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Classification of Maps on Intervals That Exhibit Dense Periodic Points

***Suja N Thomas,**P.B Vinod Kumar**
Rajagiri School of Engineering and Technology, Kochi APJ Abdul Kalam Technological University, Kerala, India.
*sujanthomas72@gmail.com
Muthoot Institute of Technology and Science, Kochi APJ Abdul Kalam Technological University, Kerala, India.
**vinodkumarpb@mgits.ac.in

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Abstract:

In this paper, continuous maps on interval with dense set of periodic points such that the periods being set of natural numbers are studied.These maps named as Periodically Rich maps could be divided into two classes on Interval, such as the maps with period four points having increasing periodic orbit and the maps with two fixed points such that image of a subset of Interval does not belong to the same subset. Notion of bigness of Periodically Rich maps is also discussed.

Keywords: Set of periods, Transitive maps, Dense Periodic points.

1. Introduction

Let X be a Hausdorff topological space and $f : X \to X$ be a continuous map. $x \in X$ such that $f(x) = x$ is called a fixed point of f. A point $x \in X$ is called a periodic point if there exists $n \in \mathbb{Z}_+$ such that $f^n(x) = x$. The smallest n for which $f^n(x) = x$ is called the period of f. The set of periodic points of period n is denoted by $Per_n(f)$ and the set of periods of f is denoted by $P(f)$, i.e.,

$Per_n(f) = \{x \in X | f^n(x) = x \text{ and } f^m(x) \neq x, \forall m < n\}$ $P(f) = \{n \in \mathbb{Z}_+ | Per_n(f) \neq \emptyset\}$.

$Per(f) = \bigcup_{n \in \mathbb{Z}_+} Per_n(f)$ is the set of periodic points of f. The forward orbit of a point $x \in X$ under the mapping f is denoted by $O_f(x)$,i.e., $O_f(x) = \{x, f(x), \ldots, f^n(x), \ldots\}$. If $x \in Per_n(f)$ then $\{x, f(x), \ldots, f^{n-1}(x)\}$ is called periodic orbit of x, in this case clearly $|O_f(x)| = n$. The periodic orbit of a point x in $I = [0, 1]$ is said to be increasing if $x < f(x) < \cdots < f^{n-1}(x)$. A map f is said to be transitive, if for every pair of non-empty open subsets U and V in X, there exists a positive integer n such that $f^n(U) \cap V \neq \varphi$.

In general, the set of periodic points $Per(f)$ can be empty, a finite set, a countable set or an uncountable set . $Per(f)$ can be even rich so that $\overline{Per(f)} = X$, which is one of the condition for f to be chaotic in the sense of Devaney [5]. If f is transitive on I then

$\overline{Per(f)} = I$ [12]. A point $x \in I$ is recurrent if for each neighborhood U of x there exists $n \in \mathbb{Z}_+$ such that $f^n(x) \in U$; the set of recurrent points is denoted by $Rec(f)$. It is proved in [3] that for map f on I, $\overline{Per(f)} = \overline{Rec(f)}$. For maps on I, $Per_1(f) \cup Per_2(f)$ is closed [4]. Considering the set of periods, $P(f)$ can also be empty, a finite or an infinite subset of $\mathbb{Z}_+$ or the whole $\mathbb{Z}_+$. Sharkovskii has proved that for maps on R or I, if $n \in P(f)$ then $m \in P(f)$ for every m which follows n in the Sharkovskii ordering $3 \prec 5 \prec 7 \prec 9 \prec \ldots \prec 2.3 \prec 2.5 \prec 2.7 \prec \ldots 2^2.3 \prec 2^2.5 \prec \ldots 2^3.3 \prec 2^3.5 \prec 2^3.7\ldots \prec 2^3 \prec 2^2 \prec 2 \prec 1.$

[10]

At this juncture a natural question arises. Is there any relation between $Per(f)$ and $P(f)$? The question prompts to think about the 'bigness' of set of periodic points. More specifically, the 'bigness' of set of periodic points is treated in two ways, (1) $Perf = X$ and (2) $P(\overline{f) = \mathbb{Z}}_+$.

To understand, some examples are listed below.

