Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 5s (2025)

Minkowski and conformal flatness of Cartan space using (α,β)-metric

Rinkal[1], Mohammad Rafee[2]
[1,2]Department of Mathematics, School of Sciences, RIMT University, India
Crossponding author Email: mohd_rafee60@yahoo.com

Article History:

Received: *09-10-2024*

Revised: *27-11-2024*

Accepted: *07-12-2024*

Abstract:

The purpose of the present paper is to study the Minkowski and conformal flatness nature of Cartan space under the condition of h-metrical d-connection of the Cartan space using (α,β)-metric $K(x,\omega)=\alpha(x,\omega)+\frac{\beta^{n+1}(x,\omega)}{\alpha^n(x,\omega)}$.

Keywords: Cartan Space, (α, β)-metric, Minkowski space, Conformally flat space.

AMS Subject Classification: 53B40, 53C60.

1. Introduction

The idea of Cartan space is due to E. Cartan [2]. Cartan was a French mathematician and geometer. Cartan space is the dual of a Finsler space [2]. This dual is defined using the functional "Legendre transformation". The relation between Cartan space and Finsler space has been studied by F. Brickell [1], H. Rund [12] and others. R. Miron ([6], [7]) introduced the theory of Hamiltonian space, He proved that Cartan space is a particular case of Hamilton space. The notion of (α,β)-metric in Cartan space was introduced by T. Igrashi ([3], [4]. He obtained the metric tensors and some invariants which characterize the special class of Cartan spaces with (α,β)-metric. H.G. Nagaraja [8], G. Shanker [13], M. Rafee ([9], [10], [11]) and Tripathi [15] have also made significant development in the theory of Cartan spaces with (α,β)-metric. The paper is organized as follows:

In Section 2, we give basic definitions and results required for subsequent sections. In Section 3, we deal with Cartan space with an (α,β)-metric, $K(x,\omega)=\alpha(x,\omega)+\frac{\beta^{n+1}(x,\omega)}{\alpha^n(x,\omega)}$, under the condition of h-metrical d-connection. In Section 4, we study the conformal change of Cartan space and find some important results.

2. Preliminaries

Consider an n-dimensional differentiable manifold M. For each point $p\in M$, we define the cotangent space at p, denoted by T_p^*M, to be dual space to the tangent space T_pM, i.e., $T_p^*M=\left(T_pM\right)^*$. The elements of T_pM are called tangent vectors located at p and the elements of T_p^*M are called tangent covectors or simply covectors located at p. Moreover, the tangent vectors are denoted by v and the tangent covectors are denoted by ω. The cotangent bundle of the manifold M is denoted by T^*M and defined as the set of all tangent covectors at all points of M. That is,

$$T^*M=\cup_{p\in M}T_p^*M$$
$$=\text{ disjoint union of all cotangent spaces } T_p^*M \text{ on } M.$$

Usually, in physical applications, a mechanical system with finite configuration space is mathematically studied using a differential manifold M of finite dimension n (that is, a mechanical system with a finite number of degrees of freedom with constraints or with holonomic constraints). According to the Newton's laws, the configuration of a system at some instant is not enough to determine its configuration at some other instant; however, the evolution of the system is fixed by the configuration and the momentum of the system at some instant. The momentum of the system corresponds to a tangent covector ω_p, at the point p of the M that represents the configuration of the system at that instant; therefore at each point of cotangent bundle T^*M determines a state of the system. When M is a configuration space, T^*M is called phase space. If $\omega_p \in T_p^*M$ represents the state of the system, there exist a unique curve in T^*M passing through ω_p describing the evolution of the state of the system.

Definition 2.1 *(Slit Cotangent Bundle)*

Let $T^*M = \bigcup_{x \in M} T_x^*M$ be a cotangent bundle of the manifold M, where T_x^*M is a cotangent space at a point $x \in M$. If we remove the zero section (i.e., zero cotangent vectors $\omega = 0$ of each cotangent spaces T_x^*M) from the cotangent bundle T^*M, we get the cotangent bundle without zero cotangent vectors. This resultant cotangent bundle is called the slit cotangent bundle and it is denoted by T^*M^0.

Using set-builder method, it can be written as $T^*M^0 = \{(x, \omega) | x \in M, \omega \in T_x^*M, \omega \neq 0\}$.

Let us first recall the definition of Cartan space:

Definition 2.2 *(Cartan space)*

A Cartan space is a pair $C = (M, K(x, \omega))$ such that the following conditions are satisfied:

Let M be a smooth manifold and T^*M be its cotangent bundle. A function $K: T^*M \to R$ is called Finsler metric or Finsler fundamental function on the cotangent bundle T^*M if it satisfies the following properties:

1. Smoothness of K:

F is C^∞ away from zero cotangent vectors of the cotangent spaces T_p^*M.

That is, F is smooth on slit cotangent bundle TM^0.

