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Generalization of Classical Summation Relation of Certain Appell's Double Hypergeometric Functions Associated with Theory of Approximation

Madhav Prasad Poudel[1],Narayan Prasad Pahari[2], Suresh Kumar Sahani*[3], Ganesh Bahadur Basnet[4], & Resham Prasad Poudel[5]

[1]School of Engineering, Pokhara University, Pokhara-30, Kaski, Nepal
[1]pdmadav@gmail.com
[2]Central Department of Mathematics, Tribhuvan University, Kirtipur, Kathmandu, Nepal
[2]nppahari@gmail.com
[*3]Department of Mathematics, Janakpur Campus, Tribhuvan University, Janakpurdham, Nepal
[3]sureshkumarsahani35@gmail.com
[4,5]Department of Mathematics, Tribhuvan University, Tri-Chandra Campus, Kathmandu, Nepal
[4]gbbmath@gmail.com, and [5]reshamprdpaudel@gmail.com
Corresponding author: Suresh Kumar Sahani[*3]
Corresponding author Email: sureshkumarsahani35@gmail.com

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Abstract:

In this research note, we have obtained some new classical summation relations of certain Appell's double hypergeometric functions associated with theory of approximation. This novel relation include, as special case, a set of well-known results. It studies the famous works of the authors [1], [2], [14], [15], [33] –[38].

Keywords: Hypergeometric functions, Srivastava's triple hypergeometric functions, Pochhammer symbol, summation, and etc.

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1. Introduction

Let $$(\delta)_g = \frac{\Gamma(\delta+g)}{\Gamma(\delta)} = \delta(\delta+1)(\delta+2)...(\delta+g-1), (\delta)_0 = 1 \tag{1}$$

and g is a positive number

and $$(\delta)_g = \frac{(-1)^\delta}{(1-\delta)_{-g}}, -g > 0 \tag{2}$$

In 1926, Appell's (see [1], [3], [4]) defined the following function

$$F_1(\alpha, \beta, \beta^1; \gamma; a, b) = \sum_{f=0}^{\infty}\sum_{g=0}^{\infty} \frac{(\alpha)_{f+g}(\beta)_g(\beta^1)_g}{(\gamma)_{f+g}} \cdot \frac{a^f}{f!} \cdot \frac{b^g}{g!} \tag{3}$$

In 1833, Lauricella (see [2]) defined n-ple hypergeometric function

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$$F_D\left(\alpha,\beta_1,\beta_2,...,\beta_g;\gamma;a_1,a_2,...,a_g\right)= \sum_{f_1,...,f_g=0}^{\infty}\frac{(\alpha)_{f_1+f_2+...+f_g}\left(\beta_1\right)_{f_1}...\left(\beta_g\right)_{f_g}}{(\gamma)_{f_1+f_2+...+f_g}}\cdot\frac{a_1^{f_1}}{\left(f_1\right)!}...\frac{a_g^{f_g}}{\left(f_g\right)!} \tag{4}$$

which corresponds to equation (3) where $g=2$.

In 1964, Srivastava (see [3], [33], [35]) defined H_A by the triple series in the following way

$$H_A\left[\alpha,\beta,\gamma;r,s;a,b,c\right]= \sum_{f,g,\ell=0}^{\infty}\frac{(\alpha)_{f+\ell}(\beta)_{f+g}(\gamma)_{g+\ell}}{(r)_g(s)_{g+\ell}}.\frac{a^f}{f!}.\frac{b^g}{g!}.\frac{c^\ell}{\ell!} \tag{5}$$

whose region of convergence is

$|a|<d,|b|<e,|c|<j,\ d+e+j=1+\ell j$ which is the generalization of [1].

