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Class of Modules for Which Strongly Hopfian Modules are Noetherian

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Abstract:
Let R be an arbitrary ring and M a left R-module. In this paper we introduce the modules M such that every strongly Hopfian module in σ[M] is Noetherian. These modules will be called SF-modules. We characterize such modules and study their properties. Relationships between SF-modules and other classes of modules are given.

Keywords: Strongly Hopfian module, SF-module, Perfect module, Locally Noetherian module; Hollow module; Semiartinian module; Π-semiartinian module.

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1. Introduction

The study of modules by properties of their endomorphisms has long been of interest. Throughout in this paper, rings are considered associative, non necessarily commutative with identity $1 \neq 0$, all modules are unitary left R-modules and R-Mod denotes the category of left unitary R-modules. We denote by $\sigma[M]$ the full subcategory of R-modules whose objects are all R-Mod subgenerated by M.

A R-module M is Noetherian (resp. artinian) if any ascending (resp. descending) chain of submodules of M is stationary. A R-module M is called Hopfian, if any surjective R-homomorphism $f: M \to M$ is an isomorphism. An object N of σ[M] is said to be strongly Hopfian, if for every R-endomorphism of N, the chain $Kerf \subseteq Kerf^2 \subseteq \cdots \subseteq Kerf^n \subseteq \cdots$ is stabilizes. A ring R is said SF-ring, if every strongly Hopfian R-module is Noetherian. Let R be a commutative ring, a R-module M is said FGS-module if every Hopfian object of $\sigma[M]$ is finitely generated. A R-module M is called endo-noetherian if for any family $(f_i)_{i\geq 1}$ of endomorphisms of M, the sequence $Ker(f_1) \subseteq Ker(f_2) \subseteq \cdots \subseteq Kerf^n \subseteq \cdots$ stabilizes. A R-module M is said $EKFN$-module if every endo-noetherian object of $\sigma[M]$ is Noetherian. A ring R is said S-ring, if every Hopfian R-module is Noetherian and an R-module M is said S-module if every Hopfian object of $\sigma[M]$ is Noetherian. An module M is hollow, if $M \neq 0$ and submodule of M is a small submodule of M .

All Noetherian module is strongly Hopfian but converse is not always true. For example, the $\mathbb{Z}$-module $M = \oplus_{p\in P}\mathbb{Z}_p$ is strongly Hopfian but it is not Noetherian, where P is the set of all primes. A module is named SF-module if every strongly object of σ[M] is noetherian
In this paper, first we present preliminary results and some fundamental properties of SF-modules. Secondly we characterize the class of finitely generated and hollow SF-modules. Additionally, we

prove that in the setting of finitely generated SF-modules, noetherian module, artinian module and semiartinian module are equivalent.

2. Some properties of SF-modules

Lemma 2.1. For a ring R we have :

1. Every Noetherian R-module is endo-noetherian.
2. Every endo-noetherian R-module is strongly Hopfian.
3. Every strongly Hopfian R-module is Hopfian.

Proposition 2.2. If M be a SF-module. Then we have the following properties:

1. Every submodule of a strongly Hopfian module in σ[M] is strongly Hopfian.
2. Every quotient of a strongly Hopfian module in σ[M] is strongly Hopfian.

Proof. 1) Let N be a submodule of strongly Hopfian module K in σ[M]. As M is an SF-module, then K is Noetherian. Since submodule of Noetherian module is Noetherian so N is Noetherian. Therefore N is strongly Hopfian beacause every Noetherian module is strongly Hopfian.
2) Result from the fact that any quotient of a Noetherian module is Noetherian. □

Proposition 2.3. Let R be a ring. The following assertions are equivalent:

1. R is SF-ring.
2. Every R-module is a SF-module.

Proof. 1) ⟹ 2). Let M a R-module and N a strongly Hopfian objet of σ[M]. Since σ[M] is the full subcategory of R-Mod then N is a strongly Hopfian R-module. As R is a SF-ring then N is Noetherian.
2) ⟹ 1) Suppose that every R-module is SF-module. Let K be a strongly Hofian R-module. Since K ∈ σ[K] then K is Noetherian. Hence R is a SF-ring. □

Remark 2.4.

1. Every S-module is SF-module.
2. Every SF-module is a EKFN-module.

Proposition 2.5. Let R be a commutative ring and M a finitely generated R-module. If M is a SF-module, then every object of σ[M] has a projective cover.