OR

Smoothness of K:

F is smooth on the slit cotangent bundle T^*M^0. That is, K is smooth on the cotangent bundle T^*M without the null section $\{(x, 0)\} \subset T^*M$, and $K \in C^0$ on the null section $\{(x, 0)\} \subset T^*M$. $K \in C^0$ means K is just continuos at every point $(x, 0) \in TM$ of the null section $\{(x, 0)\} \subset T^*M$. The regularity condition here only is desired so that we can incorporate the tools of differential calculus on the Finsler metric or Finsler fundamental function K.

2. Positivity of K: $K(x, \omega)$ is positive for all $\omega \in T_p^*M$.

3. Positive Homogeneity of K: $K(x, \omega)$ is +ve 1-homogeneous with respect to ω the cotangent bundle T^*M, i.e., $K(x, \lambda\omega) = \lambda K(x, \omega)$, $\forall\ \lambda > 0$; for any $x \in M$, $\omega \in T_x^*M$.

4. Strict Convexity of K: The hessian matrix defined by $g^{ij}(x, \omega) = \frac{1}{2} \frac{\partial^2 K^2}{\partial \omega^i \partial \omega^j}(x, \omega)$ is positive definite for all $(x, \omega) \in T^*M^0$.

Definition 2.3 *(Cartan Space)*

A differentiable manifold M equipped with a Finsler metric $K(x, \omega)$ defined on the cotangent bundle T^*M is called a Cartan space..

Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 5s (2025)

Cartan space is denoted by $C=(M,K(x,\omega))$, where $K(x,\omega)$ represents norm of the differential one form $\omega\in T_x^*M$ based at any point $x\in M$. The function $K(x,\omega)$ is called the fundamental function and $g^{ij}(x,\omega)=\frac{1}{2}\frac{\partial^2K^2}{\partial\omega_i\partial\omega_j}(x,\omega)$ is called the fundamental metric tensor of the Cartan space C. In Cartan space the metric $K\colon T^*M\to[0,\infty)$ is defined from cotangent bundle T^*M to non-negative real numbers, so at a point $x\in M$, $K(x,-)$ eats one-form $\omega\in T_p^*M$ and spits non-negative reals, amounts to saying that Cartan space is constructed on the cotangent bundle T^*M in the same way a Finsler space $(M,K(x,y))$, where $K\colon TM\to[0,\infty)$, is constructed on the tangent bundle TM.

Next we define the norm of a differential one form $\omega\in T_p^*M$ in local coordinates or in terms of fundamental metric tensor g^{ij} of the corresponding Cartan space $(M,K(x,\omega))$.

Definition 2.4 *(Norm of a Differential one Form)*

Consider $(M,K(x,\omega))$ be a Cartan space, where $K(x,\omega)$ is a Finsler metric on the cotangent bundle T^*M. Then the norm of a differential one form $\omega\in T_p^*M$ at any fixed point $x\in M$ is denoted by $K_x(\omega)$ and defined by

$$K_x^2(\omega)=\frac{1}{2}\frac{\partial^2K^2}{\partial\omega_i\partial\omega_j}(x,\omega)\omega_i\omega_j$$
$$=g^{ij}(x,\omega)\omega_i\omega_j,$$

where $g^{ij}(x,\omega)=\frac{1}{2}\frac{\partial^2K^2}{\partial\omega_i\partial\omega_j}(x,\omega)$ is the fundamental metric tensor of the Finsler metric $K(x,\omega)$ of cotangent bundle T^*M.

Proposition 2.5 (1) *Let $C=(M,K(x,\omega))$ be a Cartan space. Then the space $H=(M,K^2(x,\omega))$ determined by the Cartan space $C=(M,K(x,\omega))$ is a Hamilton space.*

Proposition 2.6 *Let $C=(M,K(x,\omega))$ be a Cartan space. Then the following properties hold:*

1. $\omega^i=\frac{1}{2}\frac{\partial K^2}{\partial\omega_i}$ is 1-homogeneous d-vector field on the dual tangent bundle T^*M^0.
2. $g^{ij}(x,\omega)=\frac{\partial\omega^i}{\partial\omega_j}=\frac{1}{2}\frac{\partial^2K^2}{\partial\omega_i\partial\omega_j}(x,\omega)$ is 0-homogeneous d-tensor field.
3. $C^{ijk}=\frac{1}{4}\frac{\partial^3K^2(x,\omega)}{\partial\omega_i\partial\omega_j\partial\omega_k}$ is -1-homogeneous symmetric d-tensor field.

Proposition 2.7 *Let $C=(M,K(x,\omega))$ be a Cartan space. Then the following properties hold:*

1. $\omega^i=g^{ij}\omega_j$ and $\omega_i=g_{ij}\omega^j$.
2. $K^2=g^{ij}\omega_i\omega_j=\omega_i\omega^j$.
3. $C^{ijk}\omega_k=0$, $C^{ikj}\omega_k=0$, $C^{kij}\omega_k=0$.