The general triple hypergeometric function series $F^{(3)}$, defined as

$$F^{(3)}\left[\begin{matrix}(\alpha)::(\beta);\left(\beta^1\right);\left(\beta^{11}\right):(\gamma):\left(\gamma^1\right):\left(\gamma^{11}\right); \\ (r)::(s);\left(s^1\right);\left(s^{11}\right):(t):\left(t^1\right):\left(t^{11}\right);\end{matrix} x,y,z\right]$$

$$\sum_{f,g,\ell=0}^{\infty}\frac{(\alpha)_{f+g+\ell}(\beta)_{f+g}\left(\beta^1\right)_{g+\ell}\left(\beta^{11}\right)_{f+\ell}}{(r)_{f+g+\ell}(s)_{f+g}\left(s^1\right)_{g+\ell}\left(s^{11}\right)_{f+\ell}}\times$$

$$\frac{(\gamma)_f\left(\gamma^1\right)_g\left(\gamma^{11}\right)_\ell}{(t)_f\left(t^1\right)_g\left(t^{11}\right)_\ell}.\frac{a^f}{f!}.\frac{b^g}{g!}.\frac{c^\ell}{\ell!} \tag{6}$$

It is clear that

$$(\alpha)_f=\prod_{\xi=1}^{A}\left(\alpha_\xi\right)_f=\prod_{\xi=1}^{A}\frac{\Gamma\left(\alpha_{\xi+f}\right)}{\Gamma\left(\alpha_\xi\right)} \tag{7}$$

$$(\beta)_g=\prod_{\varepsilon=1}^{B}\frac{\Gamma\left(\beta_\varepsilon+g\right)}{\Gamma\left(\beta_\varepsilon\right)} \tag{8}$$

and $(\gamma)_\ell=\prod_{\delta=1}^{C}\frac{\Gamma\left(\gamma_{\delta+\ell}\right)}{\Gamma\left(\gamma_\delta\right)}$ (9)

where, A of the (α) parameters,

B of the (β) parameters,

and C of the (γ) parameters.

The region of convergence of the triple series (6) (see [5] and [6]) are follows:

$A+B+B^{11}+C \le D+E+E^{11}+F,$
$A+B+B^{1}+C^{1} \le D+E+E^{1}+F^{1},$
$\mathrm{A}+\mathrm{B}^{1}+B^{11}+C^{11} \le D+E^{1}+E^{11}+F^{11},$
$A, B, C, D, E, F, A^{1}, A^{11}, B^{1}, B^{11},$

$C^{1}, C^{11}, E^{1}, E^{1}, F^{1}\, and, F^{11}$ are positive integers and

$|\alpha|, |\beta|, and\ |\gamma|<1$

but $A+B+B^{11}+C = D+E+E^{11}+F+1,$

$A+B^{1}+B^{11}+C^{11} = D+E^{1}+E^{11}+F^{11}+1,$

and $A+B+B^{1}+C^{1} = D+E+E^{1}+F^{1}+1$

In 1941, a Burchnell and Chaundy (see [5] and [6]) first time to describe the following function;

$$F\begin{matrix} A:B\ ;\ D \\ F\ :G;\ H\end{matrix}\left[\begin{matrix}(\alpha):(\beta):(r); \\ (w):(u)\ ;(v);\end{matrix}\ a,b\right] = \sum_{f,g=0}^{\infty}\frac{[\alpha]_{f+g}[\beta]_f[\gamma]_g}{[w]_{f+g}[u]_f[v]_g}\frac{a^f}{f\,!}\frac{a^g}{g\,!} \tag{10}$$

where (α) and $(\alpha)_{f+g}$ are the sequences of A parameters $\alpha_1, \alpha_2, ..., \alpha_A$ and the product $(\alpha_1)_{f+g}(\alpha_2)_{f+g}...(\alpha_A)_{f+g}$ respectively.

Many works dealing with generalized hypergeometris function (see [24, 25]) and Srivastava's triple hypergeometric functions and their associated properties has been done (see [7]-[34]).

