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Complex tangent trigonometric approach applied to (γ, τ)-rung fuzzy set using weighted averaging, geometric operators and its extension

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Abstract

This paper presents a new method that generates complex tangent trigonometric (γ, τ)-rung fuzzy sets. This article will deal with averaging, geometric, generalized weighted averaging, generalized weighted geometric using complex tangent trigonometric (γ, τ)-rung fuzzy set. We used an aggregating model to get the weighted average and geometric. Several sets with significant characteristics will be further studied using the algebraic approaches.

Keywords: (γ, τ)-rung, WA, WG, GWA, GWG.

1 Introduction

Numerous ideas have been put out to explain uncertainty, including fuzzy sets (FS),[1] which have membership grades (MG) ranging from zero to one. Atanassov.[2] For ϖ, $\kappa \in [0, 1]$ created an intuitionistic FS (IFS) in which each element has two MGs: positive ϖ and negative κ, and $0 \leq \varpi + \kappa \leq 1$. The Pythagorean FSs (PFS) idea was developed by Yager[3] and is distinguished by its MG and non-MG (NMG) with $\varpi + \kappa \geq 1$ toϖ^2+κ^2 ≤ 1. The three main concepts of the picture FS are positive MG (ϖ), neutral MG (γ), and negative MG (κ), as stated by Cuong et al.[4] It also provides more advantages than PFS and IFS with $0 \leq \varpi+\gamma+\kappa \leq 1$ sinceϖ, γ, κ $\in [0, 1]$. Expert comments such as "yes," "abstain," "no," and "refusal" will be sent, in accordance with the image FS description. Shahzaib et al.[5] used MADM to define the SFS for certain AOs. Instead of $0 \leq \varpi + \gamma + \kappa \leq 1$, SFS demands that $0 \leq \varpi^2 + \gamma^2 + \kappa^2 \leq 1$. The idea of an intelligent decision support system for SFS was initially put out by Hussain et al.[6] Both the MG and the NMG have power q in the q-rung orthogonal pair FS (q-ROFS), but their sum can never be more than one. Xu et al. developed geometric operators, including weighted, ordered weighted, and hybrid operators, that were derived from IFSs.[7] Generalized ordered weighted averaging operators (GOWs) were suggested by Li et al.[8] in 2002. Al-husband et al.[9]-[14] and[20]-[30] discussed the concept of various FS and its extension. Zeng et al.[15] explained how to compute ordered weighted distances using AOs and distance measurements. Based on the features of AOs, Peng et al. investigated a simple PFS.[16] Various algebraic structures and aggregation techniques with applications were studied by Palanikumar et al.[17–19] For the rest of my work, I will keep to the format provided here. In Section 2 deals that PFS and NS were discussed. Section 3 describes numerous methods on (γ, τ)-rung FNs. In Section 4, the AOs based on CT (γ, τ)-rung FN are discussed.

2 Background

Many important definitions that we should review for future learning are included in this section.

Definition 2 .1. Let $\mathscr{A}$ be a universal. The PFS $\Theta = \Big\{\delta, \big\langle \mathscr{M}^{\intercal}(\delta), \mathscr{M}^{\dashv}(\delta)\big\rangle \big| \delta \in \mathscr{A}\Big\}$, $\mathscr{M}^{\intercal} : \mathscr{A} \to (0,1)$ and $\mathscr{M}^{\dashv} : \mathscr{A} \to (0,1)$ called the MG and NMG of $\delta \in \mathscr{A}$ to Θ, respectively and $0 \preceq (\mathscr{M}^{\intercal}(\delta))^2 + (\mathscr{M}^{\dashv}(\delta))^2 \preceq 1$. For $\Theta = \big\langle \mathscr{M}^{\intercal}, \mathscr{M}^{\dashv}\big\rangle$ is called a Pythagorean fuzzy number (PFN).

Definition 2.2. The NS $\Theta = \Big\{\delta, \big\langle \mathscr{M}^{\intercal}(\delta), \mathscr{M}^{\sqsupset}(\delta), \mathscr{M}^{\dashv}(\delta)\big\rangle \big| \delta \in \mathscr{A}\Big\}$, where $\mathscr{M}^{\intercal}, \mathscr{M}^{\sqsupset}, \mathscr{M}^{\dashv} : \mathscr{A} \to (0,1)$ is denote the MG, IMG and NMG of $\delta \in \mathscr{A}$, respectively and $0 \leq (\mathscr{M}^{\intercal}(\delta)) + (\mathscr{M}^{\sqsupset}(\delta)) + (\mathscr{M}^{\dashv}(\delta)) \leq 2$. For $M = \big\langle \mathscr{M}^{\intercal}, \mathscr{M}^{\sqsupset}, \mathscr{M}^{\dashv}\big\rangle$ is called a neutrosophic number (-rung FN).

Definition 2.3. The Pythagorean NS $\Theta = \Big\{\delta, \big\langle \mathscr{M}^{\intercal}(\delta), \mathscr{M}^{\sqsupset}(\delta), \mathscr{M}^{\dashv}(\delta)\big\rangle \big| \delta \in \mathscr{A}\Big\}$, where $\mathscr{M}^{\intercal}, \mathscr{M}^{\sqsupset}, \mathscr{M}^{\dashv} : \mathscr{A} \to (0,1)$ is called the MG, IMG and NMG of $\delta \in \mathscr{A}$, respectively and $0 \leq (\mathscr{M}^{\intercal}(\delta))^2 + (\mathscr{M}^{\sqsupset}(\delta))^2 + (\mathscr{M}^{\dashv}(\delta))^2 \leq 2$. For $M = \big\langle \mathscr{M}^{\intercal}, \mathscr{M}^{\sqsupset}, \mathscr{M}^{\dashv}\big\rangle$ is called a Pythagorean neutrosophic number (Py-rung FN).

Definition 2.4. Let $\Theta_1 = (a_1, b_1) \in N$and $\Theta_2 = (a_2, b_2) \in N$. Then the distance between Θ_1 and Θ_2 is defined as $\mathbb{D}(\Theta_1, \Theta_2) = \sqrt{(a_1 - a_2)^2 + \frac{1}{2}(b_1 - b_2)^2}$, where N is a natural number.

3 Operations for CT (γ, τ)-rung FN

We introduce the notion of a complex tangent trigonometric, the (γ, τ)-rung FN. Consequently, $\tan \pi/2 = \eth$ and the CT (γ, τ)-rung FN and its operations were established.