Proposition 2.8 *A Cartan space $C=(M,K(x,\omega))$ is a Riemannian space if and only if d-tensor field $C^{ijk}=\frac{1}{4}\frac{\partial^3K^2}{\partial\omega_i\partial\omega_j\partial\omega_k}$ vanishes.*

Definition 2.9 *If the fundamental function $K(x,\omega)$ of a Cartan space $C=(M,K(x,\omega))$ is a function of variables $\beta(x,\omega)=\omega_ib^i(x)$, where $a^{ij}(x)$ is a Riemannian metric and $b^i(x)$ is a vector field depending only on x, then C is called Cartan space with (α,β)-metric. Here it is to be remarked that*

$K(x,\omega)$ *must satisfy all the conditions imposed on the fundamental function of a Cartan space.*

Definition 2.10 *(Minkowski space)*

Let V be a vector space of dimension n. A C^∞ function $F: V\backslash\{0\} = \{y|y \in V, y \neq 0\} \to R$ is said to be Minkowski norm if F satisfies the following properties:

1. Positivity: $F(y) \geq 0$ for all $y \in V$.
2. Positive Homogeneity: F is +ve 1-homogeneous with respect to y, i.e.,

$F(\lambda y) = \lambda F(y)$, $\forall\ \lambda > 0$; for any $y \in V\backslash\{0\}$.

3. Strong Convexity: The hessian matrix defined by $g_{ij}(y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(y)$ is positive definite for all $y \in V\backslash\{0\}$.

The vector space V quipped with Minkowski norm F is called Minkowski space and it is denoted by (V, F).

Definition 2.11 *(Conformally flat space)*

Let us consider a Cartan space $C = (M, K(x,\omega))$ with an (α,β)-metric, $K(x,\omega) = \alpha(x,\omega) + \frac{\beta^{n+1}(x,\omega)}{\alpha^n(x,\omega)}$, where $\alpha = \left(a^{ij}(x,\omega)\omega_i\omega_j\right)^{\frac{1}{2}}$ and $\beta = \omega_i b^i(x)$.

The fundamental tensor $g^{ij}(x,\omega)$ and its reciprocal tensor $g_{ij}(x,\omega)$ of the Cartan space $C = (M, K(\alpha,\beta))$ are given by [4]

$$g^{ij} = \rho a^{ij} + \rho_0 b^i b^j + \rho_{-1}(b^i\omega_j + b^j\omega_i) + \rho_{-2}\omega_i\omega_j, \tag{1}$$

where ρ, ρ_0, ρ_{-1} and ρ_{-2} are invariants which are defined and calculated as follows:

$$\rho = \frac{1}{2\alpha}K_\alpha$$

$$= 1 - n\left(\frac{\beta}{\alpha}\right)^{n+1}$$

$$\rho_0 = \frac{1}{2}K_{\beta\beta}$$

$$= \frac{n(n+1)\beta^{n-1}}{2\alpha^n}$$

$$\rho_{-1} = \frac{1}{2\alpha}K_{\alpha\beta}$$

$$= -\frac{n(n+1)\beta^n}{2\alpha^{n+2}}$$

$$\rho_{-2} = \frac{1}{2\alpha^2}\left(K_{\alpha\alpha} - \frac{1}{\alpha}K_\alpha\right)$$

$$= \frac{1}{2\alpha^2}\left[\frac{n(n+1)\beta^{n+1}}{\alpha^{n+2}} - \frac{1}{\alpha}\right]$$

and

$$g_{ij} = \sigma a_{ij} - \sigma_0 b_i b_j + \sigma_{-1}(b_i\omega_j + b_j\omega_i) + \sigma_{-2}\omega_i\omega_j, \tag{2}$$

where

$$\sigma = \frac{1}{\rho}$$

$$= \frac{2\alpha}{1-n\left(\frac{\beta}{\alpha}\right)^{n+1}}$$

$$\sigma_0 = \frac{\rho_0}{\rho\tau}$$

$$\tau = \sigma + \sigma_0 B^2 + \rho_{-1}\beta$$

$$\sigma_{-1} = \frac{\rho_{-1}}{\rho\tau}$$

$$\sigma_{-2} = \frac{\rho_{-2}}{\rho\tau},$$

where $B^2 = b^i b_j$ and B represents the norm of the differential form $\beta(x,\omega) = \omega_i b^i(x) \in T_p^*M$. The Cartan torsion tensor C^{ijk} [6] is given by

$$C^{ijk} = -\frac{1}{2}\big[r_{-1}b^ib^jb^k + \{\rho_{-1}a^{ij}b^k + \rho_{-2}a^{ij}\omega^k + r_{-2}b^ib^j\omega^k + r_{-3}b^i\omega^j\omega^k + i|j|k\} + r_{-4}\omega^i\omega^j\omega^k\big], \tag{3}$$

where its coefficients r_{-1}, r_{-2}, r_{-3} and r_{-4} are defined and calculated as follows:

$$r_{-1} = \frac{1}{2}K_{\beta\beta\beta}$$

$$= \frac{n(n-1)(n+1)\beta^{n-2}}{2\alpha^n}$$

$$r_{-2} = \frac{1}{2\alpha}K_{\alpha\beta\beta}$$

$$= -\frac{n^2(n+1)\beta^{n-1}}{2\alpha^n}$$

$$r_{-3} = \frac{1}{2\alpha^2}\left(K_{\alpha\alpha\beta} - \frac{1}{\alpha}K_{\alpha\beta}\right)$$

$$= \frac{n(n+1)(n+2)\beta^n}{2\alpha^{n+4}}$$

$$r_{-4} = \frac{1}{2\alpha^3}\left(K_{\alpha\alpha\alpha} - \frac{3}{\alpha}K_{\alpha\alpha} + \frac{3}{\alpha^2}K_\alpha\right)$$

$$= \frac{1}{\alpha^3}\left[-\frac{n(n^2+6n+2)\beta^{n+1}}{\alpha^{n+3}} + \frac{3}{\alpha^2}\right].$$