In this research note we obtain a Srivastava triple hypergeometric series $F^{(3)}$ instead of Kampe'de Feriet's double hypergeometric series

$$F\begin{bmatrix} f:g;\ \ell \\ i:\ j;\ \ k\end{bmatrix}.$$

In this research paper we prove the following theorems

$$\sum_{g=0}^{f}\frac{(-f)_g}{g\,!}F^{(3)}\left[\begin{matrix}(\alpha^1)::(\alpha);(u^1);-:-g,(r);-f+g,(u);(\gamma); \\ (\beta^1)::(\beta);(v^1);-:(s)\quad :\quad (v)\quad ;\ (w)\end{matrix}\ a,b,c\right]=$$

$$\frac{a^f(\alpha)_f(s^1)_f(r)_f}{(\beta)_f(\beta^1)_f(s)_f}F\begin{matrix}G^1:A^1+C;1+E+G \\ H^1:B^1+F;D+H\end{matrix}\left[\begin{matrix}(u^1):(\alpha^1+f),(\gamma);-f,(1-s-f),(u); \\ (v^1):(\beta^1+f),(w);(1-r-f),(v);\end{matrix}\ C,(-1)^{D-E}\frac{b}{a}\right]$$

where $x \neq 0$ (11)

Proof: We consider the series

$$\sigma=\sum_{g=0}^{\infty}\frac{(-f)_g}{g!}F^{(3)}\left[\begin{matrix}(\alpha^1)::(\alpha);(u^1);-:-g,(r);-f+g,(u);(\gamma);\\(\beta^1)::(\beta);(v^1);-:(s)\quad;\quad(v)\quad;\quad(w);\end{matrix}a,b,c\right]$$

$$=\sum_{g=0}^{\infty}\frac{(-f)_g}{g!}\sum_{\ell=0}^{\infty}\frac{(\alpha^1)_\ell(u^1)_\ell(\gamma)_\ell}{(\beta^1)_\ell(v^1)_\ell(w)_\ell}\frac{c^\ell}{\ell!}.$$

$$\sum_{j=0}^{g}\sum_{s=0}^{f-g}\frac{(\alpha)_{j+s}(\alpha^1+\ell)_{j+s}(r)_j(u)_s(u^1+\ell)_s}{(\beta)_{j+s}(\beta^1+\ell)_{j+s}(s)_j(v)_s(v^1+\ell)_s}.(-f)_j.(g-f)_s.\frac{a^j}{j!}.\frac{b^s}{s!}\tag{12}$$

with the help of finite triple series identity of Srivastava [34],we get

$$\sum_{g=0}^{f}\sum_{j=0}^{g}\sum_{k=0}^{f-g}A(f,g,j,k)=\sum_{k=0}^{f}\sum_{j=0}^{f-k}\sum_{g=0}^{f-j-k}A(f,g+j,j,k)\tag{13}$$

and Pochhammer's identities are follows:

$$\left.\begin{aligned}(\alpha)_g&=(-1)^{fA}(1-s-f)_f\\(\beta)_f&=(\beta)_k(\beta+k)_{f-k}\\(r)_{f-k}&=(r)_f\frac{(-1)^{kD}}{(1-k-f)_k}\\(-g-k)_k&=(-1)^k\frac{(g+k)!}{g!}\end{aligned}\right\}\tag{14}$$

Also, we have

$$\sigma=f!\sum_{\ell=0}^{\infty}\frac{(\alpha^1)_\ell(u^1)_\ell(\gamma)_\ell}{(\beta^1)_\ell(v^1)_\ell(w)_\ell}\frac{c^\ell}{\ell!}\sum_{k=0}^{f}\frac{(\alpha^1+\ell)_k(\alpha)_k(u^1+\ell)_k(u)_k}{(\beta^1+\ell)_k(\beta)_k(v^1+\ell)_k(v)_k}\frac{(-b)^k}{k!}$$

$$\sum_{j=0}^{f=k}\frac{(\alpha^1+\ell+k)_j(\alpha+k)_j(r)_j}{(\beta^1+\ell+k)_j(\beta+k)_j(s)_j}\frac{a^j}{j!(f-j-k)!}\sum_{g=0}^{f-j-k}(-1)^g\binom{f-j-k}{g}$$