Example 1. *A map for which $Per(f) = X$ and $P(f) \neq \mathbb{Z}_+$*

Let $f : I \longrightarrow I$ be defined as

$$f_1(x) = \begin{cases} 2x + 1/2, & 0 \leq x \leq 1/4 \\ -2x + 3/2, & 1/4 < x \leq 3/4 \\ 2x - 3/2, & 3/4 < x \leq 1 \end{cases}$$

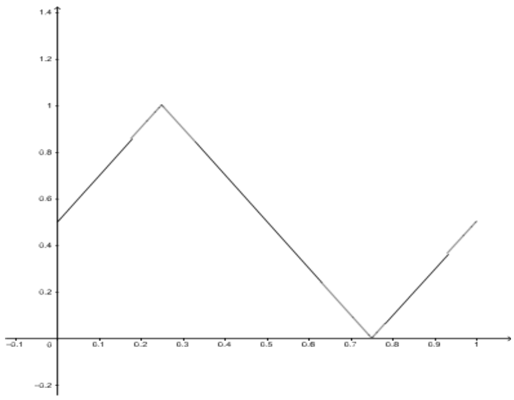

FIGURE 1. Graph of f_1

Fixed point of f_1 is at $x = 1/2$.
$f_1([0, 1/2]) = [1/2, 1]$ and $f_1([1/2, 1]) = [0, 1/2]$

$f_1^2([0, 1/2]) = f_1(f_1([0, 1/2])) = f_1([1/2, 1]) = [0, 1/2]$

$f_1^2([1/2, 1]) = f_1(f_1([1/2, 1])) = f_1([0, 1/2]) = [1/2, 1]$

$f_1^3([0, 1/2]) = f_1(f_1^2([0, 1/2])) = f_1([0, 1/2]) = [1/2, 1]$

$$f_1^3([1/2, 1]) = f_1(f_1^2([1/2, 1])) = f_1([1/2, 1]) = [0, 1/2]$$

So, $f_1^3(x) \neq x$ for all $x \in [0, 1]$ except for $x = 1/2$.

Similarly, if n is odd, $f_1^n(x) \neq x$, $\forall x \in [0, 1]$ except for $x = 1/2$.
If n is even, $f_1^n([0, 1/2]) = [0, 1/2]$ and $f_1^n([1/2, 1]) = [1/2, 1]$. *i.e.*, f_1^n has fixed points in $[0, 1]$ for even n. So there are points with even periods in $[0, 1]$. Therefore

$$P(f_1) = Z_+ \setminus \{3, 5, 7, \ldots\}.$$

On the other hand,if x is any irrational number in $[0, 1]$ then $O_{f_1}(x) = \overline{I}$. So f_1 is transitive on I [11]. Hence $\overline{Per(f_1)} = I$ [12].

Example 2. *Consider a map with* $P(f) = \mathbb{Z}_+$ *and* $\overline{Per(f)} \neq X$. *Let* $f_2 : I \to I$ *be defined as*

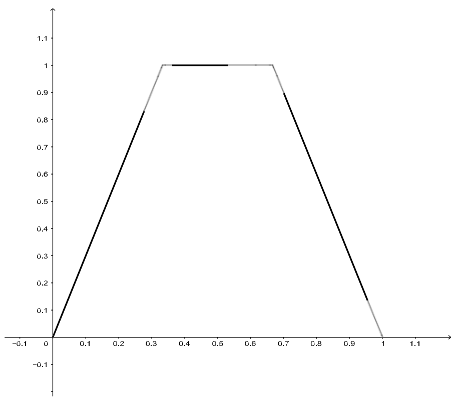

FIGURE 2. Graph of f_2

$$f_2(x) = \begin{cases} 3x, & 0 \leq x \leq 1/3 \\ 1, & 1/3 < x \leq 2/3 \\ 3 - 3x, & 2/3 < x \leq 1 \end{cases}$$

$O_{f_2}(9/28) = \{9/28, 27/28, 3/28\}$.

i.e., $9/28$ is a periodic point of period 3. Hence by proposition 1, $P(f_2) = \mathbb{Z}_+$.

Now consider open sets U and V in $(1/3, 2/3)$. $f_2^n(U) = f_2^n(V) = \{0\}$, $\forall\, n \in \mathbb{Z}_+$. So, there does not exists $n \in \mathbb{Z}_+$ such that $f_2^n(U) \cap V \neq \emptyset$. Therefore $\overline{Per(f_2)} \neq I$.

Example 3. *Tent map defined on I.*

$$f_3(x) = \begin{cases} 2x, & 0 \le x \le 1/2 \\ 2(1-x), & 1/2 < x \le 1 \end{cases}$$

$P(f_3) = \mathbb{Z}_+$ and $\overline{Per(f_3)} = I$ [9].

Example 4. *Consider* f_4 *defined on* I *by* $f_4(x) = x^2$.