Definition 3.1. The (γ, τ) NS $\Theta = \Big\{\delta, \Big\langle\Big((\eth \cdot \mathscr{R}^{\intercal})(\delta) \cdot e^{(\eth \cdot \mathscr{I}^{\intercal})(\delta)}, (\eth \cdot \mathscr{R}^{\dashv})(\delta) \cdot e^{(\eth \cdot \mathscr{I}^{\dashv})(\delta)}\Big)\Big\rangle \Big| \delta \in \mathscr{A}\Big\}$, where $(\eth \cdot \mathscr{R}^{\intercal}), (\eth \cdot \mathscr{R}^{\dashv}) : \mathscr{A} \to (0,1)$ denote the MG and NMG of $\delta \in \mathscr{A}$ to Θ, respectively and $0 \leq ((\eth \cdot \mathscr{R}^{\intercal})(\delta))^{\gamma} + ((\eth \cdot \mathscr{R}^{\dashv})(\delta))^{\tau} \leq 1$ and $0 \leq ((\eth \cdot \mathscr{I}^{\intercal})(\delta))^{\gamma} + ((\eth \cdot \mathscr{I}^{\dashv})(\delta))^{\tau} \leq 1$. For, $\Theta = \Big\langle\Big((\eth \cdot \mathscr{R}^{\intercal}) \cdot e^{(\eth \cdot \mathscr{I}^{\intercal})}, (\eth \cdot \mathscr{R}^{\dashv}) \cdot e^{(\eth \cdot \mathscr{I}^{\dashv})}\Big)\Big\rangle$ is represent a CT (γ, τ)-rung FN.

Definition 3.2. Let $\Theta = \langle((\eth \cdot \mathscr{R}^{\intercal}) \cdot e^{(\eth \cdot \mathscr{I}^{\intercal})}, ((\eth \cdot \mathscr{R}^{\dashv})) \cdot e^{(\eth \cdot \mathscr{I}^{\dashv})})\rangle$, $\Theta_1 = \langle((\eth \cdot \mathscr{R}_1^{\intercal}) \cdot e^{(\eth \cdot \mathscr{I}_1^{\intercal})}, (\eth \cdot \mathscr{R}_1^{\dashv}) \cdot e^{(\eth \cdot \mathscr{I}_1^{\dashv})})\rangle$, $\Theta_2 = \langle((\eth \cdot \mathscr{R}_2^{\intercal}) \cdot e^{(\eth \cdot \mathscr{I}_2^{\intercal})}, (\eth \cdot \mathscr{R}_2^{\dashv}) \cdot e^{(\eth \cdot \mathscr{I}_2^{\dashv})})\rangle$ be any three CT (γ, τ)-rung FNs, and $(\gamma, \tau) > 0$. Then

1. $$\Theta_1 \bigoplus \Theta_2 = \begin{bmatrix} \sqrt[\gamma]{\begin{matrix} ((\eth \cdot \mathscr{R}_1^{\intercal}))^{\gamma} + ((\eth \cdot \mathscr{R}_2^{\intercal}))^{\gamma} \\ -((\eth \cdot \mathscr{R}_1^{\intercal}))^{\gamma} \cdot ((\eth \cdot \mathscr{R}_2^{\intercal}))^{\gamma} \end{matrix}} \cdot e^{\sqrt[\gamma]{\begin{matrix} ((\eth \cdot \mathscr{I}_1^{\intercal}))^{\gamma} + ((\eth \cdot \mathscr{I}_2^{\intercal}))^{\gamma} \\ -((\eth \cdot \mathscr{I}_1^{\intercal}))^{\gamma} \cdot ((\eth \cdot \mathscr{I}_2^{\intercal}))^{\gamma} \end{matrix}}}, \\ ((\eth \cdot \mathscr{R}_1^{\dashv}))^{\tau}((\eth \cdot \mathscr{R}_2^{\dashv}))^{\tau} \cdot e^{((\eth \cdot \mathscr{I}_1^{\dashv}))^{\tau}((\eth \cdot \mathscr{I}_2^{\dashv}))^{\tau}} \end{bmatrix},$$
2. $$\Theta_1 \oslash \Theta_2 = \begin{bmatrix} ((\eth \cdot \mathscr{R}_1^{\intercal}))^{\gamma}((\eth \cdot \mathscr{R}_2^{\intercal}))^{\gamma} \cdot e^{((\eth \cdot \mathscr{I}_1^{\intercal}))^{\gamma}((\eth \cdot \mathscr{I}_2^{\intercal}))^{\gamma}}, \\ \sqrt[\tau]{\begin{matrix} ((\eth \cdot \mathscr{R}_1^{\dashv}))^{\tau} + ((\eth \cdot \mathscr{R}_2^{\dashv}))^{\tau} \\ -((\eth \cdot \mathscr{R}_1^{\dashv}))^{\tau} \cdot ((\eth \cdot \mathscr{R}_2^{\dashv}))^{\tau} \end{matrix}} \cdot e^{\sqrt[\tau]{\begin{matrix} ((\eth \cdot \mathscr{R}_1^{\dashv}))^{\tau} + ((\eth \cdot \mathscr{I}_2^{\dashv}))^{\tau} \\ -((\eth \cdot \mathscr{R}_1^{\dashv}))^{\tau} \cdot ((\eth \cdot \mathscr{I}_2^{\dashv}))^{\tau} \end{matrix}}} \end{bmatrix}$$
3. $$\partial \cdot \Theta = \begin{bmatrix} \sqrt[\gamma]{1 - \big(1 - (\eth \cdot (\mathscr{R}^{\intercal})^{\gamma}\big)^{\partial}} \cdot e^{\sqrt[\gamma]{1 - \big(1 - (\eth \cdot (\mathscr{I}^{\intercal})^{\gamma}\big)^{\partial}}}, \\ ((\eth \cdot (\mathscr{R}^{\dashv})^{\tau})^{\partial} \cdot e^{((\eth \cdot (\mathscr{I}^{\dashv})^{\tau})^{\partial}} \end{bmatrix}$$
4. $$\Theta^{\partial} = \begin{bmatrix} ((\eth \cdot (\mathscr{R}^{\intercal})^{\gamma})^{\partial} \cdot e^{((\eth \cdot (\mathscr{I}^{\intercal})^{\gamma})^{\partial}}, \\ \sqrt[\tau]{1 - \big(1 - (\eth \cdot (\mathscr{R}^{\dashv})^{\tau}\big)^{\partial}} \cdot e^{\sqrt[\tau]{1 - \big(1 - (\eth \cdot (\mathscr{I}^{\dashv})^{\tau}\big)^{\partial}}} \end{bmatrix}.$$

