\varphi) = \varphi + 1,\ C(\varphi) \approx \frac{1}{\varphi^2(\varphi+1)(\varphi+2)},\ D(\varphi) \approx \frac{\varphi+1}{\varphi},\ E(\varphi) \approx \frac{1}{\varphi^3(\varphi+2)}.$

Now

$$\sum_{s=1}^{\infty} \frac{1}{a(s)} = \sum_{s-1}^{\infty} \frac{1}{s^2} < \infty, \quad \sum_{s=1}^{\infty} \frac{1}{b^{\frac{1}{\kappa}}(s)} = \sum_{s=1}^{\infty} \frac{1}{s} = \infty,$$

and the auxiliary second order equation (4) becomes,

$$\Delta\big(\varphi^2 \Delta z(\varphi)\big) - \frac{(2\varphi+1)(\varphi+1)}{(\varphi+2)} z(\varphi+1) = 0. \qquad (23)$$

It has a nonoscillatory solution $z(\varphi) = \varphi + 1$. Moreover, the positive sequence $\rho(\varphi) = l$ which implies that $\Delta\rho(\varphi) = 1$ and

$$H(\varphi, \varphi) = 0, \varphi \geq \varphi_0 \ ,$$

$$H(\varphi, j) = \varphi - j > 0, \varphi > j \geq \varphi_0,$$

$$\Delta_2[H(\varphi, j)] = H(\varphi, j+1) - H(\varphi, j) = -1 < 0, \varphi > j \geq \varphi_0,$$

$$h(\varphi, j) = \frac{1}{\varphi+1}[(\varphi - j) - (\varphi + 1)].$$

From the above values, (15) becomes

$$lim\,sup_{\varphi\to\infty}\frac{1}{H(\varphi,M)}\sum_{j=M}^{\varphi-1}\left[j(\varphi-j)(j+1)-\frac{3j^3(2(j+1)+(\varphi-j)}{8j(3j+1)}\right]=\infty,$$

and (16) becomes

$$\sum_{t=\varphi}^{\infty}\left(\frac{t+1}{t}\sum_{j=t}^{\infty}\frac{1}{j^2(j+2)(j+1)}\sum_{i=j}^{\infty}(i+1)\right)=\infty.$$

Therefore, conditions of theorem 2.8 are satisfied, so every solution of (22) is oscillatory.

Example 3.2. Look over the equation

$$\Delta\big(\varphi^2\Delta(\varphi(\Delta y(\varphi)))\big)-\frac{(2\varphi+1)(\varphi+1)}{\varphi+2}(\Delta(\varphi+1))+\frac{8\varphi^2}{\varphi+1}f(y(3\varphi)))=0,\varphi\geq 1 \tag{24}$$

Here $a(\varphi)=\varphi^2,\ b(\varphi)=\varphi,\ p(\varphi)=\frac{(2\varphi+1)(\varphi+1)}{\varphi+2},\ q(\varphi)=\frac{8\varphi^2}{\varphi+1},\ \kappa=1,$

$\sigma(\varphi)=\varphi+1,\ f(y)=y^{\kappa},\ c(\varphi)=\varphi^2(\varphi+1)(\varphi+2),\ d(\varphi)=\frac{\varphi}{\varphi+1},$

$Q(\varphi)=\varphi+1,\ C(\varphi)\approx\frac{1}{\varphi^2(\varphi+1)(\varphi+2)},\ D(\varphi)\approx\frac{\varphi+1}{\varphi},\ E(\varphi)\approx\frac{1}{\varphi^3(\varphi+2)}.$

Now

$$\sum_{s=1}^{\infty}\frac{1}{a(s)}=\sum_{s-1}^{\infty}\frac{1}{s^2}<\infty,\quad \sum_{s=1}^{\infty}\frac{1}{b^{\frac{1}{\alpha}}(s)}=\sum_{s=1}^{\infty}\frac{1}{s}=\infty,$$

and equation (4) becomes,

$$\Delta\big(\varphi^2\Delta z(\varphi)\big)-\frac{(2\varphi+1)(\varphi+1)}{(\varphi+2)}z(\varphi+1)=0. \tag{25}$$