$$=\sum_{\ell=0}^{\infty}\sum_{k=0}^{f}\frac{(u^1)_{\ell+k}(\gamma)_\ell(u)_k}{(v^1)_{\ell+k}(w)_\ell(v)_k}\frac{(c)^\ell}{\ell!}\frac{(b)^k}{k!}(-1)^k\sum_{j=0}^{f-k}\frac{(\alpha^1)_{\ell+k+j}(\alpha)_{k+j}(r)_j}{(\beta^1)_{\ell+k+j}(\beta)_{k+j}(s)_j}\frac{(\alpha)^j}{j!}\times(-1)^g\binom{f-j-k}{g}\tag{15}$$

now, using the following identity

$$\sum_{g=0}^{t}(-1)^g\binom{t}{g}=\begin{cases}1 & for\ t=0\\0 & for\ t\geq 1\end{cases}\tag{16}$$

in equation (15), without loss of generality we may assume that all terms or all values of *j* summation vanishing and then put $j=f-k$ in also equation (15) we obtain

$$\sigma=\sum_{\ell=0}^{\infty}\sum_{k=0}^{f}\frac{\left(u^1\right)_{\ell+k}\left(\gamma\right)_{\ell}\left(u\right)_{\ell}}{\left(v^1\right)_{\ell+k}\left(w\right)_{\ell}\left(v\right)j}\frac{\left(c\right)^{\ell}}{\ell!}\frac{\left(b\right)^{k}}{k!}\left(-1\right)^{k}\frac{\left(\alpha^1\right)_{\ell+f}\left(\alpha\right)_{f}\left(r\right)_{f-k}\left(a\right)^{f-k}f!}{\left(\beta^1\right)_{\ell+f}\left(\beta\right)_{f}\left(s\right)_{f-k}\left(f-k\right)!} \tag{17}$$

Thus,

$$\left(r\right)_{f-k}=\frac{\left(r_1\right)_{f-k}\ldots\left(r_D\right)_{f-k}}{1!}$$

$$=\frac{\left(r_1\right)\left(-1\right)^{k}}{\left(1-r_1-f\right)_{k}}\cdots\frac{\left(r_D\right)_{f}\left(-1\right)^{k}}{\left(1-r_D-f\right)_{k}}$$

$$=\frac{\left(-1\right)^{kD}\left(r\right)}{\left(1-r-f\right)_{k}} \tag{18}$$

Thus,

$$\sigma=\frac{\left(\alpha\right)_{f}\left(\alpha^1\right)_{f}\left(r\right)_{f}}{\left(\beta\right)_{f}\left(\beta^1\right)_{f}\left(s\right)_{f}}\sum_{\ell=0}^{\infty}\sum_{k=0}^{f}\frac{\left(u^1\right)_{\ell+k}\left(\alpha^1+f\right)_{\ell}\left(\gamma\right)_{\ell}\left(-f\right)_{k}\left(1-s-f\right)_{k}\left(u\right)_{k}}{\left(v^1\right)_{\ell+k}\left(\beta^1+f\right)_{\ell}\left(w\right)_{\ell}\left(1-r-f\right)_{k}\left(v\right)_{k}}\frac{\left(-1\right)^{k(D,E)}}{k!}$$

$\left(\frac{b}{a}\right)^{k}\frac{\left(c\right)^{\ell}}{\ell!}$ and our result follows.

For cases reducibility:

Put $b=0$ and $A=B=G^1=H^1=0$ in our main theorem *i.e.* in equation (11), we get

$$\sum_{g=0}^{\infty}\frac{\left(-f\right)_{g}}{g!}F\begin{matrix}A^1:D+1;C\\B^1:E;\mathrm{F}\end{matrix}\left[\begin{matrix}\left(\alpha^1\right):-g,\left(r\right);\left(\gamma\right);\\\left(\beta^1\right):\quad\left(s\right);\ \left(w\right);\end{matrix}a,\ c\right]$$

$$=\frac{\left(\alpha^1\right)_{f}\left(r\right)_{f}\left(a\right)^{f}}{\left(\beta^1\right)_{f}\left(s\right)_{f}}A^1+C^{F}B^{1}+F\left[\begin{matrix}\left(\alpha^1+f\right),\left(\gamma\right);\\\left(\beta^1+f\right),\left(w\right);\end{matrix}c\right] \tag{19}$$