Proof. If M is SF-module, then by remark 2.4. M is EKFN-module and by Proposition 3.4. of [5], every object of σ[M] has a projective cover. □

Proposition 2.6. For a R-module M, the following properties are equivalent:

1. M is an SF-module.
2. Every module in σ[M] is an SF-module.

Proof. 1)⟹ 2): Let N ∈ σ[M] then σ[N] is the smallest category of σ[M] containing N and it is a full subcategory of σ[M]. If K is a strongly Hopfian object of σ[N], then K ∈ σ[M] and since M is an SF -module then K is Noetherian.
2)⟹ 1): it's obvious because M ∈ σ[M]. ☐

Proposition 2.7. Over R-artinian ring, all R-module is SF-module.

Proof. Let M be a R-module and let N be a strongly Hopfian module in σ[M]. Since R is artinian ring then according to 31.5 of [11], σ[M] = R/Ann(M)-Mod. Hence every module in σ[M] is an R/Ann(M)-module therefore N is a ideal of R/Ann(M). As R is artinian then R/Ann(M) is artinian and so N is finitely generated. Since over artinian ring, finitely generated and Noetherian are equivalent then N is Noetherian. ☐

Proposition 2.8. Let M be a R-module. If every module in σ[M] is injective, then M is an SF-module.

Proof. Suppose that every module in σ[M] is injective. Let K be a strongly hopfian object of σ[M] then K is hopfian. Since by hypothesis K is injective then according to Theorem 3.5. of [9] , K is Noetherian and therefore M is an SF-module. ☐

3. Characterization of SF-modules

Definition 3.1. Let M be an R-module.
A module N in σ[M] is semiperfect in σ[M] if every factor module of N has a projective cover in σ[M].

Definition 3.2. N is perfect in σ[M] if, for every index set Λ, the sum $N^{(\Lambda)}$ is semiperfect in σ[M].

Lemma 3.3. (see 43.11 in [11]). Let R be a commutative ring, M a finitely generated, self-projective R-module. Then the following statements are equivalent:

1. M perfect in σ[M]
2. $\bar{R} =$ R/An(M) is a perfect ring.

Theorem 3.4. Let R be a commutative ring, M a finitely generated, self-projective R-module. Then the following statements are equivalent:

1. M is a SF-module;
2. M perfect in σ[M];
3. All M-generated flat module in σ[M] is projective in σ[M].

Proof. According to 43.8 in [11], we have the equivalence of assertions (2) and (3). Now let's prove that 1) is equivalent to 2).

1)⟹ 2): M finitely generated SF-module implies σ[M] = R/An(M)-Mod and M ≅ R/An(M) is an artinian principal ideal ring. Since every artinian ring is perfect ring then M is perfect and so R/An(M) is a perfect ring. Referring to 43.11 in [11], M is perfect in σ[M].
2)⟹ 1) If M perfect in σ[M] then by 43.11 of [11] , R/An(M) is a perfect ring. Since every perfect

ring is semiperfect and every semiperfect ring is semilocal, then R/Ann(M) is a semilocal ring. By theorem 3.2 in [4], we deduce that R/Ann(M) is an SF-ring. As M is finitely generated σ[M] = R/Ann(M)-Mod and so every module in σ[M] is a R/Ann(M)-module. If N is a strongly Hopfian module in σ[M] then N is Noetherian because R/Ann(M) is an SF-ring and N is a module of R/Ann(M). Therefore M is an SF-module. ☐

NB: We denote by Max(M), the set of maximal submodules of a module M.

Corollary 3.5. Let R be a commutative ring and M a self-projective hollow module and Max(M) ≠ ∅. If M is a SF-module, then $S = End_R(M)$ satisfies the descending chain conditions for cyclic ideals.

Proof. Assume M a projective hollow module and Max(M) ≠ ∅ then according to Theorem 2.2 of [2], M is a finitely generated local module. Then M is finitely generated self-projective. Hence if M is a SF-module then by Theorem 3.4. M perfect in σ[M] and referring to 43.4 of [11], $S = End_R(M)$ satisfies the descending chain conditions for cyclic ideals. ☐

Definition 3.6. A module M is called semiartinian if every nonzero homomorphic image of M has nonzero socle.

Definition 3.7. A module M is called Π-semiartinian if the direct product M^I is a semiartinian module for every non empty set I.

Definition 3.8. The ring R is called strongly π-regular if for each a ∈ R, there is an integer n ≥ 1 and b ∈ R such that $a^n = a^{n+1}b$. M is called Fitting module if every endomorphism of M satifies Fitting's lemma (i.e., there exists an integer n ≥ 1 such that $M = Kerf^n \oplus Imf^n$).