Fixed points are $0, 1$ and $f_4^n(x) \to 0$, $\forall x \in (0, 1)$. So no periodic points other than fixed points.

Here neither $\overline{Per(f_4)} \neq I$ nor $P(f_4) \neq \mathbb{Z}_+$.

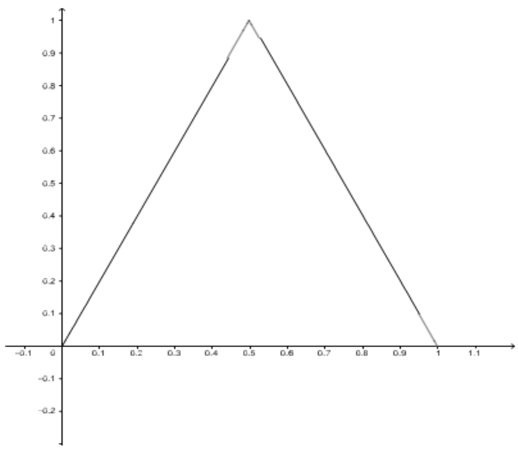

FIGURE 3. Graph of f_3

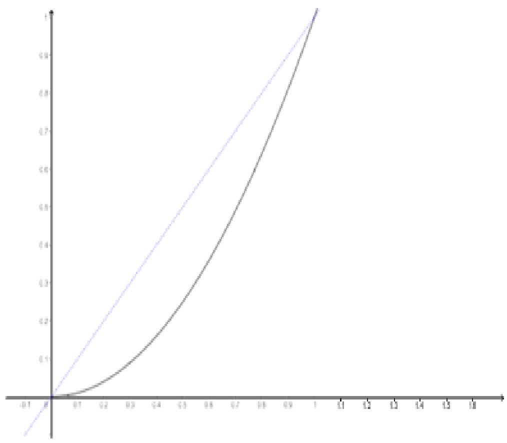

FIGURE 4. Graph of f_4

Example 5. *Let* $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ *and* $f_5 : S^1 \longrightarrow S^1$ *be defined as* $f_5(z) = z^2$.

$\overline{Per(f_5)} = S^1$ and $|Per_n(f_5)| = 2^n - 1, n \ge 1$ [6]

Example 6. *Let* $f_6 : \mathbb{R} \longrightarrow \mathbb{R}$ *be defined as* $f_6(x) = \frac{x^2-7}{2}$.

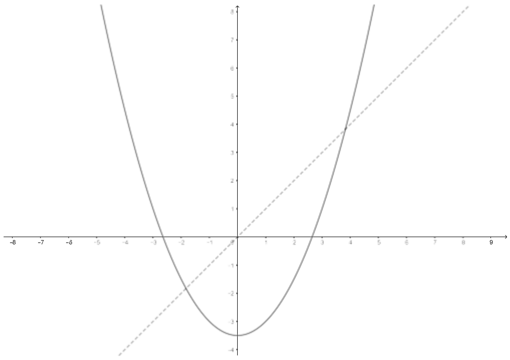

FIGURE 5. Graph of f_6

$Per_3(f_6) \neq \emptyset$ and therefore $P(f_6) = \mathbb{Z}_+$ [10].

Also $Per(f_6) \cap \mathbb{Q} = \emptyset$ and $\overline{Per(f_6)} = \mathbb{R}$.

Example 7. *Let* $f_7 : \mathbb{C} \longrightarrow \mathbb{C}$ *be defined as* $f_7(z) = z^2 - z$. $P(f_7) = \mathbb{Z}_+ \setminus \{2\}$ [2]. *Here* $\overline{Per(f_7)} \neq \mathbb{C}$.

From the examples given above, it is observed that there is no implication between $P(f) = \mathbb{Z}_+$ and $\overline{Per(f)} = X$.

Definition 1. *Let f be a continuous map defined on a topological space X. We say that f is Periodically Rich if f has both a dense set of periodic points and the set of periods is equal to $\mathbb{Z}+$.*

i.e., f is Periodically Rich if (1) $\overline{Per(f)} = X$ and (2) $P(f) = \mathbb{Z}_+$.
Let $PR(X)$ be the set of continuous maps on X which are Periodically Rich. If $PR(X) \neq \emptyset$, we say that X admits Periodically Rich maps. For example Interval I admits Periodi- cally Rich maps since tent map satisfies both the conditions of definition.

2. PERIODICALLY RICH MAPS ON INTERVAL

The following proposition is a corollary to Sharkovskii's result.