Definition 3.3. For any two CT (γ, τ)-rung FNs $\Theta_1 = \langle(((\eth \cdot \mathscr{R}_1^{\intercal}), (\eth \cdot \mathscr{R}_1^{\dashv})))\rangle$ and $\Theta_2 = \langle(((\eth \cdot \mathscr{R}_2^{\intercal}), (\eth \cdot \mathscr{R}_2^{\dashv})))\rangle$. Then

$$\mathbb{D}_E(\Theta_1, \Theta_2) = \sqrt{\frac{1}{2}\left[\begin{bmatrix} 1 + ((\eth \cdot \mathscr{R}_1^{\intercal}))^2 - ((\eth \cdot \mathscr{R}_1^{\dashv}))^2 \\ -\big(1 + ((\eth \cdot \mathscr{R}_2^{\intercal}))^2 - ((\eth \cdot \mathscr{R}_2^{\dashv}))^2\big) \end{bmatrix}^2 + \begin{bmatrix} ((\eth \cdot \mathscr{I}_1^{\intercal}))^2 - ((\eth \cdot \mathscr{I}_1^{\dashv}))^2 \\ -\big(((\eth \cdot \mathscr{I}_2^{\intercal}))^2 - ((\eth \cdot \mathscr{I}_2^{\dashv}))^2\big) \end{bmatrix}^2\right]}$$

where $\mathbb{D}_E(\Theta_1, \Theta_2)$ is called the ED between Θ_1 and Θ_2.

$$\mathbb{D}_H(\Theta_1, \Theta_2) = \frac{1}{2}\left[\left|\begin{array}{c} 1 + ((\partial \cdot \mathscr{R}_1^{\intercal}))^2 - ((\partial \cdot \mathscr{R}_1^{\dashv}))^2 \\ -\left(1 + ((\partial \cdot \mathscr{R}_2^{\intercal}))^2 - ((\partial \cdot \mathscr{R}_2^{\dashv}))^2\right) \end{array}\right| + \left|\begin{array}{c} ((\partial \cdot \mathscr{I}_1^{\intercal}))^2 - ((\partial \cdot \mathscr{I}_1^{\dashv}))^2 \\ \left(-((\partial \cdot \mathscr{I}_2^{\intercal}))^2 - ((\partial \cdot \mathscr{I}_2^{\dashv}))^2\right) \end{array}\right|\right]$$

where $\mathbb{D}_H(\Theta_1, \Theta_2)$ is called the HD between Θ_1 and Θ_2.

4 AOs based on CT (γ, τ)-rung FN

We use CT (γ, τ)-rung FNWA, CT (γ, τ)-rung FNWG, GCT (γ, τ)-rung FNWA, and GCT (γ, τ)-rung FNWG to describe the AOs.

4.1 CT $(\gamma, \tau)NWA$

Definition 4.1. Let $\Theta_i = \langle(((\partial \cdot \mathscr{R}_i^{\intercal}) \cdot e^{(\partial \cdot \mathscr{I}_i^{\intercal})}, (\partial \cdot \mathscr{R}_i^{\dashv}) \cdot e^{(\partial \cdot \mathscr{I}_i^{\dashv})}))\rangle$ be the CT (γ, τ)-rung FNs, $W = (\omega_1, \omega_2, ..., \omega_\ell)$ be the weight of Θ_i, $\omega_i \geq 0$ and $\bigoplus_{i=1}^{\ell} \omega_i = 1$. Then CT (γ, τ)-rung FNWA $(\Theta_1, \Theta_2, ..., \Theta_\ell) = \bigoplus_{i=1}^{\ell} \omega_i \Theta_i$.

Theorem 4.2. *Let* $\Theta_i = \left\langle(((\partial \cdot \mathscr{R}_i^{\intercal}) \cdot e^{(\partial \cdot \mathscr{I}_i^{\intercal})}, (\partial \cdot \mathscr{R}_i^{\dashv}) \cdot e^{(\partial \cdot \mathscr{I}_i^{\dashv})}))\right\rangle$ *be the CT* (γ, τ)*-rung FNs. Then CT* $(\gamma, \tau)NWA(\Theta_1, \Theta_2, ..., \Theta_\ell)$

$$= \left[\begin{array}{c} \sqrt[\gamma]{1 - \bigotimes_{i=1}^{\ell}\left(1 - ((\partial \cdot \mathscr{R}_i^{\intercal}))^{\gamma}\right)^{\omega_i}} \cdot e^{\sqrt[\gamma]{1 - \bigotimes_{i=1}^{\ell}\left(1 - ((\partial \cdot \mathscr{I}_i^{\intercal}))^{\gamma}\right)^{\omega_i}}}, \\ \bigotimes_{i=1}^{\ell}(((\partial \cdot \mathscr{R}_i^{\dashv}))^{\tau})^{\omega_i} \cdot e^{\bigotimes_{i=1}^{\ell}(((\partial \cdot \mathscr{I}_i^{\dashv}))^{\tau})^{\omega_i}} \end{array}\right].$$

Proof. If $\ell = 2$, then CT (γ, τ)-rung FNWA$(\Theta_1, \Theta_2) = \omega_1\Theta_1 \bigoplus \omega_2\Theta_2$, where

$$\omega_1\Theta_1 = \left[\begin{array}{c} \sqrt[\gamma]{1 - \left(1 - ((\partial \cdot \mathscr{R}_1^{\intercal}))^{\gamma}\right)^{\omega_1}} \cdot e^{\sqrt[\gamma]{1 - \left(1 - ((\partial \cdot \mathscr{I}_1^{\intercal}))^{\gamma}\right)^{\omega_1}}}, \\ (((\partial \cdot \mathscr{R}_1^{\dashv}))^{\tau})^{\omega_1} \cdot e^{(((\partial \cdot \mathscr{I}_1^{\dashv}))^{\tau})^{\omega_1}} \end{array}\right]$$

$$\omega_2\Theta_2 = \left[\begin{array}{c} \sqrt[\gamma]{1 - \left(1 - ((\partial \cdot \mathscr{R}_2^{\intercal}))^{\gamma}\right)^{\omega_2}} \cdot e^{\sqrt[\gamma]{1 - \left(1 - ((\partial \cdot \mathscr{I}_2^{\intercal}))^{\gamma}\right)^{\omega_2}}}, \\ (((\partial \cdot \mathscr{R}_2^{\dashv}))^{\tau})^{\omega_2} \cdot e^{(((\partial \cdot \mathscr{I}_2^{\dashv}))^{\tau})^{\omega_2}} \end{array}\right].$$