Let '|' denote the covariant differentiation with respect to Christoffel symbols γ^i_{jk} constructed from a_{ij}. Whenever we talk about Christoffel symbols γ^i_{jk} constructed from a_{ij}, we mean $\gamma^i_{jk} = \frac{1}{2}a^{li}\left(\frac{\partial a_{kl}}{\partial x^j} + \frac{\partial a^{lj}}{\partial x^k} - \frac{\partial a^{jk}}{\partial x^l}\right)$. Since $\omega_{i|k} = 0$ and from Ricci's theorem of tensor calculus [15] we have $a^{ij}_{|k} = 0$, if $b^i_{|k} = 0$, then $g^{ij}_{|k} = 0$. Also, let $\Gamma^i_{jk}(p) = \frac{1}{2}g^{ir}(\partial_j g_{rk} + \partial_k g_{jr} - \partial_r g_{jk})$ be the Christoffel symbols constructed from fundamental metric tensor $g_{ij}(x,\omega)$ of the Cartan space $(M, K(x,\omega))$. Now, for the Cartan space $(M, K(x,\omega))$, we state canonical d-connection is a triplet given by

$$D\Gamma = (N_{jk}, H^i_{jk}, C^{jk}_i),$$

where

$$N_{ij} = \Gamma^k_{ij}\omega_k - \frac{1}{2}\Gamma^k_{hr}\omega_k\omega^r\dot{\partial}^h g_{ij} \tag{4}$$

$$H^i_{jk} = \frac{1}{2} g^{ir} (\partial_j g_{rk} + \partial_k g_{jr} - \partial_r g_{jk}) \tag{5}$$

$$C_i^{jk}(x, \omega) = -\frac{1}{2} g_{ir}(x, \omega) \frac{\partial g^{jk}(x,\omega)}{\partial \omega^r} = g_{ir}(x, \omega) C^{rjk}(x, \omega) \tag{6}$$

are respectively called canonical N-connection, Christoffel symbols and d-tensor field of type (2,1).

Let us use the D-connection to find the h-covariant derivative $D\Gamma$. We use the symbol $'|_{h'}$ to indicate h-covariant with respect to D-connection $D\Gamma$. Let us define the meaning of h-metrical d-connection in the Cartan space.

Definition 2.12 (10) *An h-metrical d-connection on a Cartan space $C = (M, K(\alpha(x, \omega), \beta(\omega))$ with (α, β)-metric is a d-connection, $D\Gamma$ on C, satisfying the following properties:*

1. $g^{ij}_{|h} = 0$
2. $a^{ij}_{|h} = 0$
3. h-deflection tensor $D_{ij} (= \omega_{i|j}) = 0.$

3. Minkowski nature of Cartan spaces with (α, β)-metric $K(x, \omega) = \alpha(x, \omega) + \frac{\beta^{n+1}(x,\omega)}{\alpha^n(x,\omega)}$

Consider the Cartan space with (α, β)-metric $K(x, \omega) = \alpha(x, \omega) + \frac{\beta^{n+1}(x,\omega)}{\alpha^n(x,\omega)}$. First we calculate the h-covariant derivative of the (α, β)-metric as follows and then we impose the h-metrical d-connection $D\Gamma$ on the Cartan space with the given (α, β)-metric:

$$K(x, \omega) = \alpha(x, \omega) + \frac{\beta^{n+1}(x,\omega)}{\alpha^n(x,\omega)}$$

$$g^{ij}(\omega_i \omega_{j|h} + \omega_j \omega_{i|h}) + \omega_i \omega_j g^{ij}_{|h} = \alpha_{|h} + \frac{(n+1)\alpha^n \beta \beta_{|h} - n\beta^{n+1}\alpha^{n-1}\alpha_{|h}}{\alpha^{2n}}.$$