It has a nonoscillatory solution $z(\varphi)=\varphi+1$. Moreover, (8) becomes

$$\sum_{s=\varphi_3}^{\varphi-1}\frac{1}{\varphi^2(\varphi+1)(\varphi+2)}\sum_{t=s}^{\infty}8t(t+2)=\infty,$$

and from Corollary 2.7, (14) becomes

$$lim\,sup_{\varphi\to\infty}\Big[(\varphi+1)^2(\varphi+2)(\varphi+3)\sum_{s=\varphi_1}^{\varphi-1}(\varphi+1)^2(\varphi+2)(\varphi+3)+\sum_{t=\varphi+1}^{\varphi-1}\left(\frac{1}{(t+1)(t+3)}\right)+\frac{(\varphi+1)^3(\varphi+3)}{\varphi+2}\sum_{t=\varphi}^{\infty}t+2\Big]>1.$$

So every solution of (24) are oscillatory.

4 Application

Samuelson is the originator of the business cycle model [12]. This model is built on a set of underlying assumptions.

Yearly income $y(\varphi)$ is equal to the sum of Capital Investment $G(\varphi)$, Consumption $C(\varphi)$ and Private Investment $I(\varphi)$

$$y(\varphi)=G(\varphi)+C(\varphi)+I(\varphi). \tag{26}$$

Capital Investment $G(\varphi)$ remains constant.

$$G(\varphi)=G. \tag{27}$$

Consumption $C(\varphi)$ depends on the previous year income and on marginal tendency to consume, it is denoted as α.

$$C(\varphi) = \alpha y(\varphi - 1), \tag{28}$$

where the multiplier parameter $0 < \alpha < 1$.

Private Investment I(r) depends on consumption changes and on the accelerator factor β, where

$\beta > 0$

$$I(\varphi) = \beta(C(\varphi) - C(\varphi - 1)) = \alpha\beta(y(\varphi - 1) - y(\varphi - 2)). \tag{29}$$

Substitute (27), (28) and (29), the yearly income $y(\varphi)$ can be determined as a second-order difference equation

$$y(\varphi) = G + \alpha y(\varphi - 1) + \alpha\beta y(\varphi - 1) - \alpha\beta y(\varphi - 2). \tag{30}$$

The model developed by Paul Samuelson is a cornerstone in the field of economics. It delineates five possible paths or trends that economic activities might follow. These trajectories are shaped by the interplay of two fundamental parameters: the marginal propensity to consume (α) and the accelerator coefficient (β).

The marginal propensity to consume (α) quantifies the proportion of additional income consumers spend rather than save. Conversely, the accelerator coefficient (β) is a metric that gauges the sensitivity of investment expenditures to shifts in GDP.

Each unique pairing of α and β yields a specific pattern of economic activity, which can range from stable growth to cyclical oscillations. This spectrum of outcomes underscores the intricate relationship between consumption, investment, and aggregate economic activity.

Samuelson's model offers a holistic blueprint for deciphering the mechanics of economic cycles. It equips economists with the tools to scrutinize and forecast the impact of alterations in consumer behavior and investment on the course of the economy. This model is particularly instrumental in policy formulation, as it aids in devising strategies to navigate economic cycles effectively.

In essence, Samuelson's model is a potent instrument furnishing invaluable insights into economic cycle dynamics. It is indispensable in economic prognostication and policy design, contributing significantly to our understanding of economic phenomena. This model's versatility and predictive power make it an essential tool in the economist's toolkit. Its ability to capture the nuanced interdependencies within an economy underscores its enduring relevance in economic analysis and policy planning.