Proposition 1. *For continuous maps $f : I \to I$ if $Per_3(f) \neq \emptyset$ then $P(f) = \mathbb{Z}_+$.*

Proposition 2. *Let $f : I \longrightarrow I$ be a continuous map. If f is transitive and $Per_3(f) \neq \emptyset$ then $f \in PR(I)$.*

Proof. If f is transitive then $\overline{Per(f)} = I$ [12] and $Per_3(f) \neq \emptyset$ implies $P(f) = \mathbb{Z}_+$ by proposition

1. Hence $f \in PR(I)$.

Proposition 3. *Let $f : I \longrightarrow I$ be a transitive map. If there exists $x \in Per_4(f)$ with increasing orbit then $f \in PR(I)$.*

Since f is transitive, $\overline{Per(f) = I}$ [12]. We use the following lemmas from the literature [1] to show that $Per_3(f) \neq \varphi$.

Lemma 1. *Let $f : I \to I$ be a continuous map and let $J, K \subset I$ be closed intervals with $f(J)$ m K, then there exists a closed interval $H \subset J$ with $f(H) = K$.*

Lemma 2. *Let $f : I \to I$ be a continuous map. Suppose H and K are closed intervals with $H \subset K \subset I$ and $f(H) = K$, then f has a fixed point in H.*

Proof of Proposition 3: Let $x \in Per_4(f)$ with increasing orbit, *i.e.*,

$$x < f(x) < f^2(x) < f^3(x).$$

$$f([x, f(x)]) \ \mathrm{m} \ [f(x), f^2(x)]$$

$$f([f(x), f^2(x)]) \ \mathrm{m} \ [f^2(x), f^3(x)]$$

$$f([f^2(x), f^3(x)]) \ \mathrm{m} \ [x, f(x)]$$

So by lemma 2, there exists

$$F_1 \subset [x, f(x)] \text{ such that } f(F_1) = [f(x), f^2(x)]$$

$$F_2 \subset [f(x), f^2(x)] \text{ such that } f(F_2) = [f^2(x), f^3(x)]$$

$$F_3 \subset [f^2(x), f^3(x)] \text{ such that } f(F_3) = [x, f(x)].$$

Now, $f^2(F_3) = f(F_1)$; $f^2(F_1) = f(F_2)$; $f^2(F_2) = [x, f(x)]$.Therefore $[x, f(x)] = f^3(F_1)$. So by lemma 2, f^3 has a fixed point in F_1, say y. Now $y \notin Per_2(f)$, since $f^2(y) \in [f(x), f^2(x)]$. Therefore $y \in Per_3(f)$.

So $Per_3(f) \neq \emptyset$. Now, by Proposition 2, $f \in PR(I)$.

Proposition 4. *Let f be a continuous map such that*

(i) $f(x) = x$ *for some* $x \in (0, 1)$

(ii) $f(1) = 1$

(iii) *For every* $J \subset I$, $f(J) \not\subseteq J$

(iv) f *is transitive. Then* $f \in PR(I)$.

Before proving proposition, we prove a lemma.

Lemma 3. *Let* $f : I \to I$ *be a map and there exists* $a < b < c$ *in* $(0, 1)$ *such that*

(i) $f(a) = a$

(ii) $f(c) \leq a$

(iii) $f(b) \geq c$

then $P(f) = \mathbb{Z}_+$.

Proof. Let $a < b < c$ and $f(a) = a$. Let $f(b) = b'$, $f(c) = c'$. For given conditions, $c' \leq a$ and $c \leq b'$. So $c \in [c', b']$. By Intermediate value theorem, there exists $x \in [b, c]$ such that $f(x) = c$. Now since $x \in (a, b')$, $f(a) = a$,and $f(b) = b'$ by Intermediate value theorem, there exists $y \in (a, b)$ with $f(y) = x$. Therefore $c' = f(c) < y < x = f(y) < c = f(x)$. By standard result [8], $P(f) = \mathbb{Z}_+$

Proof of Proposition 4: By condition (i), there exists $a \in (0, 1)$ such that $f(a) = a$ and

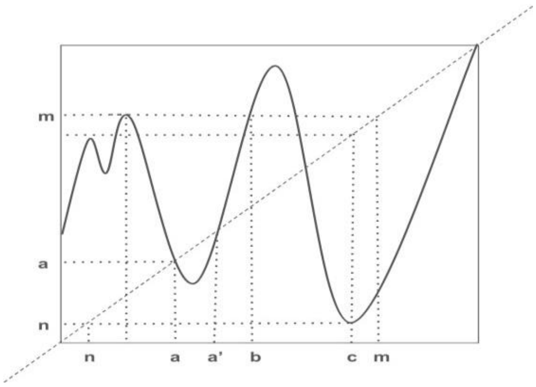

FIGURE 6. Case 1 of Proposition 4

by condition (ii), $f(1) = 1$. Let us take one of the two possibilities $f(0) \neq 1$ *or* $f(0) = 1$.

