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Principal Intersection Graph of Commutative Rings

Lamine Ngom[1], Mankagna Albert Diompy [2]

[1,2]Department of Mathematics and Computer Science, University of Cheikh Anta Diop, Dakar, Senegal

Email Id: laminengo89@hotmail.fr[1]

Corresponding author: albertdiompy@yahoo.fr

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Abstract:

Let R be a commutative ring. The principal intersection graph of a commutative ring R, noted $G_c(R)$, consist of all proper ideals of R as vertices. Two distinct vertices I and J are adjacent if $I \cap J \neq 0$ and either I or J is a principal (cyclic) ideal. In this paper, we investigate some properties from graph theory of $G_c(R)$ and its algebraic properties where R is a ring.

Keywords: Principal intersection graph, Principal ideal domain, Ore domain, Bezout domain, Connected graph, Complete graph, Hamiltonian graph.

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1. Introduction

The intersection graph of ideals of a ring R is the graph having the set of all ideals as its set of vertices. Two distinct vertices I and J are adjacent if and only if their intersection is non-zero idea and either I or J is a principal (cyclic) ideal. Intersection graph were introduced by Bosak in 1964 [6]. Since, particular intersection graph like small intersection graph, prime intersection graph, semisimple intersection graph are studied respectively in [3, 1, 11, 5, 7]. Recently, several properties of these kinds of graphs were investigated by many authors as Ansari-Toroghy, Nikmehr - Soleymanzadeh and Alwan in 2016; 2017 and 2023 respectively.

In this paper, R is a commutative ring with identity (or eventually a domain). Here, we introduce a particular intersection graph $G_c(R)$ named Principal Intersection Graph, whose set of vertices is the proper ideals of R. We will study the algebraic properties of $G_c(R)$ and also its properties when seen as a graph.

This paper is organized as follow: in the first section, we recall some properties of rings and graph theory. In the second section, we study connectedness, completeness, k-partite and Hamiltonian properties of this intersection graph. We gave a characterization of the connectedness, completeness and Hamiltonian properties of $G_c(R)$ as a principal ideal domain, an Ore domain and a Bezout domain.

2. Definitions and preliminary results

Definition 2.1.: Definitions from ring theory

- An ideal I of commutative ring R is principal written as $I = aR$ for some $a \in R$, if it is generated by one element.
- A ring R is principal if every proper ideal is a principal ideal.

- A ring R is an Ore ring if it satisfies the Ore Condition. That is: For all elements a and b in R, $aR \cap bR \neq \{0\}$.
- A ring R is a domain if it has no zero-divisor.
- A ring R is an Ore ring if for all elements a and b in R, $aR \cap bR \neq \{0\}$.
- A principal ideal domain is a domain such that every ideal of R is a principal ideal.
- Bezout ring is a domain in which for any to elements $a, b \in R$, there is $n \geq 0$ such that $Ra^n + Rb^n$ is a principal ideal.

Definition 2.2.: Definitions from graph theory

- Let I and J two distinct vertices, $I - J$ means that I and J are adjacent.
- The degree of a vertex I of graph $G_c(R))$ which denoted by $\deg(I)$ is the number of edges incident on I.
- If $|V(G_c(R)))| > 2$, a path from I to J is a sequence of adjacent vertices $I - I_1 - I_2 - \cdots - I_n - J$, where $I_i \in V(G_c(R)$.
- The length a path graph of a graph is the number of edges in this path.
- A path using k distinct vertices has length $k - 1$.
- The distance between two distinct vertices I and J is denoted by $d(I; J)$ is the length of the shortest path connecting I and J.
- If there is not a path between I and J, $d(I; J) = 0$.
- The number of vertices of $G_c(R)$ is the order of the graph.
- The diameter of a graph $G_c(R))$ is $\mathrm{diam}(G_c(R))) = \sup\{d(I; J)/I; J \in V(G_c(R)))\}$.
- A graph $G_c(R))$ is connected, if for any vertices I and J of $G_c(R))$ there is a path between I and J. If not, $G_c(R))$ is disconnected.
- A closed path $I - I_1 - I_2 - \cdots - I_n - I$ is a cycle.
- The girth of $G_c(R))$ is the length of the shortest cycle in $G_c(R))$.
- A Hamiltonian cycle is a cycle that contains every vertex of the graph.
- A Hamiltonian graph is graph containing a Hamiltonian cycle.
- A graph with no loop or multiple edges is a simple graph.