Now, $\omega_1\Theta_1 \bigoplus \omega_2\Theta_2$

$$= \left[\begin{array}{c} \sqrt[\gamma]{\begin{array}{c} \left(1 - \left(1 - ((\partial \cdot \mathscr{R}_1^{\intercal}))^{\gamma}\right)^{\omega_1}\right) + \\ \left(1 - \left(1 - ((\partial \cdot \mathscr{R}_2^{\intercal}))^{\gamma}\right)^{\omega_2}\right) \\ -\left(1 - \left(1 - ((\partial \cdot \mathscr{R}_1^{\intercal}))^{\gamma}\right)^{\omega_1}\right) \cdot \\ \left(1 - \left(1 - ((\partial \cdot \mathscr{R}_2^{\intercal}))^{\gamma}\right)^{\omega_2}\right), \end{array}} \cdot e^{\sqrt[\gamma]{\begin{array}{c} \left(1 - \left(1 - ((\partial \cdot \mathscr{I}_1^{\intercal}))^{\gamma}\right)^{\omega_1}\right) + \\ \left(1 - \left(1 - ((\partial \cdot \mathscr{I}_2^{\intercal}))^{\gamma}\right)^{\omega_2}\right) \\ -\left(1 - \left(1 - ((\partial \cdot \mathscr{I}_1^{\intercal}))^{\gamma}\right)^{\omega_1}\right) \cdot \\ \left(1 - \left(1 - ((\partial \cdot \mathscr{I}_2^{\intercal}))^{\gamma}\right)^{\omega_2}\right), \end{array}}} \\ (((\partial \cdot \mathscr{R}_1^{\dashv}))^{\tau})^{\omega_1} \cdot (((\partial \cdot \mathscr{R}_2^{\dashv}))^{\tau})^{\omega_2} \cdot e^{(((\partial \cdot \mathscr{I}_1^{\dashv}))^{\tau})^{\omega_1} \cdot (((\partial \cdot \mathscr{I}_2^{\dashv}))^{\tau})^{\omega_2}} \end{array}\right]$$

$$= \begin{bmatrix} \sqrt[\gamma]{1-\left(1-((\partial\cdot\mathscr{R}_1^{\intercal}))^{\gamma}\right)^{\omega_1}\left(1-((\partial\cdot\mathscr{R}_2^{\intercal}))^{\gamma}\right)^{\omega_2}}\cdot \\ e^{\sqrt[\gamma]{1-\left(1-((\partial\cdot\mathscr{I}_1^{\intercal}))^{\gamma}\right)^{\omega_1}\left(1-((\partial\cdot\mathscr{I}_2^{\intercal}))^{\gamma}\right)^{\omega_2}}}, \\ (((\partial\cdot\mathscr{R}_1^{\dashv}))^{\tau})^{\omega_1}\cdot(((\partial\cdot\mathscr{R}_2^{\dashv}))^{\tau})^{\omega_2}\cdot \\ e^{(((\partial\cdot\mathscr{I}_1^{\dashv}))^{\tau})^{\omega_1}\cdot(((\partial\cdot\mathscr{I}_2^{\dashv}))^{\tau})^{\omega_2}} \end{bmatrix}$$

Hence, CT $(\gamma,\tau)NWA(\Theta_1,\Theta_2)$

$$= \begin{bmatrix} \sqrt[\gamma]{1-\bigotimes_{i=1}^{2}\left(1-((\partial\cdot\mathscr{R}_i^{\intercal}))^{\gamma}\right)^{\omega_i}}\cdot e^{\sqrt[\gamma]{1-\bigotimes_{i=1}^{2}\left(1-((\partial\cdot\mathscr{I}_i^{\intercal}))^{\gamma}\right)^{\omega_i}}}, \\ \bigotimes_{i=1}^{2}(((\partial\cdot\mathscr{R}_i^{\dashv}))^{\tau})^{\omega_i}\cdot e^{\bigotimes_{i=1}^{2}(((\partial\cdot\mathscr{I}_i^{\dashv}))^{\tau})^{\omega_i}} \end{bmatrix}.$$

It valid for $\ell \geq 3$,
Thus, CT $(\gamma,\tau)NWA(\Theta_1,\Theta_2,...,\Theta_\ell)$

$$= \begin{bmatrix} \sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-((\partial\cdot\mathscr{R}_i^{\intercal}))^{\gamma}\right)^{\omega_i}}\cdot e^{\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-((\partial\cdot\mathscr{I}_i^{\intercal}))^{\gamma}\right)^{\omega_i}}}, \\ \bigotimes_{i=1}^{\ell}(((\partial\cdot\mathscr{R}_i^{\dashv}))^{\tau})^{\omega_i}\cdot e^{\bigotimes_{i=1}^{\ell}(((\partial\cdot\mathscr{I}_i^{\dashv}))^{\tau})^{\omega_i}} \end{bmatrix}.$$

If $\ell=\ell+1$, then CT (γ,τ)-rung FNWA $(\Theta_1,\Theta_2,...,\Theta_\ell,\Theta_{\ell+1})$

$$= \begin{bmatrix} \sqrt[\gamma]{\begin{array}{c}\bigoplus_{i=1}^{\ell}\left(1-\left(1-((\partial\cdot\mathscr{R}_i^{\intercal}))^{\gamma}\right)^{\omega_i}\right)+\left(1-\left(1-(\mathscr{R}_{\ell+1}^{\intercal})^{\gamma}\right)^{\omega_{\ell+1}}\right) \\ -\bigotimes_{i=1}^{\ell}\left(1-\left(1-((\partial\cdot\mathscr{R}_i^{\intercal}))^{\gamma}\right)^{\omega_i}\right)\cdot\left(1-\left(1-(\mathscr{R}_{\ell+1}^{\intercal})^{\gamma}\right)^{\omega_{\ell+1}}\right)\end{array}}\cdot \\ e^{\sqrt[\gamma]{\begin{array}{c}\bigoplus_{i=1}^{\ell}\left(1-\left(1-((\partial\cdot\mathscr{I}_i^{\intercal}))^{\gamma}\right)^{\omega_i}\right)+\left(1-\left(1-(\mathscr{I}_{\ell+1}^{\intercal})^{\gamma}\right)^{\omega_{\ell+1}}\right) \\ -\bigotimes_{i=1}^{\ell}\left(1-\left(1-((\partial\cdot\mathscr{I}_i^{\intercal}))^{\gamma}\right)^{\omega_i}\right)\cdot\left(1-\left(1-(\mathscr{I}_{\ell+1}^{\intercal})^{\gamma}\right)^{\omega_{\ell+1}}\right)\end{array}}}, \\ \bigotimes_{i=1}^{\ell}(((\partial\cdot\mathscr{R}_i^{\dashv}))^{\tau})^{\omega_i}\cdot((\mathscr{R}_{\ell+1}^{\dashv})^{\tau})^{\omega_{\ell+1}}\cdot e^{\bigotimes_{i=1}^{\ell}(((\partial\cdot\mathscr{I}_i^{\dashv}))^{\tau})^{\omega_i}\cdot((\mathscr{I}_{\ell+1}^{\dashv})^{\tau})^{\omega_{\ell+1}}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell+1}\left(1-((\partial\cdot\mathscr{R}_i^{\intercal}))^{\gamma}\right)^{\omega_i}}\cdot e^{\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell+1}\left(1-((\partial\cdot\mathscr{I}_i^{\intercal}))^{\gamma}\right)^{\omega_i}}}, \\ \bigotimes_{i=1}^{\ell+1}(((\partial\cdot\mathscr{R}_i^{\dashv}))^{\tau})^{\omega_i}\cdot e^{\bigotimes_{i=1}^{\ell+1}(((\partial\cdot\mathscr{I}_i^{\dashv}))^{\tau})^{\omega_i}} \end{bmatrix}.$$