As we have considered that the d-connection $D\Gamma$ of the Cartan space is h-metrical, so by definition 2.12 of h-metrical d-connection $D\Gamma$, we get

$$\omega_{j|h} = 0, \omega_{i|h} = 0, \alpha_{|h} = 0, g^{ij}_{|h} = 0.$$

Using these values in above expression, we get

$$\beta_{|h} = 0 \quad (\because \alpha \neq 0, \beta \neq 0) \tag{7}$$

$$(\omega_i b^i(x))_{|h} = 0 \quad (\because \beta(x, \omega) = \omega_i b^i(x))$$

$$\omega_i b^i(x)_{|h} + b^i(x) \omega_{i|h} = 0.$$

As we have stipulated the d-connection $D\Gamma$ of the Cartan space is h-metrical, therefore by definition 2.12, we have

$$\omega_{i|h} = 0.$$

Using these values in above expression, we get

$$\omega_i b^i(x)_{|h} + b^i(x) \times 0 = 0$$

$$\omega_i b^i(x)_{|h} = 0$$

$$b^i(x)_{|h} = 0. \tag{8}$$

Now, we find h-covariant derivatives of the coefficients of metric tensor g^{ij} and then use conditions of h-metrical d-connection $D\Gamma$ of Cartan space as follows, we get

$$\because \rho = 1 - n\left(\frac{\beta}{\alpha}\right)^{n+1}$$

$$\therefore \rho_{|h} = 0. \tag{9}$$

$$\because \rho_0 = \frac{n(n+1)\beta^{n-1}}{2\alpha^n}$$

$$\therefore \rho_{0|h} = 0. \tag{10}$$

$$\because \rho_{-1} = -\frac{n(n+1)\beta^n}{2\alpha^{n+2}}$$

$$\therefore \rho_{-1|h} = 0. \tag{11}$$

$$\because \rho_{-2} = \frac{1}{2\alpha^2}\left[\frac{n(n+1)\beta^{n+1}}{\alpha^{n+2}} - \frac{1}{\alpha}\right]$$

$$\therefore \rho_{-2|h} = 0. \tag{12}$$

The h-covariant differentiation of the equation (1) gives

$$g^{ij}_{|h} = \rho a^{ij}_{|h} + a^{ij}\rho_{|h} + \rho_0(b^i b^j)_{|h} + b^i b^j \rho_0 + \rho_{-1}(b^i\omega^j + b^j\omega^i)_{|h} +$$
$$(b^i\omega^j + b^j\omega^i)\rho_{-1|h} + \rho_{-2}(\omega^i\omega^j)_{|h} + \omega^i\omega^j\rho_{-2|h}$$

$$g^{ij}_{|h} = \rho a^{ij}_{|h} + a^{ij}\rho_{|h} + \rho_0\left(b^i b^j_{|h} + b^j b^i_{|h}\right) + b^i b^j \rho_{0|h} + \rho_{-1}\left(b^i\omega^j_{|h} + \omega^i b^i_{|h} + b^j\omega^i_{|h} + \omega^i b^j_h\right)$$
$$\rho_{-1|h}\left(b^i\omega^j + b^j\omega^i\right) + \rho_{-2|h}\left(\omega^i\omega^j_{|h} + \omega^j\omega^i_{|h}\right) + \omega^i\omega^j\rho_{-2|h}.$$

Using the conditions of h-metrical d-connection $D\Gamma$ of Cartan space and equations (8), (9), (10), (11) and (12), above equation reduces to $g^{ij}_{|h} = 0$.

Thus, allowing d-connection $D\Gamma$ of the Cartan space to be h-metrical, it gives two important quantities namely $a^{ij}_{|h} = 0$ (by definition of h-metrical d-connection) and $g^{ij}_{|h} = 0$, i.e., h-covariant derivatives of fundamental metric tensors of associated Riemannian space and Cartan space vanishes.

Now, since $a^{ij}_{|h} = 0$ and $g^{ij}_{|h} = 0$, therefore there corresponding Chritoffel symbols will also be same, i.e., $H^i_{jh} = \gamma^i_{jh}$ and its equivalent condition is given by

$$b^i_{|k} = 0. \tag{13}$$

Now, since $H^i_{jh} = \gamma^i_{jh}$ therefore the curvature tensor D^i_{hjk} of $D\Gamma$ coincides with the curvature tensor R^i_{hjk} of Riemannian connection $R\Gamma = (\gamma^i_{jk}, \gamma^i_{jk}y_i, 0)$, i.e.,

$$D^i_{hjk} = R^i_{hjk}.$$

If the Riemannian curvature tensor vanishes, i.e., $R^i_{hjk} = 0$, the curvature tensor of d-connection also vanishes, i.e., $D^i_{hjk} = 0$. This discussion can be summarized as follows:

Proposition 3.1 *A Cartan space C with the (α, β)-metric $K(x,\omega) = \alpha(x,\omega) + \frac{\beta^{n+1}(x,\omega)}{\alpha^n(x,\omega)}$ admitting a h-metrical d-connection is locally flat if and only if the associated Riemannian space is locally flat.*