Proposition 2.3. Let R be a ring. $G_c(R)$ is an empty graph if and only if R is a field.

Proof:

- $\Rightarrow$) It is obvious that if $G_c(R)$ is empty graph then $G_c(R)$ has no vertices. Since vertices of $G_c(R)$ are the proper ideals of R, then R has proper ideal, that is R is a field.
- $\Leftarrow$) Conversely, if R is a field, R has no proper ideal. ▫

Lemma 2.4. If the graph $G_c(R)$ is a null graph, then for all $(a; b) \in R \setminus \{1\} \times R \setminus \{1\}$, $ab = 0$.

Proof: Assume that $G_c(R)$ is a null graph. Let $a \neq 1$ and $b \neq 1$ be two elements of R. Since $G_c(R)$ is a null graph, then $aR \cap bR = \{0\}$. Therefore, we have that, $ab \in aR \cap bR$.

Thus, $ab = 0$

Proposition 2.5. Let R be a commutative nonzero ring. The graph $G_c(R)$ is a null graph if and only if $R = \{0; 1\}$.

Proof:

- $\Leftarrow$) It is clear that if $R = \{0; 1\}$, then $G_c(R)$ is a null graph.
- $\Rightarrow$) Let $1 \neq x \in R$ and $G_c(R)$ a null graph. Take $y \in R$ such that $y \neq 1$ and $1 - y \neq 1$, then $xy = 0$. By lemma 2.4, $x = x - 0 = x - xy = x(1 - y) = 0$. Since R is commutative nonzero ring, then $R = \{0; 1\}$.

Example 2.6. If p is prime integer, the graph $G_c\left(\mathbb{Z}_p\right)$ is null graph.

Lemma 2.7. If R is a domain, then $G_c(R)$ is a connected.

Proof: Let I and J to vertices of $G_c(R)$. Since I and J are proper ideals of R, there exists $a \neq 0$ and $b \neq 0$ in I and J, respectively, such that $ab \neq 0$. So, we have $ab \in I \cap J$ implies that $I \cap J \neq 0$. If one of the ideals I or J is principal, then I and J are adjacent. Moreover, I and J are not principal ideals. Since $I \cap J \neq 0$, let $0 \neq c \in I \cap J$ and put $K = cR$. Then $I \cap K \neq 0$ and $K \cap J \neq 0$. Thus $I - K - J$ is a path between I and J.

3. Connectedness, Completeness, Hamiltonian graph

Lemma 3.1. If R is a domain, every connected graph $G_c(R)$ is complete.

Theorem 3.2. Let R be a domain. The followings statements are equivalents:

1. $G_c(R)$ is a connected graph;
2. $G_c(R)$ is a complete graph;
3. R is an Ore domain.

Proof:

1. $\Rightarrow$(2) Follows from Lemma 3.1
2. $\Rightarrow$(3) Let a and b to non-zero elements in R. Put on $I = aR$ and $J = bR$. Since $G_p(R$ is connected, I and J principal ideals, then $I \cap J \neq 0$. Hence $aR \cap bR \neq 0$ and $ab \neq 0$. That is R is an Ore domain.
3. $\Rightarrow$ (1) follows from Lemma 2.7

Proposition 3.3. Let R be domain. If $G_c(R)$ is a connected graph, then $\mathrm{diam}\left(G_c(R)\right) \leq 2$.

Proof: Let I and J be to vertices of $G_c(R)$.

- If $I \cap J \neq 0$, such that at least of of them is principal, then $I - J$. Thus $d(I, J) = 1$.
- If $I \cap J \neq 0$, I and J both none principal, there are non-zero elements a and b such that $I - cR - J$ with $c = ab$. Thus $d(I, J) = 2$.
- If $I \cap J = 0$ for all nonzero $a \in I$ and $b \in J$, $aR \cap bR = 0$ which contradicts Theorem 3.2.