□

Theorem 4.3. *Let* $\Theta_i = \left\langle (((\partial\cdot\mathscr{R}_i^{\intercal})\cdot e^{(\partial\cdot\mathscr{I}_i^{\intercal})}, (\partial\cdot\mathscr{R}_i^{\dashv})\cdot e^{(\partial\cdot\mathscr{I}_i^{\dashv})}))\right\rangle$ *be the CT* (γ,τ)*-rung FNs. Then CT* (γ,τ)*-rung FNWA* $(\Theta_1,\Theta_2,...,\Theta_\ell)=\Theta$ *(idempotency property).*

Proof. Since $(\eth\cdot\mathscr{R}_i^{\intercal})=(\eth\cdot\mathscr{R}^{\intercal})$, $(\eth\cdot\mathscr{R}_i^{\dashv})=(\eth\cdot\mathscr{R}^{\dashv})$ and $(\eth\cdot\mathscr{I}_i^{\intercal})=(\eth\cdot\mathscr{I}^{\intercal})$, $(\eth\cdot\mathscr{I}_i^{\dashv})=(\eth\cdot\mathscr{I}^{\dashv})$ and $\bigoplus_{i=1}^{\ell}\omega_i=1$. Now, CT $(\gamma,\tau)NWA(\Theta_1,\Theta_2,...,\Theta_\ell)$

$$\begin{aligned}
&= \left[\begin{array}{c}\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-((\eth\cdot\mathscr{R}_i^{\intercal}))^{\gamma}\right)^{\omega_i}}\cdot e^{\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-((\eth\cdot\mathscr{I}_i^{\intercal}))^{\gamma}\right)^{\omega_i}}},\\ \bigotimes_{i=1}^{\ell}(((\eth\cdot\mathscr{R}_i^{\dashv}))^{\tau})^{\omega_i}\cdot e^{\bigotimes_{i=1}^{\ell}(((\eth\cdot\mathscr{I}_i^{\dashv}))^{\tau})^{\omega_i}}\end{array}\right]\\
&= \left[\begin{array}{c}\sqrt[\gamma]{1-\left(1-(\eth\cdot(\mathscr{R}^{\intercal})^{\gamma}\right)^{\bigoplus_{i=1}^{\ell}\omega_i}}\cdot e^{\sqrt[\gamma]{1-\left(1-(\eth\cdot(\mathscr{I}^{\intercal})^{\gamma}\right)^{\bigoplus_{i=1}^{\ell}\omega_i}}},\\ ((\eth\cdot(\mathscr{R}^{\dashv})^{\tau})^{\bigoplus_{i=1}^{\ell}\omega_i}\cdot e^{((\eth\cdot(\mathscr{I}^{\dashv})^{\tau})^{\bigoplus_{i=1}^{\ell}\omega_i}}\end{array}\right]\\
&= \left[\begin{array}{c}\sqrt[\gamma]{1-\left(1-(\eth\cdot(\mathscr{R}^{\intercal})^{\gamma}\right)}\cdot e^{\sqrt[\gamma]{1-\left(1-(\eth\cdot(\mathscr{I}^{\intercal})^{\gamma}\right)}},\\ (\eth\cdot(\mathscr{R}^{\dashv})^{\tau}\cdot e^{(\eth\cdot(\mathscr{I}^{\dashv})^{\tau}}\end{array}\right]\\
&= \Theta.
\end{aligned}$$

□

Theorem 4.4. *Let* $\Theta_i=\left\langle(((\eth\cdot\mathscr{R}_i^{\intercal})\cdot e^{(\eth\cdot\mathscr{I}_i^{\intercal})},(\eth\cdot\mathscr{R}_i^{\dashv})\cdot e^{(\eth\cdot\mathscr{I}_i^{\dashv})}))\right\rangle$ *be the CT* (γ,τ)*-rung FNs. Then CT* (γ,τ)*-rung FNWA*$(\Theta_1,\Theta_2,...,\Theta_\ell)$*, where* $\overleftarrow{(\eth\cdot\mathscr{R}^{\intercal})}=\min(\eth\cdot\mathscr{R}_{ij}^{\intercal})$, $\widehat{(\eth\cdot\mathscr{R}^{\intercal})}=\max(\eth\cdot\mathscr{R}_{ij}^{\intercal})$, $\overleftarrow{(\eth\cdot\mathscr{R}^{\dashv})}=\min(\eth\cdot\mathscr{R}_{ij}^{\dashv})$, $\widehat{(\eth\cdot\mathscr{R}^{\dashv})}=\max(\eth\cdot\mathscr{R}_{ij}^{\dashv})$ *and* $\overleftarrow{(\eth\cdot\mathscr{I}^{\intercal})}=\min(\eth\cdot\mathscr{I}_{ij}^{\intercal})$, $\widehat{(\eth\cdot\mathscr{I}^{\intercal})}=\max(\eth\cdot\mathscr{I}_{ij}^{\intercal})$, $\overleftarrow{(\eth\cdot\mathscr{I}^{\dashv})}=\min(\eth\cdot\mathscr{I}_{ij}^{\dashv})$, $\widehat{(\eth\cdot\mathscr{I}^{\dashv})}=\max(\eth\cdot\mathscr{I}_{ij}^{\dashv})$ *and where* $1\le i\le n,\ j=1,2,...,i_j$*. Then,*

$$\begin{aligned}
&\left\langle\overleftarrow{(\eth\cdot\mathscr{R}^{\intercal})\cdot e^{(\eth\cdot\mathscr{I}^{\intercal})}},(\eth\cdot\widehat{\mathscr{R}^{\dashv})\cdot e^{(\eth\cdot\mathscr{I}^{\dashv})}}\right\rangle\\
&\le CT(\gamma,\tau)NWA(\Theta_1,\Theta_2,...,\Theta_\ell)\\
&\le \left\langle(\eth\cdot\widehat{\mathscr{R}^{\intercal})\cdot e^{(\eth\cdot\mathscr{I}^{\intercal})}},\overleftarrow{(\eth\cdot\mathscr{R}^{\dashv})\cdot e^{(\eth\cdot\mathscr{I}^{\dashv})}}\right\rangle.
\end{aligned}$$

(Boundedness property).