Now, we find h-covariant derivatives of the coefficients of Cartan torsion tensor C^{ijk} and then use conditions of h-metrical d-connection $D\Gamma$ of Cartan space and equation (7) as follows:

$$\because r_{-1} = \frac{n(n-1)(n+1)\beta^{n-2}}{2\alpha^n}$$

$$\therefore r_{-1|h} = 0 \tag{14}$$

$$\because r_{-2} = -\frac{n^2(n+1)\beta^{n-1}}{2\alpha^n}$$

$$\therefore r_{-2|h} = 0 \tag{15}$$

$$\because r_{-3} = \frac{n(n+1)(n+2)\beta^n}{2\alpha^{n+4}}$$

$$\therefore r_{-3|h} = 0 \tag{16}$$

$$\because r_{-4} = \frac{1}{\alpha^3}\left[-\frac{n(n^2+6n+2)\beta^{n+1}}{\alpha^{n+3}} + \frac{3}{\alpha^2}\right]$$

$$\therefore r_{-4|h} = 0. \tag{17}$$

Now we calculate the value of h-covariant derivative of d-tensor field C_i^{jk} of type (2,1) under the assumption of h-metrical d-connection as follows:

$$\because C_k^{ij} = g_{kr}C^{rij}$$

$$\therefore C_{k|h}^{ij} = \left(g_{kr}C^{rij}\right)_{|h}$$

$$= g_{kr} \times C_{|h}^{rij} + C^{rij} \times 0g_{kr|h}$$

$$= g_{kr}C_{|h}^{rij}$$

$$= -g_{kr}\frac{1}{2}[r_{-1}b^r b^i b^j + r_{-2}b^r b^i \omega^j + r_{-3}b^r \omega^i \omega^j + r_{-4}\omega^r \omega^i \omega^j + \rho_{-1}a^{ri}b^j +$$
$$\rho_{-2}a^{ri}\omega^j + r|i|j]_{|h}$$

$$= -g_{kr}\frac{1}{2}[r_{-1} \times (b^r b^i b^j)_{|h} + b^r b^i b^j \times 0r_{-1|h} + r_{-2} \times (b^r b^i \omega^j)_{|h} + b^r b^i \omega^j \times$$
$$0r_{-2|h} + r_{-3} \times (b^r \omega^i \omega^j)_{|h} + b^r \omega^i \omega^j \times 0r_{-3|h} + r_{-4} \times (\omega^r \omega^i \omega^j)_{|h} + \omega^r \omega^i \omega^j \times$$
$$0r_{-4|h} + \rho_{-1} \times (a^{ri}b^j)_{|h} + a^{ri}b^j \times 0\rho_{-1|h} + \rho_{-2} \times (a^{ri}\omega^j)_{|h} + a^{ri}\omega^j \times$$
$$0\rho_{-2|h} + (r|i|j)_{|h}]$$

$$= -g_{kr}\frac{1}{2}[r_{-1}(b^r b^i b^j)_{|h} + r_{-2}(b^r b^i \omega^j)_{|h} + r_{-3}(b^r \omega^i \omega^j)_{|h} + r_{-4}(\omega^r \omega^i \omega^j)_{|h} +$$
$$\rho_{-1}(a^{ri}b^j)_{|h} + \rho_{-2}(a^{ri}\omega^j)_{|h} + (r|i|j)_{|h}]$$

$$= -g_{kr}\frac{1}{2}[r_{-1}(b^r b^i 0b_{|h}^j + b^r b^j 0b_{|h}^i + b^i b^j 0b_{|h}^r) + r_{-2}(b^r b^i 0\omega_{|h}^j + b^r \omega^j 0b_{|h}^i + b^i \omega^j 0b_{|h}^r) +$$
$$r_{-3}(b^r \omega^i 0\omega_{|h}^j + b^r \omega^j 0\omega_{|h}^i + \omega^i \omega^j 0b_{|h}^r) + r_{-4}(\omega^r \omega^i 0\omega_{|h}^j + \omega^r \omega^j 0\omega_{|h}^i + \omega^i \omega^j 0\omega_{|h}^r) +$$
$$\rho_{-1}(a^{ri}0b_{|h}^j + b^j 0a_{|h}^{ri}) + \rho_{-2}(a^{ri}0\omega_{|h}^j + \omega^j 0a_{|h}^{ri}) + 0(r|i|j)_{|h}]$$

$$C_{k|h}^{ij} = 0. \tag{18}$$

One knows that a Cartan space C is Berwald space if and only if $C^{ij}_{k|h} = 0$ [13]. Hence from equation (18), we have the following proposition:

Proposition 3.2 *A Cartan space C with the (α,β)-metric $K(x,\omega) = \alpha(x,\omega) + \frac{\beta^{n+1}(x,\omega)}{\alpha^n(x,\omega)}$ admitting h-metrical d-connection is a Berwald space.*

In [13], it is deduced that a locally Minkowski space is a Berwald space in which curvature tensor vanishes. Hence, from the Propositions 3.1 and 3.2, we have following theorem:

Theorem 3.3 *A Cartan space C with the (α,β)-metric $K(x,\omega) = \alpha(x,\omega) + \frac{\beta^{n+1}(x,\omega)}{\alpha^n(x,\omega)}$ admitting h-metrical d-connection is locally Minkowski space if and only if the associated Riemannian space is locally flat.*

4. Conformal flatness of Cartan space with (α,β)-metric $K(x,\omega) = \alpha(x,\omega) + \frac{\beta^{n+1}(x,\omega)}{\alpha^n(x,\omega)}$

In this section our aim is to conformally transform a Cartan space $(M, K(x,\omega))$ to another Cartan space $(M, \bar{K}(x,\omega))$ and then to determine the nature of curvature tensor $\bar{D}^i_{hjk}$ in the conformally transformed space $(M, \bar{K}(x,\omega))$ under the influence of h-metrical d-connection on the original Cartan space $(M, K(x,\omega))$. That is, we are going to determine the shape of conformally transformed space $(M, \bar{K}(x,\omega))$ under the stipulation of h-metrical d-connection on $(M, K(x,\omega))$.