Case 1: If $f(0) \neq 1$.

Let $m = \sup\{f(x)|x \in (0, a]\}$. Clearly $m > a$. There are two possibilities either $m \neq 1$ *or* $m = 1$.

Consider $m \neq 1$. Let $b = \inf\{x \in (a, m]|f(x) > m\}$, then for all $x \in [a, b], f(x) \leq m$, and $f(b) > m$. Let $a' = \sup\{x \in [a, b]|f(x) = x\}$, $a' \in [a, b)$. Now $f(x) > x$ for all $x \in (a', b]$ and $f([a', b]) = [a', m]$. Let $n = min\{f(x)|x \in [b, 1]\}$ and $c = \inf\{x \in [b, 1]|f(x) = n\}$.

Now $a' > f(c) = n$.

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Claim: There exists $d \in (a^r, c)$ such that $f(d) > c$.

If $c < m$, take $d = b$. If $c \geq m$, there exists $y \in (b, c)$ such that $f(y) > c$, take $d = y$. Claim is proved.

So we have, $a^r < d < c$ with $f(a^r) = a^r$, $f(c) \leq f(a^r)$ and $c < f(d)$. By lemma 3, $P(f) = \mathbb{Z}_+$.

Now consider the other case $m = 1$.

Let $x \in [0, a]$ such that $f(x) = m = 1$. Let $n = min\{f(x)|x \in [a, 1]\}$ and let

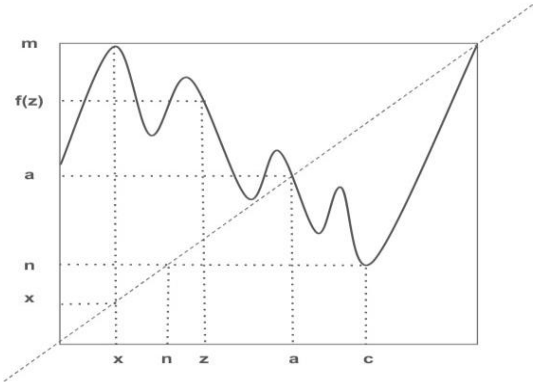

FIGURE 7. when $m = 1, n \geq x$

$f(c) = n$, $c \in [a, 1]$. Either $n \neq 0$ or $n = 0$. If $n \neq 0$ and $n \geq x$,then there exists $z \in (x, a)$ such that $f(z) \geq x$. So $x < z < a$ with $f(x) \geq f(a)$ and $x \leq f(z)$. By lemma 3, $P(f) = \mathbb{Z}_+$.

If $n \neq 0$ and $n < x$, then there exists $z \in (n, a)$ such that $f(z) > n$. Here $z \in (n, x) \cup (x, a)$. If $z \in (n, x)$ then $z < x < c$ with $f(z) > z$ and $f(x) > x$. De- fine $g(y) = f(y) — y$. By Intermediate value theorem, there exists $s \in (z, x)$ such that $g(s) = 0$ and $g(z) < 0$, $g(x) > 0$. Therefore $s \in (z, x)$ is fixed for f . So $s < x < c$ with fixed s, $f(c) \leq f(s)$ and $c \leq f(x)$. By lemma 3, $P(f) = \mathbb{Z}_+$.

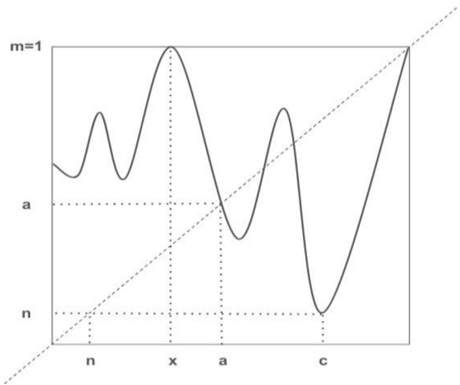

FIGURE 8. when $m = 1, n < x$

Case 2: If $f(0) = 1$

Let $c \in (0, 1)$ such that $f(c) = 0$. If such c doesn't exist, $p = \inf\{f(x) | x \in I\} > 0$. Hence $[p, 0]$ is invariant under f.