Proof. Since, $\overleftarrow{(\eth\cdot\mathscr{R}^{\intercal})}=\min(\eth\cdot\mathscr{R}_{ij}^{\intercal})$, $\widehat{(\eth\cdot\mathscr{R}^{\intercal})}=\max(\eth\cdot\mathscr{R}_{ij}^{\intercal})$ and $\overleftarrow{(\eth\cdot\mathscr{R}^{\intercal})}\le(\eth\cdot\mathscr{R}_{ij}^{\intercal})\le\widehat{(\eth\cdot\mathscr{R}^{\intercal})}$ and $\overleftarrow{(\eth\cdot\mathscr{I}^{\intercal})}=\min(\eth\cdot\mathscr{I}_{ij}^{\intercal})$, $\widehat{(\eth\cdot\mathscr{I}^{\intercal})}=\max(\eth\cdot\mathscr{I}_{ij}^{\intercal})$ and $\overleftarrow{(\eth\cdot\mathscr{I}^{\intercal})}\le(\eth\cdot\mathscr{I}_{ij}^{\intercal})\le\widehat{(\eth\cdot\mathscr{I}^{\intercal})}$.

Now $\overleftarrow{(\eth\cdot\mathscr{R}^{\intercal})}\cdot e^{\overleftarrow{(\eth\cdot\mathscr{I}^{\intercal})}}$

$$\begin{aligned}
&= \sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-(\overleftarrow{(\eth\cdot\mathscr{R}^{\intercal})})^{\gamma}\right)^{\omega_i}}\cdot e^{\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-(\overleftarrow{(\eth\cdot\mathscr{I}^{\intercal})})^{\gamma}\right)^{\omega_i}}}\\
&\le \sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-(((\eth\cdot\mathscr{R}_{ij}^{\intercal})))^{\gamma}\right)^{\omega_i}}\cdot e^{\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-(((\eth\cdot\mathscr{I}_{ij}^{\intercal})))^{\gamma}\right)^{\omega_i}}}\\
&\le \sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-(\widehat{(\eth\cdot\mathscr{R}^{\intercal})})^{\gamma}\right)^{\omega_i}}\cdot e^{\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-(\widehat{(\eth\cdot\mathscr{I}^{\intercal})})^{\gamma}\right)^{\omega_i}}}\\
&= \widehat{(\eth\cdot\mathscr{R}^{\intercal})}.
\end{aligned}$$

Since, $\overleftarrow{(\eth\cdot(\mathscr{R}^{\dashv})^{\tau})}=\min((\eth\cdot\mathscr{R}_{ij}^{\dashv}))^{\tau}$, $(\eth\cdot\widehat{(\mathscr{R}^{\dashv}})^{\tau})=\max((\eth\cdot\mathscr{R}_{ij}^{\dashv}))^{\tau}$ and $\overleftarrow{(\eth\cdot(\mathscr{R}^{\dashv})^{\tau})}\le((\eth\cdot\mathscr{R}_{ij}^{\dashv}))^{\tau}\le(\eth\cdot\widehat{(\mathscr{R}^{\dashv}})^{\tau})$ and $\overleftarrow{(\eth\cdot(\mathscr{I}^{\dashv})^{\tau})}=\min((\eth\cdot\mathscr{I}_{ij}^{\dashv}))^{\tau}$, $(\eth\cdot\widehat{(\mathscr{I}^{\dashv}})^{\tau})=\max((\eth\cdot\mathscr{I}_{ij}^{\dashv}))^{\tau}$ and $\overleftarrow{(\eth\cdot(\mathscr{I}^{\dashv})^{\tau})}\le((\eth\cdot\mathscr{I}_{ij}^{\dashv}))^{\tau}\le(\eth\cdot\widehat{(\mathscr{I}^{\dashv}})^{\tau})$.

We have,

$$
\begin{aligned}
\overleftarrow{(\eth\cdot(\mathscr{R}^{\dashv})^{\tau}} &= \bigotimes_{i=1}^{\ell}\overleftarrow{(\eth\cdot(\mathscr{R}^{\dashv})^{\tau})}^{\omega_i}\cdot e^{\bigotimes_{i=1}^{\ell}\overleftarrow{(\eth\cdot(\mathscr{I}^{\dashv})^{\tau})}^{\omega_i}}\\
&\leq \bigotimes_{i=1}^{\ell}(((\eth\cdot\mathscr{R}^{\dashv}_{ij}))^{\tau})^{\omega_i}\cdot e^{\bigotimes_{i=1}^{\ell}(((\eth\cdot\mathscr{I}^{\dashv}_{ij}))^{\tau})^{\omega_i}}\\
&\leq \bigotimes_{i=1}^{\ell}\left(\eth\cdot\widehat{(\mathscr{R}^{\dashv}})^{\tau}\right)^{\omega_i}\cdot e^{\bigotimes_{i=1}^{\ell}(\eth\cdot\widehat{(\mathscr{I}^{\dashv}})^{\tau})^{\omega_i}}\\
&= (\eth\cdot\widehat{(\mathscr{R}^{\dashv}})^{\tau})\cdot e^{(\eth\cdot\widehat{(\mathscr{I}^{\dashv}})^{\tau})}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{2}\times\left[\begin{array}{l}\left[\left(\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-(\overleftarrow{(\eth\cdot\mathscr{R}^{\intercal})})^{\gamma}\right)^{\omega_i}}\right)^2+1-\left(\bigotimes_{i=1}^{\ell}(((\widehat{\eth\cdot\mathscr{R}^{\dashv}}))^{\tau})^{\omega_i}\right)^2\right]\\ +\left[\left(\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-(\overleftarrow{(\eth\cdot\mathscr{I}^{\intercal})})^{\gamma}\right)^{\omega_i}}\right)^2-\left(\bigotimes_{i=1}^{\ell}(((\widehat{\eth\cdot\mathscr{I}^{\dashv}}))^{\tau})^{\omega_i}\right)^2\right]\end{array}\right]\\
\leq\;&\frac{1}{2}\times\left[\begin{array}{l}\left[\left(\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-((\eth\cdot(\eth\cdot\mathscr{R}^{\intercal}_{ij})))^{\gamma}\right)^{\omega_i}}\right)^2+1-\left(\bigotimes_{i=1}^{\ell}(((\eth\cdot\mathscr{R}^{\dashv}_{ij}))^{\tau})^{\omega_i}\right)^2\right]\\ +\left[\left(\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-((\eth\cdot(\eth\cdot\mathscr{I}^{\intercal}_{ij})))^{\gamma}\right)^{\omega_i}}\right)^2-\left(\bigotimes_{i=1}^{\ell}(((\eth\cdot\mathscr{I}^{\dashv}_{ij}))^{\tau})^{\omega_i}\right)^2\right]\end{array}\right]\\
\leq\;&\frac{1}{2}\times\left[\begin{array}{l}\left[\left(\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-((\widehat{\eth\cdot\mathscr{R}^{\intercal}}))^{\gamma}\right)^{\omega_i}}\right)^2+1-\left(\bigotimes_{i=1}^{\ell}(\overleftarrow{(\eth\cdot\mathscr{R}^{\dashv})})^{\tau})^{\omega_i}\right)^2\right]\\ +\left[\left(\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-((\widehat{\eth\cdot\mathscr{I}^{\intercal}}))^{\gamma}\right)^{\omega_i}}\right)^2-\left(\bigotimes_{i=1}^{\ell}(\overleftarrow{(\eth\cdot\mathscr{I}^{\dashv})})^{\tau})^{\omega_i}\right)^2\right]\end{array}\right].
\end{aligned}
$$