Hence, $\mathrm{diam}\left(G_c(R)\right) \leq 2$

Proposition 3.4. Let R be a ring. The following statements are equivalents.

1. $G_c(R)$ is a complete graph;
2. R is essential and R has at most one non-principal ideal.

Proof:

- (1)⇒ (2). Assume that $G_c(R)$ is complete and let I be a proper ideal of R. By definition, the vertex I is adjacent to any others vertex, that is I is essential ideal. Then R is essential ring.
Assume again that R has at least two proper ideals which are not principal. Let I_1 and I_2 be two non-principal ideals of R. The vertices I_1 and I_2 can not be adjacent; that is $G_c(R)$ is not complete. Then R has at most one non-principal ideal.
- (2)⇒ (1). Let J and K be two vertices of $G_c(R)$. Since R is essential, then $J \cap K \neq 0$. Since R has at most one non-principal ideal, we have two possible cases: either J and K are principal, or exactly one between J and K is principal

– If J and K are principal, J and K adjacent vertices.

– If one between J and K is principal, J and K are adjacent vertices. ▫

The result follows.

Lemma 3.5. The graph $G_c(R)$ of a principal ideal domain R is a complete graph.

Proof: Let I and J two proper ideals of R. Since I and J nonzero ideals, there is non-zero elements a and b in R such that $a \in I$ and $b \in J$. Hence, $0 \neq ab \in aR \cap bR \subseteq I \cap J$ implies that I and J are adjacent.

Example 3.6. The graph $G_c(\mathbb{Z})$ is complete because for all $n, m \in \mathbb{Z}$, $n\mathbb{Z} \cap m\mathbb{Z} \neq 0$.

Corollary 3.7. If R a field, then $G_c(R[x])$ is a complete graph.

Lemma 3.8. The graph $G_c(R)$ of an köthe ring R is a complete graph.

Proof: Let I and J be two proper ideals of R. Since R is a köthe ring, there is non-zero elements a and b in R such that $a \in I$ and $b \in J$. Hence $aR \cap bR \subset I \cap J$ implies that I and J are adjacent.

Lemma 3.9.

1. $G_c(R)$ is a complete graph if and only if R is an essential domain which has at most one non-principal ideal.
2. If R has more than one non-principal ideal, then $G_c(R)$ is a disconnected graph.

Proof:

1. If $G_c(R)$ is a complete graph, by proposition 3.4, it has at most one non-principal ideal. For all a and b two non-zero elements in R, $aR \cap bR = abR$. Since $G_c(R)$ is a complete, then $ab \neq 0$. Conversely, if R is an essential domain which has at most one non-principal ideal, then $G_c(R)$ is complete by proposition 3.4
2. It is clear that two non-principal ideals of R can not be adjacent vertices of the graph $G_c(R)$. ▫

Theorem 3.10. Let R be a Bezout ring. The followings statements are equivalents:

1. $G_c(R)$ is a complete graph;
2. R is a principal ideal domain.

Proof:

1. $\Rightarrow$(2). Since $G_c(R)$ is a complete graph, there is at most one non-principal ideal. Let I this ideal. For all $a_1 \in I$, $I_1 = Ra_1$ is adjacent I, that is $I_1 \cap I \neq 0$. If $I = I_1$, I is principal. Otherwise, there exist $a_2 \in I \setminus I_1$ and let $I_2 = Ra_1 + Ra_2$. If $I \neq I_2 = Ra_1 + Ra_2$, there exists $a_3 \in I \setminus I_2$ and let $I_3 = Ra_1 + a_2 + Ra_3$. Inductively, let $I_n = Ra_1 + \cdots + Ra_n$. If $I \neq I_n$, we choose $a_{n+1} \in I \setminus I_n$. Since $G_c(R)$ is complete, the chain $I_1 - I_2 - \cdots - I_n$ must be finite. Moreover, the ideal $I_n = Ra_1 + \cdots + Ra_n$ is principal because R is Bezout domain. This is a contradiction. Then R is a principal ideal domain.