Theorem 3.9. Let R be a ring and M a finitely generated R-module. If M is SF-module then the following statements are equivalent:

1. M is artinian;
2. M is semiartinian module;
3. Every module in σ[M] is semiartinian ;
4. M is Π-semiartinian module;
5. M is Noetherian.

Proof. 1)⟹ 2): It's obvious.

2)⟺ 3): If M is semiartinian module, then by Corollary 2.13. in [8], R/An(M) is a semiartinian ring. Let N an object of σ[M] then since M is finitely generated σ[M] = R/Ann(M)-Mod and therefore N is a module R/Ann(M)-module. It is well know a ring R is semiartinian if and only if every R-module is semiartinian . Since R/Ann(M) is a semiartinian ring then every R/Ann(M)-module is semiartinian and hence N is semiartinian. The converse is trivial.

2)⟺ 4) Result from Corollary 3.3. of [8]

2)⟹ 5) If M is semiartinian then according to Corollary 2.13. in [8], $\mathrm{End}_R(M)$ is a strongly π-regular ring. Therefore by proposition 2.7 of [6], M is Fitting module and so an strongly Hopfian module. Since by hypothesis M is a SF-module so M is Noetherian.

5)⟹ 1) M finitely generated and SF-module implies $M \cong R/An(M)$ is artinian principal ideal and over artinian principal ideal ring Noetherian module and artinian module coincide. ▢

Lemma 3.10. (see proposition 14 of hollow and semihollow modules). Let N be a proper submodule of a module M. If M is a hollow module and M/N is finitely generated, then M is finitely generated.

Theorem 3.11. Let R be a commutative ring and M a hollow module. We suppose that for every proper submodule N of M, M/N is finitely generated. Then the following conditions are equivalent:

1. M is a SF-module;
2. M is a locally Noetherian module;
3. M is Noetherian module;

Proof. 1)⟹ 2): Let M be a SF-module then by remark 2.4. M is a EKFN-module. By hypothesis, it results from lemma 3.10. that M is finitely generated. Hence by Theorem 3. of [5], M is a locally Noetherian module.
2) ⇔ 3) Result from Corollary 2.3. in [7]
Now we prove that 2) ⟹ 1): Let $N \in \sigma[M]$ a strongly Hopfian module. Since M is locally Noetherian then according to Corollary 2.3. in [7], $R/Ann(M)$ is a Noetherian ring. Since M is finitely generated $\sigma[M] = R/Ann(M)$-Mod and $M \cong R/Ann(M)$ is finitely generated and Noetherian. So $N \in \sigma[M]$ implies that N is an ideal of $R/Ann(M)$ and therefore a submodule of M. It's well know over Noetherian ring, every submodule of finitely generated module is finitely generated. Hence N is Noetherian because over Noetherian ring, finitely generated and Noetherian module coincide. ▢

Corollary 3.12. Let R be a commutative ring and M a hollow module. We suppose that for every proper submodule N of M, M/N is finitely generated. Then the following conditions are equivalent:

1. M is a SF-module,
2. Every finitely generated module in $\sigma[M]$ is Noetherian.
3. Every finitely generated module is finitely presented in $\sigma[M]$.
4. Every direct sum of M-injective module in $\sigma[M]$ is M-injective.

Proof. By hypothesis, it results from Lemma 3.10. that, M is finitely generated and according to the theorem 3.11. M is a SF-module if and only if, M is a locally Noetherian module; and referring to 27.3 of [11], we have the result. ▢

Theorem 4. Let M be a local R-module, then the following are equivalent:

1. M is a S-module;
2. M is a SF-module;
3. M is of finite length and every submodule of M is cyclic;
4. M is of finite representation type;

5. M is FGS-module;

Proof. 1) $\Longrightarrow$ 2). Result from remark 2.4.
2) $\Leftrightarrow$ 3) $\Leftrightarrow$ 4) Since M is local SF-module then M is a finite generated SF-module. By Lemma 2.1. and Lemma 3.3. M is isomorphic to $R/Ann(M)$ who is a principal ideal ring. This double equivalence result from Theorem 9 in [10].
4) $\Longrightarrow$ 5) Result from Theorem 1 in [3].
5) $\Longrightarrow$ 1) Let N be a Hopfian module in σ[M]. Since M is a FGS-module then N is finite generated. From Proposition 3 in [3] N is Noetherian. Therefore M is a S-module. □

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Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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