Table 1: Five Different Path Cycle

Case	Values	Behavior of the Cycles
1	$\alpha = 0.5, \beta = 0$	Cycle less path
2	$\alpha = 0.5, \beta = 1$	Damped Fluctuations
3	$\alpha = 0.5, \beta = 2$	Fluctuation of constant Amplitude
4	$\alpha = 0.5, \beta = 3$	Explosive Cycles
5	$\alpha = 0.5, \beta = 4$	Cycle less Explosive Path

Samuelson's model, a cornerstone in economic theory, illustrates the behavior of economic cycles based on varying values of the marginal propensity to consume (α) and the accelerator coefficient (β).

1. When α=0.5 and β=0, the economy follows a **cycle-less path**, indicating stability without fluctuations. This suggests a balanced economy where changes in income do not significantly affect consumption or investment patterns.
2. With α=0.5 and β=1, the economy experiences **damped fluctuations**, suggesting cycles gradually diminishing over time. This indicates an economy that, while initially reactive to changes in income, eventually stabilizes.
3. For α=0.5 and β=2, the economy undergoes **fluctuations of constant amplitude**, indicating regular cyclical patterns. This represents an economy with consistent cycles, reflecting a direct, proportional relationship between income changes and consumption or investment.
4. When α=0.5 and β=3, the economy exhibits **explosive cycles**, suggesting rapid and potentially unstable growth. This scenario might occur when an increase in income significantly boosts consumption and investment, leading to accelerated economic growth.
5. Lastly, with α=0.5 and β=4, the economy follows a **cycle-less explosive path**, indicating steady, rapid growth without cyclical fluctuations. This could represent an economy where income changes lead to increased consumption and investment but without the cyclical patterns seen in other scenarios.

These patterns provide valuable insights into how changes in consumption and investment behaviors, driven by shifts in income, can impact the cyclical behavior of an economy. Understanding these dynamics is crucial for economists and policymakers to effectively manage economic stability and growth. Thus, Samuelson's model is a powerful tool in economic analysis and policy planning.

Based on these assumptions, the nonoscillatory second-order difference equation (25) is

$$y(\varphi) = G + \left(1 + \frac{(2\varphi-3)(\varphi-1)}{\varphi}\right) y(\varphi-1) + \frac{(\varphi-2)^2}{(\varphi-1)^2} y(\varphi-1) - \frac{(\varphi-2)^2}{(\varphi-1)^2} y(\varphi-2). \qquad (31)$$

It is a corresponding equation of (30). Here $y(\varphi)$ denotes the current year income of the business and

$$\alpha = \left(1 + \frac{(2\varphi-3)(\varphi-1)}{\varphi}\right) = \ TheMultiplier\ ,$$

and

$$\beta = \left(\frac{\varphi(\varphi-2)^2}{[\varphi+(2\varphi-3)(\varphi-1)](\varphi-1)^2}\right) = \ TheAccelerator\ ,$$

Illustration: Let's consider an illustration to understand this equation better. Suppose a steel company owner invests an autonomous investment of Rs. 2 crore. The previous two years' income is unknown, and we refer to the Path Cycle table to determine the values of autonomous investment, current consumption, induced investment, and yearly income for the next five years and from the Path cycle table we take $\alpha = 0.5, \beta = 1$

Table 2: Business Income

Time Period (φ)	Autonomous Investment $G(\varphi)$	Current Consumption $C(\varphi)$	Induced Investment $I(\varphi)$	Yearly Income $Y(\varphi)$
1	2,00,00,000	0	0	2,00,00,000
2	2,00,00,000	1,00,00,000	1,00,00,000	4,00,00,000
3	2,00,00,000	2,00,00,000	1,00,00,000	5,00,00,000
4	2,00,00,000	2,50,00,000	50,00,000	5,00,00,000
5	2,00,00,000	2,50,00,000	0	4,50,00,000

The data represents a business's income over five years. The autonomous investment, G(φ), remains constant at 2,00,00,000. In the first year, no consumption or induced investment leads to a yearly income, Y(φ), of 2,00,00,000. In the second year, consumption, C(φ), increases to 1,00,00,000 and induced investment, I(φ), to 1,00,00,000, doubling the yearly income to 4,00,00,000. In the third year, consumption doubles, and the yearly income increases to 5,00,00,000. In the fourth year, consumption increases further, but induced investment decreases, keeping the yearly income steady. In the fifth year, induced investment drops to zero, reducing the yearly income to 4,50,00,000. This suggests a need for strategies to maintain or increase induced investment.