2. $\Rightarrow$(1) follows from Lemma 3.5

Corollary 3.11. Let R be a Bezout domain. The followings statements are equivalents:

1. $G_c(R)$ is a complete graph;
2. R is a principal ideal domain.
3. For two ideals I and J, $I \cap J = 0$ implies $I = 0$ or $J = 0$.
4. For all $(a; b) \in R^2$, $aR \cap bR = 0$ implies $a = 0$ or $b = 0$.
5. Every non-zero ideal of R is indecomposable.

Remark 3.12.

1. If R is Bezout domain, for all vertices $I_1, I_2, \cdots, I_n \in G_c(R)$, $I_1 - I_2 - \cdots - I_n - I_1$ is a cycle.
2. $\text{girth}\big(G_c(R)\big) = 3$

Proposition 3.13. If R is Bezout domain, N and K two vertices of $G_c(R)$ such that $K \subset N$, then $\deg(K) \leq \deg(N)$.

Proof: Let N and K two vertices of $G_c(R)$ such that $K \subset N$. If J is another vertex of $G_c(R)$ then $J \cap K \neq 0$. Since R is Bezout principal ideal domain and $J \cap K \subset J \cap J \cap N$, then $J \cap N \neq 0$.

Theorem 3.14. The followings statements are equivalents in a Bezout domain R.

1. $G_c(R)$ is a complete;
2. R is an integral domain and has at most one non-principal ideal;
3. R is a principal ideal domain.

Proof:

- (1)$\Leftrightarrow$ (2) follows from Lemma 3.9
- (3)$\Leftrightarrow$ (1) follows from Theorem 3.10

Corollary 3.15. The graph $G_c(R)$ of a Bezout domain is a regular graph.

Proof: Since R is a Bezout domain, $G_c(R)$ is complete in view of Theorem 3.14; Then $G_c(R)$ is regular graph.

Proposition 3.16. If R is an Ore domain with k proper principal ideals, then the clique number $w\big(G_c(R)\big) = k$.

Proof: Let I_1, I_2, ..., I_k the k proper principal ideals, $I_{k+1}, I_{k+2}, \dots, I_n$ the $n - k$ proper non-principal ideals of R. Since R is an Ore domain, for every vertex I_i for $i \in \{1,2,\dots,k\}$ $I_i \cap I_j \neq 0$ for $j > k$. That is I_i for $i \in \{1,2,\dots,k\}$ adjacent to each other vertex in the graph. Then the graph induced by the path $\{I_{k+1}, I_{k+2}, \dots, I_n\}$ is complete. Thus $w\big(G_c(R)\big) = k$. Here we recall a result from [5]

Theorem 3.17. Let R a domain with k proper non-principal ideals and k′ proper principal ideals of R. The followings statements are equivalents.

1. $G_c(R)$ is a simple graph and $k' \geq k$;
2. $G_c(R)$ is Hamiltonian graph.

Proof:

$(1) \Rightarrow (2)$. The order of $G_c(R)$ is $n = k' + k$. Let L_k the set of non-principal ideals, $L_k{}'$ the set of principal ideals, (I, J) a pair of non-adjacent vertices of $G_c(R)$. Three cases are possibles:

– case 1: If I and J are non - principal ideals, then $\deg(I) + \deg(J) = 2k' \geq k' + k = n$.

– case 2: If I and J are principal ideals, then $\deg(I) + \deg(J) = 2(n-1) \geq k' + k = n$.

– case 3: If exactly one ideal between I or J is principal, $\deg(I) + \deg(J) = k' + 2(n-1) \geq n$.

By Ore Theorem, $G_c(R)$ is a Hamiltonian graph.

$(2) \Rightarrow (1)$. Assume that $G_c(R)$ is not a simple graph or $k' < k$;

– If $G_c(R)$ is not a simple graph clearly $G_c(R)$ is not Hamiltonian graph.

– If $k' < k$, since $n = k + k'$ there is no cycle containing every vertex of $G_c(R)$. That is, there is no Hamiltonian cycle. Then $G_c(R)$ is not Hamiltonian.

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Compliance with ethical standards

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