Hence, $\left\langle\overleftarrow{(\eth\cdot\mathscr{R}^{\intercal})\cdot e^{(\eth\cdot\mathscr{I}^{\intercal})}},(\eth\cdot\widehat{\mathscr{R}^{\dashv})\cdot e^{(\eth\cdot\mathscr{I}^{\dashv})}}\right\rangle\leq CT(\gamma,\tau)NWA(\Theta_1,\Theta_2,...,\Theta_\ell)$
$\leq\langle(\eth\cdot\widehat{\mathscr{R}^{\intercal})\cdot e^{(\eth\cdot\mathscr{I}^{\intercal})}},\overleftarrow{(\eth\cdot\mathscr{R}^{\dashv})\cdot e^{(\eth\cdot\mathscr{I}^{\dashv})}}\rangle.$ □

Theorem 4.5. *Let* $\Theta_i=\langle((\eth\cdot\mathscr{R}^{\intercal}_{t_{ij}})\cdot e^{(\eth\cdot\mathscr{I}^{\intercal}_{t_{ij}})},(\eth\cdot\mathscr{R}^{\dashv}_{t_{ij}})\cdot e^{(\eth\cdot\mathscr{I}^{\dashv}_{t_{ij}})})\rangle$
and $W_i=\langle((\eth\cdot\mathscr{R}^{\intercal}_{h_{ij}})\cdot e^{(\eth\cdot\mathscr{I}^{\intercal}_{h_{ij}})},(\eth\cdot\mathscr{R}^{\dashv}_{h_{ij}})\cdot e^{(\eth\cdot\mathscr{I}^{\dashv}_{h_{ij}})})\rangle$*, be the CT* (γ,τ)*-rung FNWAs. For any* i*, if there is* $(\eth\cdot\mathscr{R}^{\intercal}_{t_{ij}})^2\leq(\eth\cdot\mathscr{R}^{\intercal}_{h_{ij}})^2$ *and* $(\eth\cdot\mathscr{R}^{\dashv}_{t_{ij}})^2\geq(\eth\cdot\mathscr{R}^{\dashv}_{h_{ij}})^2$ *and* $(\eth\cdot\mathscr{I}^{\intercal}_{t_{ij}})^2\leq(\eth\cdot\mathscr{I}^{\intercal}_{h_{ij}})^2$ *and* $(\eth\cdot\mathscr{I}^{\dashv}_{t_{ij}})^2\geq(\eth\cdot\mathscr{I}^{\dashv}_{h_{ij}})^2$*or* $\Theta_i\leq W_i$*. Prove that CT* $(\gamma,\tau)NWA(\Theta_1,\Theta_2,...,\Theta_\ell)\leq CT(\gamma,\tau)NWA(W_1,W_2,...,W_\ell)$*, where* $(i=1,2,...,\ell);(j=1,2,...,i_j)$ *(monotonicity property).*

Proof. For any i, $(\eth\cdot\mathscr{R}^{\intercal}_{t_{ij}})^2\leq(\eth\cdot\mathscr{R}^{\intercal}_{h_{ij}})^2$.
Therefore, $1-((\eth\cdot\mathscr{R}^{\intercal}_{t_i}))^2\geq1-((\eth\cdot\mathscr{R}^{\intercal}_{h_i}))^2$.
Hence, $\bigotimes_{i=1}^{\ell}\left(1-((\eth\cdot\mathscr{R}^{\intercal}_{t_i}))^2\right)^{\omega_i}\geq\bigotimes_{i=1}^{\ell}\left(1-((\eth\cdot\mathscr{R}^{\intercal}_{h_i}))^2\right)^{\omega_i}$
and $\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-((\eth\cdot\mathscr{R}^{\intercal}_{t_i}))^{\gamma}\right)^{\omega_i}}\leq\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-((\eth\cdot\mathscr{R}^{\intercal}_{h_i}))^{\gamma}\right)^{\omega_i}}$.
Similarly, $(\eth\cdot\mathscr{I}^{\intercal}_{t_{ij}})^2\leq(\eth\cdot\mathscr{I}^{\intercal}_{h_{ij}})^2$.
Therefore, $1-((\eth\cdot\mathscr{I}^{\intercal}_{t_i}))^2\geq1-((\eth\cdot\mathscr{I}^{\intercal}_{h_i}))^2$.
Hence, $\bigotimes_{i=1}^{\ell}\left(1-((\eth\cdot\mathscr{I}^{\intercal}_{t_i}))^2\right)^{\omega_i}\geq\bigotimes_{i=1}^{\ell}\left(1-((\eth\cdot\mathscr{I}^{\intercal}_{h_i}))^2\right)^{\omega_i}$

and $\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-((\partial\cdot\mathscr{I}_{t_i}^{\intercal}))^{\gamma}\right)^{\omega_i}} \leq \sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-((\partial\cdot\mathscr{I}_{h_i}^{\intercal}))^{\gamma}\right)^{\omega_i}}$.

For any i, $\left((\partial\cdot\mathscr{R}_{t_{ij}}^{\dashv})\right)^2 \geq \left((\partial\cdot\mathscr{R}_{h_{ij}}^{\dashv})\right)^2$ and $\left((\partial\cdot\mathscr{R}_{t_{ij}}^{\dashv})\right)^{\tau} \geq \left((\partial\cdot\mathscr{R}_{h_{ij}}^{\dashv})\right)^{\tau}$.
Therefore, $1-\left(\bigotimes_{i=1}^{\ell}(\partial\cdot\mathscr{R}_{t_{ij}}^{\dashv})\right)^{\tau} \leq 1-\left(\bigotimes_{i=1}^{\ell}(\partial\cdot\mathscr{R}_{h_{ij}}^{\dashv})\right)^{\tau}$.
Similarly, for any i,
$\left((\partial\cdot\mathscr{I}_{t_{ij}}^{\dashv})\right)^2 \geq \left((\partial\cdot\mathscr{I}_{h_{ij}}^{\dashv})\right)^2$ and $\left((\partial\cdot\mathscr{I}_{t_{ij}}^{\dashv})\right)^{\tau} \geq \left((\partial\cdot\mathscr{I}_{h_{ij}}^{\dashv})\right)^{\tau}$.
Therefore, $-\left(\bigotimes_{i=1}^{\ell}(\partial\cdot\mathscr{I}_{t_{ij}}^{\dashv})\right)^{\tau} \leq -\left(\bigotimes_{i=1}^{\ell}(\partial\cdot\mathscr{I}_{h_{ij}}^{\dashv})\right)^{\tau}$.