So there exists $z \in (c, 1)$ such that $f(z) = c$. Therefore $P(f) = \mathbb{Z}_+$.

By condition (iv), $\overline{Per(f)} = I$. Therefore $f \in PR(I)$. Hence the proof.

We summarize the above propositions 3 and 4 as given below

Proposition 5. *Let f be transitive map on I. Then $f \in PR(I)$ if f belongs to anyone of the following two classes.*

Class 1.

There exists $x \in Per_4(f)$ with increasing periodic orbit.

Class 2.

(i) $f(x) = x$ *for some* $x \in (0, 1)$

(ii) $f(1) = 1$

(iii) *For every* $J \subset I$, $f(J) \not\subseteq J$.

Now, we shall give an example to show that transitivity of f is not a necessary condition for f to be in $PR(X)$.

Example 8. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined as*

$$f(x) = \begin{cases} 3x, & 0 \le x \le 1/3 \\ -3x + 2, & 1/3 \le x < 2/3 \\ 3x - 2, & 2/3 < x \le 1 \\ f(x - 1) + 1, & x \ge 1 \\ f(x + 1) - 1, & x \le 0 \end{cases}$$

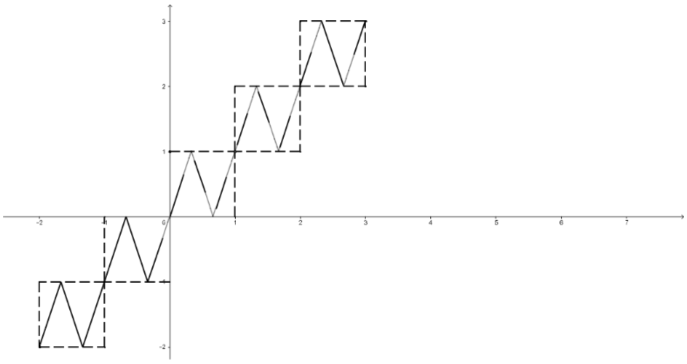

FIGURE 9. Graph of f

f is not transitive since $f(n, n+1) = ([n, n+1])$, for all $n \in \mathbb{R}$

$$O_f\left(\frac{1}{7}\right) = \left\{\frac{1}{7}, \frac{3}{7}, \frac{5}{7}\right\}. \quad \therefore \frac{1}{7} \in Per_3(f).$$

Hence $P(f) = \mathbb{Z}_+$.

$f|_{[n,n+1]}$ has one fixed point in $(n, n+1)$

$f^i|_{[n,n+1]}$ has $3^i - 2$ fixed points in $(n, n+1)$.

If x and y are fixed points of f^i in $(n, n+1)$, then

$$|f^i(x) - f^i(y)| < (1/3)^{i-1}.$$

Therefore $\overline{Per(f)} = \mathbb{R}$.

Hence $f \in PR(\mathbb{R})$. Here f is not transitive.

Note: Let $C(I)$ be the space of all continuous function defined on I and let

(i) $T(I) = \{f \in C(I) | f$ is transitive$\}$

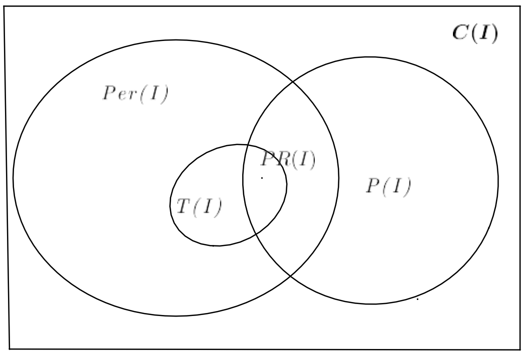

FIGURE 10. The relationship between $Per(I)$, $P(I)$, $T(I)$, and $PR(I)$.

(ii) $Per(I) = \{f \in C(I) | \overline{Per(f)} = I\}$

(iii) $P(I) = \{f \in C(I) | P(f) = \mathbb{Z}_+\}$.

Then we have, $T(I) \subset Per(I)$ and $PR(I) = Per(I) \cap P(I)$. Also $T(I) \cap P(I) \neq \emptyset$ (see figure 10).

It is proved that $T(I) \cong T(I) \cong \overline{I_2}$ [7].
Now, we shall prove the following proposition.