Put $c=0$ and $A^1=B^1=G^1=H^1=0$ in our main theorem *i.e.*, in equation (11), we get

$$\sum_{g=0}^{f}\frac{\left(-f\right)_{g}}{g!}F\begin{matrix}A:D+1;\ G+1\\B:\quad E\quad;\quad F\end{matrix}\left[\begin{matrix}\left(\alpha\right):-g,\left(r\right);-f+g,\left(u\right);\\\left(\beta\right):\quad\left(s\right)\quad;\qquad\left(v\right);\end{matrix}a,b\right]$$

$$\frac{\left(\alpha\right)_{f}\left(r\right)_{f}a^{f}}{\left(\beta\right)_{f}\left(s\right)_{f}}1+E+G^{F}D+H\left[\begin{matrix}-f,\left(1-s-f\right),\left(u\right);\\\left(1-r-f\right),\quad\left(v\right);\end{matrix}\left(-1\right)^{D-E}\left(\frac{b}{a}\right)\right] \tag{20}$$

Without loss of generality we may assume $A=E=H=1$ and $B=D=G=0$ in above equation (20), making suitable arrangements of parameters and also using the arbitrary definition of Jacobi polynomials, we get

$$\sum_{g=0}^{\infty}\frac{(-\beta-f)_g(1+\beta)_{f-g}}{g!(f-g)!}\,F_2\left[r,-g,\,g-f;1+\alpha,1+\beta;\,a,b\right]$$

$$=\frac{(r)_f(-1)^f}{(1+\alpha)_f}(a+b)^f\,L_f^{(\alpha,\beta)}\left(\frac{b-a}{b+a}\right) \tag{21}$$

where F_2 is Appell's polynomial of second kind, given by

$F_2\left[\alpha;\beta,\gamma;\,r;\,a,b\right]=\sum_{f,g=0}^{\infty}\frac{(\alpha)_{f+g}(\beta)_f(\gamma)_g}{(r)_g(s)_g}\,\frac{a^f}{f!}\,\frac{b^g}{g!}$ which is the result of [28].

Similarly, putting $A=B=0$ in equation (20), we get

$$\sum_{g=0}^{f}\frac{(-f)_g}{g!}\,D+1_E^F\left[\begin{matrix}-g,(r); \\ (s);\end{matrix}\,a\right]G+1_H^F\left[\begin{matrix}-f+g,(u); \\ (v);\end{matrix}\,b\right]$$

$$=\frac{(r)_f\,a^f}{(s)_f}1+E+G_D^F+H\left[\begin{matrix}-f,(1-s-f),\,(u); \\ (1-r-f)\quad,\quad(v);\end{matrix}(-1)^{D-E}\,\frac{b}{a}\right] \tag{22}$$

which is the result of [27].

Putting $D=E=H=G=1$ in (22), we get

$$\sum_{g=0}^{\infty}\frac{(-f)_g}{g!}\,{}_2F_1\left[\begin{matrix}-g,\alpha; \\ \beta\;\;;\end{matrix}\,a\right]{}_2F_1\left[\begin{matrix}-f+g,\,\gamma; \\ r\quad;\end{matrix}\,b\right]$$

$$=\frac{(\alpha)_f\,a^f}{(\beta)_f}\,{}_3F_2\left[\begin{matrix}-f,(1-\beta-f); & \gamma; \\ (1-\alpha-f), & r;\end{matrix}\,\frac{b}{a}\right] \tag{23}$$

which is the result of [29].