For that, consider an n-dimensional Cartan space $C = (M, K(x,\omega))$ equipped with a real smooth n-manifold M and the (α,β)-metric $K(x,\omega) = \alpha(x,\omega) + \frac{\beta^{n+1}(x,\omega)}{\alpha^n(x,\omega)}$, where $\alpha = (a^{ij}(x,\omega)\omega_i\omega_j)^{\frac{1}{2}}$ and $\beta = \omega_i b^i(x)$. By a conformal change $\sigma: K \to \bar{K}$ such that $\bar{K}(\bar{\alpha},\bar{\beta}) = e^\sigma K(\alpha,\beta)$, we have the another Cartan space $\bar{C}^n = (M, \bar{K}(\bar{\alpha},\bar{\beta}))$, where $\bar{\alpha} = e^\sigma\alpha$ and $\bar{\beta} = e^\sigma\beta$.

Putting $\alpha = (a^{ij}(x,\omega)\omega_i\omega_j)^{\frac{1}{2}}$ and $\beta = \omega_i b^i(x)$ in the above relations, we get

$$\bar{\alpha} = e^\sigma\alpha$$

$$\bar{\alpha} = e^\sigma(a^{ij}(x,\omega)\omega_i\omega_j)^{\frac{1}{2}}$$

$$\bar{\alpha} = (\underline{e^{2\sigma}a^{ij}(x,\omega)}\omega_i\omega_j)^{\frac{1}{2}}$$

$$\bar{\alpha} = (\underline{\bar{a}^{ij}}\omega_i\omega_j)^{\frac{1}{2}}$$

$$\bar{a}^{ij} = e^{2\sigma}a^{ij}(x,\omega)$$

and

$$\bar{\beta} = e^\sigma\beta$$

$$\bar{\beta} = e^\sigma\omega_i b^i(x)$$

$$\bar{\beta} = \omega_i\underline{e^\sigma b^i(x)}$$

$$\bar{\beta} = \omega_i\underline{b^i}$$

$$\bar{b}^i = e^\sigma b^i(x).$$

Now we calculate the Christoffel symbols $\bar{\gamma}^p_{rk}$ in conformally transformed space $(M, \bar{K}(x,\omega))$ as follows:

We know from Riemannian geometry Christoffel symbols of second kind γ^p_{rk} from fundamental metric tensor $a^{pq}(x,\omega)$ can be defined as

$$\gamma^p_{qk} = \frac{1}{2}a^{lp}\left(\frac{\partial a_{kl}}{\partial x^q} + \frac{\partial a^{lq}}{\partial x^k} - \frac{\partial a^{qk}}{\partial x^l}\right).$$

Similarly, we can also define the Christoffel symbols $\bar{\gamma}^p_{rk}$ in conformally transformed space $(M, \bar{K}(x,\omega))$ as

$$\begin{aligned}
\bar{\gamma}^p_{qk} &= \frac{1}{2}\bar{a}^{lp}\left(\frac{\partial \bar{a}_{kl}}{\partial x^q} + \frac{\partial \bar{a}_{lq}}{\partial x^k} - \frac{\partial \tilde{a}_{qk}}{\partial x^l}\right) \\
&= \frac{1}{2}e^{2\sigma}a^{lp}(x,\omega)\left(\frac{\partial e^{2\sigma}a_{kl}(x,\omega)}{\partial x^q} + \frac{\partial e^{2\sigma}a_{lq}(x,\omega)}{\partial x^k} - \frac{\partial e^{2\sigma}a_{qk}(x,\omega)}{\partial x^l}\right) \\
&= \frac{1}{2}e^{2\sigma}a^{lp}\left[\left(e^{2\sigma}\frac{\partial a_{kl}}{\partial x^q} + a_{kl}\frac{\partial e^{2\sigma}}{\partial x^q}\right) + \left(e^{2\sigma}\frac{\partial a_{lq}}{\partial x^k} + a_{lq}\frac{\partial e^{2\sigma}}{\partial x^k}\right) - \left(e^{2\sigma}\frac{\partial a_{jq}}{\partial x^l} + a_{qk}\frac{\partial e^{2\sigma}}{\partial x^l}\right)\right] \\
&= \frac{1}{2}e^{2\sigma}a^{lp}\left[\left(e^{2\sigma}\frac{\partial a_{kl}}{\partial x^q} + 2e^{2\sigma}a_{kl}\frac{\partial \sigma}{\partial x^q}\right) + \left(e^{2\sigma}\frac{\partial a_{lq}}{\partial x^k} + 2e^{2\sigma}a_{lq}\frac{\partial \sigma}{\partial x^k}\right) - \right. \\
&\quad \left.\left(e^{2\sigma}\frac{\partial a_{qk}}{\partial x^l} + 2e^{2\sigma}a_{qk}\frac{\partial \sigma}{\partial x^l}\right)\right] \\
&= \frac{1}{2}e^{4\sigma}a^{lp}\left[\left(\frac{\partial a_{kl}}{\partial x^q} + \frac{\partial a_{lq}}{\partial x^k} - \frac{\partial a_{qk}}{\partial x^l}\right) + \left(2a_{kl}\frac{\partial \sigma}{\partial x^q} + 2a_{lq}\frac{\partial \sigma}{\partial x^k} - 2a_{qk}\frac{\partial \sigma}{\partial x^l}\right)\right] \\
&= e^{4\sigma}\left[\frac{1}{2}a^{lp}\left(\frac{\partial a_{kl}}{\partial x^q} + \frac{\partial a_{lq}}{\partial x^k} - \frac{\partial a_{qk}}{\partial x^l}\right) + \left(a^{lp}a_{kl}\sigma_q + a^{lp}a_{lq}\sigma_k - a^{lp}a_{qk}\sigma_l\right)\right] \\
&= e^{4\sigma}\left[\gamma^p_{qk} + \left(\delta^p_k\sigma_q + \delta^p_q\sigma_k - a_{qk}\sigma^i\right)\right].
\end{aligned}$$