The nonoscillatory second-order difference equation (25) can be written as a corresponding equation. Here, the notation " φ " denotes the current year's income of a business, " φ -1" represents the income of the previous year, and " φ -2" represents the income of the year before that.

As per the table, in the first year, there is no autonomous investment or consumption, and hence, the yearly income is Rs. 2 crore. In the second year, the autonomous investment is still Rs. 2 crore, but there is a consumption of Rs. 3 crore, resulting in an induced investment of Rs. 1.5 crore. This brings the yearly income to Rs. 4.5 crore. Similarly, in the third year, with a consumption of Rs. 4 crore, the induced investment is Rs. 4 crore, leading to a yearly income of Rs. 10 crore.

By using the second-order difference equation, we can determine the yearly income for the remaining two years based on the values of consumption and induced investment. This equation helps to provide a better understanding of the dynamics of economic cycles and how changes in consumption and investment can impact them.

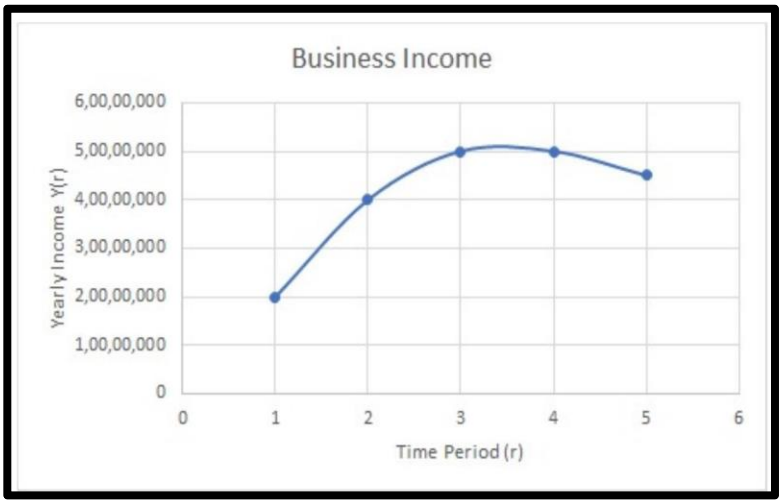

Figure 1 :

This graph indicates the five-year income of the business. it will be changed based on the Consumption.

5 Conclusion

The article shows that Property A solution is closely related to Samuelson's business cycle model, which is a prominent economic theory of the fluctuations in output and employment. The authors illustrate the importance of their main results by giving some examples and comparing them with existing literature. They also suggest some possible directions for future research on this topic. The research paper discussed the oscillatory behavior and Property A condition for a positive middle term. This equation has applications in economics, physics, and mathematical biology. The article establishes new oscillatory behavior and Property A conditions for the equation, which can be considered an improvement to previous results. The authors use a generalized Riccati transformation technique and some auxiliary second-order difference equations. Additionally, the article derives a solution for the equation that possesses Property A and Oscillatory.

The article also highlights the connection between their Property A solution and Samuelson's business cycle model. This model is a well-known economic theory that explains the fluctuations in output and employment. By drawing on this connection, the authors illustrate the importance of their main results by giving some examples and comparing them with existing literature.

The article also suggests possible directions for future research on this topic, highlighting the potential for further exploration and applications of the results in various fields. The application of Samuelson's business cycle model in a second-order difference equation (2) is presented, and the yearly income is calculated based on the model. This demonstrates the practical relevance of the research and its potential implications for real-world scenarios.

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