Putting $A=A^1=B^1=C=E=H=1$ and $B=D=F=G=G^1=H^1=0$ in our main theorem *i.e.*, in equation (11), arranging parameters and using the definition of Jacobi polynomial, we get

$$\sum_{g=0}^{f}\frac{(-f)_g}{g!}\,F^{(3)}\left[\begin{matrix}r::\alpha;-;-;-g;-f+g;\gamma; \\ \beta::-;-;-;1+\mathrm{b};1+\mathrm{v};-;\end{matrix}\,a,\,b,\,c\right]$$

$$=\frac{(f)!(-1)^f(A)_f(r)_f}{(1+v)_f(\beta)_f(1+b)_f}(b+a)^f\,{}_2F_1\left[\begin{matrix}r+f, & r; \\ \beta+f & ;\end{matrix}\,c\right]L_F^{(b,\mathrm{v})}\left(\frac{b-a}{b+a}\right) \tag{24}$$

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which is the result of [30].

Putting $A^1 = B^1 = C = A = E = H = 1$ and $B = G^1 = H^1 = D = G = F = 0$, replacing g, f into f, g in our main theorem and making suitable arrangement of variables of parameters, we get

$$\sum_{f=0}^{g} \frac{(-g)_f}{f!} F^{(3)} \left[\begin{array}{c} \alpha :: \beta; -; -; -f; f-g; \beta-\gamma; \\ \gamma :: -; -; -; 1+s; 1+v; -, \quad \frac{a}{1+a}, \frac{-a}{1+a}, \frac{c}{1+c} \end{array} \right]$$

$$= \frac{(1+s+v)_{2g} (\alpha)_g (\beta)_g}{(1+s)_g (1+v)_g (1+s+v)_g (\gamma)_g} \left(\frac{a}{1+c} \right)^g {}_2F_1 \left[\begin{array}{c} \alpha+g, \gamma-\beta; \\ \gamma+g; \end{array} \frac{c}{1+c} \right] \tag{25}$$

which is the result of [31].

Also, putting $c = 0$ and $c = a$ in (25), we get

$$F^{(3)} \left[\begin{array}{c} \alpha :: \beta; -; -; u-w; r; \gamma-\beta; \\ \gamma :: -; -; -; u; s; -; \frac{a}{c+1}, \frac{b}{c+1}, \frac{c}{c+1} \end{array} \right] = (c+1)^{\alpha} F^{(3)} \left[\begin{array}{c} \alpha, \beta :: -; -; -; \mathrm{r}; -; \mathrm{w}; \\ r :: -; -; - : s; -; u; \end{array} b, c-a, -a \right] \tag{26}$$

which is the result of [31].

Putting $A = E = H = 0$ and $B = D = G = 1$ in equation (20), we get

$$\sum_{g=0}^{f} \frac{(f)_g}{g!} F_3(-g, -f+g; r, u; \beta, a, b) = \frac{(r)_f a^f}{(\beta)_f} {}_2F_1 \left[\begin{array}{c} -f, u; \\ 1-r-f; \end{array} \frac{-b}{a} \right] \tag{27}$$

Where F_3 is Appell's polynomial of third kind given by

$$F_3[\alpha, \beta; \gamma, r; s; a, b] = \sum_{f,g=0}^{\infty} \frac{(\alpha)_f (\beta)_g (\gamma)_f (r)_g}{(s)_{f+g}} \frac{a^f}{f!} \frac{b^g}{g!}$$

Putting $A = B = D = G = 0, E = H - 1$ in equation (20), applying the definitions of Laguerrl polynomials and Jacobi polynomials, we get

$$\sum_{g=0}^{f} \frac{(-v-f)_g}{(1+s)_g} M_g^{(s)}(a)\, L_{f-g}^{(v)}(b) = \frac{(-1)^f}{(1+b)_f} (a+b)^f L_f^{(s,v)} \left(\frac{b-a}{b+a} \right) \tag{28}$$

(Where $M_g^{(b)}(a)$ represents generalized Laguerrl polynomials) which is the result of [25].

In equation (20), setting $= G = H = 2, D = 0$ and using definition of Rice polynomials $H_f^{(\alpha,\beta)}(s, v, \theta)$, we get

$$\sum_{g=0}^{f} \frac{(-f)_g}{g!} F_{B:0:2}^{A:1:3} \left[\begin{array}{l} (\alpha) : -g; -f+g, 1+\alpha+\beta+f, u; \\ (\beta) : -; \quad 1+\alpha \quad ; \quad v, \quad ; \end{array} a, b \right]$$

$$=\frac{f\,!\,a^{f}\,(\alpha)_{f}}{(1+\alpha)_{f}\,\,(\beta)_{f}}\,H_{f}^{(\alpha,\beta)}\left(u,\,v,\,\frac{b}{a}\right) \tag{29}$$

which is the result of [33].