Proposition 6. $\overline{P(I)} = C(I)$.
Proof. Let c be a fixed point of f, i.e., $f(c) = c$. Given an $\epsilon > 0$, we will construct a $g \in P(I)$ such that $P(g) = \mathbb{Z}_+$.

Case 1: $c \neq 1$.

Since f is continuous at c, there exists a $\delta > 0$ (choose $\delta < \frac{\epsilon}{2}$) such that

$$|f(x) - c| < \frac{\epsilon}{2}, \text{ for all } x \in [c, c + \delta].$$

Let $\vartheta = \frac{\delta}{4}$, A be $[c, c + 2\vartheta]$ and B be $[c + 2\vartheta, c + 3\vartheta]$.

Define $g_1 : A \to A$ as

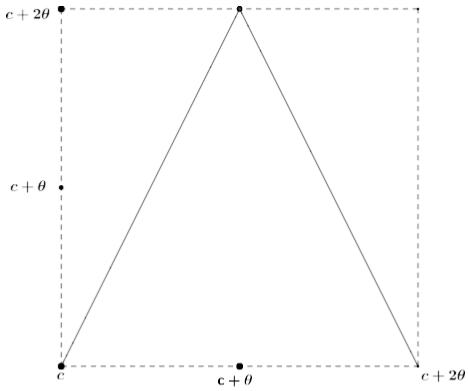

FIGURE 11. Graph of g_1

$$g_1(x) = \begin{cases} 2x - c, & x \in [c, c + \vartheta] \\ -2x + 3c + 4\vartheta. & x \in [c + \vartheta, c + 2\vartheta]. \end{cases}$$

(see figure 11)

g_1 on $[c, c + 2\vartheta]$ is topologically conjugate to the tent map on $[0, 1]$ defined by

$$T(x) = \begin{cases} 2x, & \leq x \leq 1/2 \\ 2(1 - x), & 1/2 \leq x \leq 1 \end{cases}$$

via the homeomorphism $h : [c, c + 2\vartheta] \to [0, 1]$ defined as $h(x) = \frac{x-c}{2\theta}$.

Therefore $\overline{Per(g_1)} = [c, c + 2\vartheta]$.

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Let $g_2 : B \to [0, 1]$ be defined as

$$g_2(x) = c + (x - c - 2\vartheta)\,\frac{f(c + 3\vartheta) - c}{\vartheta}.$$

i.e., the graph of g_2 is a line segment joining $(c + 2\vartheta, c)$ and $(c + 3\vartheta, f(c + 3\vartheta))$ (see figure 12).

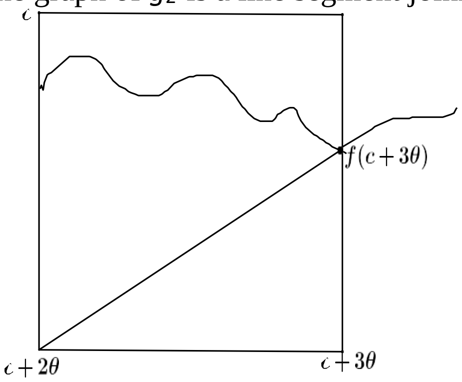

FIGURE 12. Graph of g_2

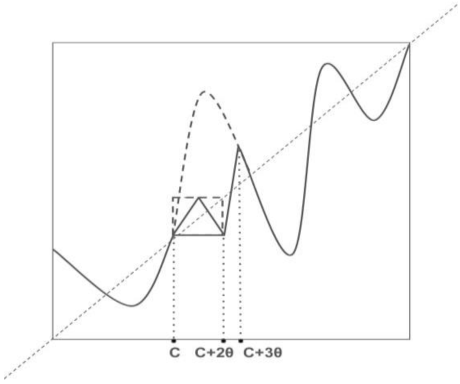

FIGURE 13. Graph of g at $c \neq 1$

Now, define $g : [0, 1] \to [0, 1]$ as

$$g(x) = \begin{cases} f(x), & x \in [0, 1] \setminus (A \cup B) \\ g_1(x), & x \in A \\ g_2(x), & x \in B \end{cases}$$

(see figure 13).

$$|f(x) - g(x)| = \begin{cases} 0, & x \in [0, 1] \setminus (A \cup B) \\ |f(x) - g_1(x)|, & x \in A \\ |f(x) - g_2(x)|, & x \in B \end{cases}$$