Hence, the components of Christoffel symbols $\bar{\gamma}^p_{qk}$, constructed from $\bar{a}^{pq}$, in conformally transformed space are given by

$$\bar{\gamma}^p_{qk} = \gamma^p_{qk} + B^p_{qk}, \tag{19}$$

where $B^p_{qk} = \sigma_k\delta^p_q + \sigma_q\delta^p_k - a_{kq}\sigma^p$, $\sigma^p = \sigma_q a^{pq}$.

The covariant derivative of $\bar{b}^p$ with respect to $\bar{\gamma}^p_{rk}$, yields

$$\bar{b}^p_{|k} = e^{\sigma}\left(b^p_{|k} + 2\sigma_k b^p + b^r\sigma_r\delta^p_k - \sigma_p b^r a_{rk}\right). \tag{20}$$

Transvecting the equation (20) by $\bar{b}^k$, and putting

$$M^p = \frac{1}{B^2}\left(b^k b^p_{:k} - \frac{b^r_{:r}b^p}{n+4}\right), \tag{21}$$

we have $\sigma^p = \bar{M}^p - M^p$, from which we get $\sigma_p = \bar{M}^p - M_p$. Substituting the values of σ_p and σ^p in equation (19) and using $D^p_{hq} = \gamma^p_{hq} + \delta^p_h M_q + \delta^p_h M_q + \delta^p_q M_h - M^p a_{hq}$, we find

$$\bar{D}^p_{hq} = D^p_{hq}. \tag{22}$$

Here D^p_{hq} is a symmetric and conformally invariant linear connection on M.

The whole discussion can be summarized in the following proposition.

Proposition 4.1 *Let $C = (M, K(x,\omega))$ be a Cartan space the (α,β)-metric $K(x,\omega) = \alpha(x,\omega) + \frac{\beta^{n+1}(x,\omega)}{\alpha^n(x,\omega)}$. Then, there exists a conformally invariant symmetric linear connection D^p_{qk} on M.*

Next, if we denote the curvature tensor of D^p_{qk} by D^p_{hqk}, then from the equation (22), we get

$$\overline{D}^p_{hqk} = D^p_{hqk}. \tag{23}$$

Since $b^p_{|k} = 0$, from equation (21), we get $M^i = 0$. Hence, we deduce that $D^p_{qk} = \gamma^p_{qk}$ and $D^p_{hqk} = R^p_{hqk}$.

Thus we have the following proposition:

Proposition 4.2 *Let $C = (M, K)$ be a Cartan space the (α, β)-metric $K(x, \omega) = \alpha(x, \omega) + \frac{\beta^{n+1}(x,\omega)}{\alpha^n(x,\omega)}$ admitting h-metrical d-connection. Then, there exists a conformally invariant symmetric linear connection D^p_{qh} such that $D^p_{qk} = \gamma^p_{qk}$ and it's curvature tensor $D^p_{hqk} = R^p_{hqk}$.*

Next, if the associated Riemannian space (M, α) is locally flat, that is, $R^p_{hqk} = 0$, then from Proposition 4.2 and equation (23), we deduce that $\overline{D}^p_{hqk} = 0$, that is, the space C is conformally flat. Thus we have the following theorem:

Theorem 4.3 *Let $C = (M, K)$ be a Cartan space the (α, β)-metric $K(x, \omega) = \alpha(x, \omega) + \frac{\beta^{n+1}(x,\omega)}{\alpha^n(x,\omega)}$ admitting h-metrical d-connection. Then the space C is conformally flat if and only if the associated Riemannian space is locally flat.*

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