Putting $D=E=0, H=G=1$ in equation (20), using suitable arrangement of variables and parameters and then applying the definition of Gegenbaur's polynomials $C_{f}^{u}(a)$, we get

$$\sum_{g=0}^{f}\frac{(-f)_{g}}{g!}F\,\begin{matrix}A:1;2\\B:0;1\end{matrix}\left[\begin{matrix}(\alpha);-g;\,f+g, & 2u+f;\\(\beta):-;\mathrm{u}+\frac{1}{2} & ;\end{matrix}\;a,\,b\right]=\frac{f\,!\mathrm{a}^{f}\,(\alpha)_{f}}{(2u)_{f}\,(\beta)_{f}}C_{f}^{u}\left(\frac{a-2b}{a}\right) \tag{30}$$

Also, putting $=G=E=D=0,\ H=1$ in given equation (20) and applying the famous definitions of Shively's Pseudo-Laguerre polynomials $R_{f}\,(u,\,\theta)$, we get

$$\sum_{g=0}^{f}\frac{(-f)_{g}}{g!}F\,\begin{matrix}A:1;1\\B:0;1\end{matrix}\left[\begin{matrix}(\alpha):-g;-f+g\\(\beta):-;\,f+g;\end{matrix}\;a,\,b\right]=\frac{f\,!a^{f}\,(v)_{f}\,(\alpha)_{f}}{(v)_{2f}\,(\beta)_{f}}\,R_{f}\left(v,\frac{b}{a}\right) \tag{31}$$

Putting $D=G=E=0,\ H=2$ in equation (20), arranging variables and parameters and then applying the definition of Batemans' polynomials $J_{f}^{(u,v)}(\theta)$, we get

$$\sum_{g=0}^{f}\frac{(-f)_{g}}{g!}F\,\begin{matrix}A:1;1\\B:0;2\end{matrix}\left[\begin{matrix}(\alpha):-g;\,f+g & ;\\(\beta):-;u+1,v+\frac{u}{2}+1; & \end{matrix}\;a^{2},b^{2}\right]$$

$$=\frac{a^{2f+u}}{b^{u}}\,\frac{(\alpha)_{f}\,\,f\,!\Gamma(u+1)\Gamma\left(u+\frac{u}{2}+1\right)}{(\beta)_{f}\,\Gamma\left(u+\frac{u}{2}+f+1\right)}\,J_{f}^{(v,\,v)}\left(\frac{b}{a}\right) \tag{32}$$

Putting $D=E=G=H=1$ in equation (20), arranging parameters suitably and applying the definition of Rainville's polynomials $\psi_{f}\,(u,v,a)$, we obtain

$$\sum_{g=0}^{f}\frac{(-f)_{g}}{g!}F\,\begin{matrix}A:2;2\\B:1;1\end{matrix}\left[\begin{matrix}(\alpha):-g,\frac{1}{2}+\frac{u}{2}; & -f+g,\frac{1}{2}-\frac{u}{2}; \\(\beta) \quad\quad V; & V \quad\quad ;\end{matrix}\;a,\,b\right]=\frac{(-1)^{f}\,(\alpha)_{f}}{(\beta)_{f}}\,a^{f}\psi_{f}\left(v,\,u,\frac{b}{a}\right) \tag{33}$$

Conclusion

Hypergeometric series in one and more variables occur frequently in a wide variety of problems in theoretical physics, applied mathematics, engineering sciences, statistics and operation research.In our research note, we obtained a finite summation of triple hypergeometric series in terms of Kampe de Feriet's double hypergeometric series. A number of finite sums of Kampe de Feriets' double polynomials of higher order are obtained.

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