Hence

$$\|f - g\| = \max\{\max_{x\in A} |f(x) - g_1(x)|, \max_{x\in B} |f(x) - g_2(x)|\}$$

$$\leq \max\left\{ \max_{x\in A} |f(x) - c| + \max_{x\in A} |c - g_1(x)|,\ \max_{x\in B} |f(x) - c| + \max_{x\in B} |c - g_2(x)| \right\}$$

$$< \max\{\frac{\epsilon}{2} + \frac{\epsilon}{4}, \frac{\epsilon}{2} + \frac{\epsilon}{2}\} = \max\{\frac{3\epsilon}{4}, \epsilon\} = \epsilon.$$

Let $x_0 = c + \frac{4}{7}\vartheta$.

$$g(x_0) = g(c + \frac{4}{7}\vartheta) = c + \frac{8}{7}\vartheta$$
$$g(c + \frac{8}{7}\vartheta) = c + \frac{12}{7}\vartheta \text{ and } g(c + \frac{12}{7}\vartheta) = x_0$$

Therefore $x_0 \in Per_3(g)$. Hence $P(g) = \mathbb{Z}_+$.

Case 2: If $c = 1$.

Let A be $[1 - 2\vartheta, 1]$ and B be $[1 - 3\vartheta, 1 - 2\vartheta]$.

Let $g_1 : A \to A$ be defined as

$$g(x) = \begin{cases} 2x-1, & x \in [1 - \vartheta, 1] \\ 3 - 2x - 4\vartheta, & x \in [1 - 2\vartheta, 1 - \vartheta] \end{cases}$$

Let g2 : B → [0, 1] be defined as

$$g2(x) = f(1 - 3\theta) + \frac{(x - 1 + 3\theta)(1 - f(1 - 3\theta))}{\theta}$$

Let $g : [0, 1] \to [0, 1]$ be such that

$$g(x) = \begin{cases} f(x), & x \in [0, 1] \setminus (A \cup B) \\ g_2(x), & x \in B \\ g_1(x), & x \in A \end{cases}$$

(see figure 14)

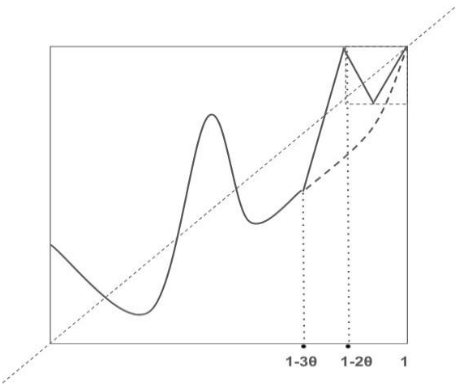

FIGURE 14. Graph of g at $c = 1$

Then as in the earlier case, given $\epsilon > 0$,

$$\|f - g\| < \epsilon.$$

$(1 - 4/7\vartheta) \in Per_3(g)$ since

$$O_g(1-\frac{4}{7}\theta) = \{1-\frac{4}{7}\theta, 1-\frac{8}{7}\theta, 1-\frac{12}{7}\theta\}.$$

Hence $P(g) = \mathbb{Z}_+$.

Hence the proposition.

Corollary 1. *Let* $f \in C(I)$. *Then*

(i) *Given* $\epsilon > 0$, *there exists* $g \in P(I)$ *such that* $|f - g| < \epsilon$ *and* $g/_J \in PR(J)$ *for some interval* $J \subset I$.

(ii) $|\{g : I \to I|$ *given* $\epsilon > 0$, $|f - g| < \epsilon$ *and* $g/_J \in PR(J)\}| = |Fix(f)|$.

Proof. (i)Take

$$J = \begin{cases} [c, c+2\vartheta] & \text{if } c \neq 1 \\ [1-2\vartheta, 1] & \text{if } c = 1. \end{cases}$$

$g/_J \cong T/_{[0,1]}$. Therefore $g \in PR(J)$.

(ii) The cardinality of set of all g's in Corollary (i) is the number of fixed points of f. Because at each fixed points of f we can construct g as above.

Note: If h is a homeomorphism on I, then

$h \notin PR(I)$ as $Per(h) = Per_1(h) \cup Per_2(h)$. Here $P(h) = \{1, 2\}$.

3. Conclusion

Studies of Periodically Rich maps on various topological spaces would contribute to literature of the chaotic dynamical systems.The set of periodic points is a critical con- cept in studying chaotic maps as it captures the recurrence and regularity in the otherwise seemingly erratic dynamics.The set of periods reflects the variety and richness of periodic behavior in chaotic systems, often being dense in natural numbers, indicating that the sys- tem exhibits periodic behavior at every scale.

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