Hence,

$$\frac{1}{2}\times\left[\left[\begin{array}{c}\left(\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-((\partial\cdot\mathscr{R}_{ti}^{\intercal}))^{\gamma}\right)^{\omega_i}}\right)^2\\ +1-\left(\bigotimes_{i=1}^{\ell}((\partial\cdot\mathscr{R}_{t_i}^{\dashv}))^{\tau}\right)^2\end{array}\right]+\left[\begin{array}{c}\left(\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-((\partial\cdot\mathscr{I}_{ti}^{\intercal}))^{\gamma}\right)^{\omega_i}}\right)^2\\ -\left(\bigotimes_{i=1}^{\ell}((\partial\cdot\mathscr{I}_{t_i}^{\dashv}))^{\tau}\right)^2\end{array}\right]\right]$$

$$\leq \frac{1}{2}\times\left[\left[\begin{array}{c}\left(\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-((\partial\cdot\mathscr{R}_{hi}^{\intercal}))^{\gamma}\right)^{\omega_i}}\right)^2\\ +1-\left(\bigotimes_{i=1}^{\ell}((\partial\cdot\mathscr{R}_{h_i}^{\dashv}))^{\tau}\right)^2\end{array}\right]+\left[\begin{array}{c}\left(\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-((\partial\cdot\mathscr{I}_{hi}^{\intercal}))^{\gamma}\right)^{\omega_i}}\right)^2\\ -\left(\bigotimes_{i=1}^{\ell}((\partial\cdot\mathscr{I}_{h_i}^{\dashv}))^{\tau}\right)^2\end{array}\right]\right].$$

Hence, CT $(\gamma,\tau)NWA\,(\Theta_1,\Theta_2,...,\Theta_\ell) \leq CT(\gamma,\tau)NWA\,(W_1,W_2,...,W_\ell)$. □

4.2 CT (γ,τ)-rung FNWG

Definition 4.6. Let $\Theta_i = \left\langle\left(((\partial\cdot\mathscr{R}_i^{\intercal})\cdot e^{(\partial\cdot\mathscr{I}_i^{\intercal})}, (\partial\cdot\mathscr{R}_i^{\dashv})\cdot e^{(\partial\cdot\mathscr{I}_i^{\dashv})})\right)\right\rangle$ be the CT (γ,τ)-rung FNs. Then (γ,τ)-rung FNWG $(\Theta_1,\Theta_2,...,\Theta_\ell) = \bigotimes_{i=1}^{\ell}\Theta_i^{\omega_i}$.

Corollary 4.7. *Let* $\Theta_i = \left\langle\left(((\partial\cdot\mathscr{R}_i^{\intercal})\cdot e^{(\partial\cdot\mathscr{I}_i^{\intercal})}, (\partial\cdot\mathscr{R}_i^{\dashv})\cdot e^{(\partial\cdot\mathscr{I}_i^{\dashv})})\right)\right\rangle$ *be the CT* (γ,τ)*-rung FNs. Then CT* (γ,τ)*-rung FNWG* $(\Theta_1,\Theta_2,...,\Theta_\ell)$

$$=\left[\begin{array}{c}\bigotimes_{i=1}^{\ell}(((\partial\cdot\mathscr{R}_i^{\intercal}))^{\gamma})^{\omega_i}\cdot e^{\bigotimes_{i=1}^{\ell}(((\partial\cdot\mathscr{I}_i^{\intercal}))^{\gamma})^{\omega_i}}\\ \sqrt[\tau]{1-\bigotimes_{i=1}^{\ell}\left(1-((\partial\cdot\mathscr{R}_i^{\dashv}))^{\tau}\right)^{\omega_i}}\cdot e^{\sqrt[\tau]{1-\bigotimes_{i=1}^{\ell}\left(1-((\partial\cdot\mathscr{I}_i^{\dashv}))^{\tau}\right)^{\omega_i}}}\end{array}\right].$$

Corollary 4.8. *(i) Let* $\Theta_i = \left\langle\left(((\partial\cdot\mathscr{R}_i^{\intercal})\cdot e^{(\partial\cdot\mathscr{I}_i^{\intercal})}, (\partial\cdot\mathscr{R}_i^{\dashv})\cdot e^{(\partial\cdot\mathscr{I}_i^{\dashv})})\right)\right\rangle$ *be the CT* (γ,τ)*-rung FNs and all are equal. Then* (γ,τ)*-rung FNWG*$(\Theta_1,\Theta_2,...,\Theta_\ell)=\Theta$.
(ii) It has other properties, including boundedness and monotonicity, as well as having (γ,τ)*-rung FNWG.*

4.3 Generalized CT (γ,τ)-rung FNWA (GCT(γ,τ)-rung FNWA)

Definition 4.9. Let $\Theta_i = \left\langle\ (((\partial\cdot\mathscr{R}_i^{\intercal}),(\partial\cdot\mathscr{R}_i^{\dashv})))\ \right\rangle$ be the CT (γ,τ)-rung FN. Then GCT (γ,τ)-rung FNWA $(\Theta_1,\Theta_2,...,\Theta_\ell) = \left(\bigoplus_{i=1}^{\ell}\omega_i\Theta_i^{\partial}\right)^{1/\partial}$.

Theorem 4.10. *Let* $\Theta_i = \left\langle (((\eth\cdot\mathscr{R}_i^{\mathsf{T}}),(\eth\cdot\mathscr{R}_i^{\dashv}))) \right\rangle$ *be the CT* (γ,τ)*-rung FNs. Then GCT* (γ,τ)*-rung FNWA* $(\Theta_1,\Theta_2,...,\Theta_\ell)$

$$
= \left[\begin{array}{c} \left(\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-\left(((\eth\cdot\mathscr{R}_i^{\mathsf{T}}))^{\gamma}\right)^{\gamma}\right)^{\omega_i}}\right)^{1/\gamma}\cdot e^{\left(\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-\left(((\eth\cdot\mathscr{I}_i^{\mathsf{T}}))^{\gamma}\right)^{\gamma}\right)^{\omega_i}}\right)^{1/\gamma}}, \\ \sqrt[\tau]{1-\left(1-\left(\bigotimes_{i=1}^{\ell}\left(\sqrt[\tau]{1-\left(1-((\eth\cdot\mathscr{R}_i^{\dashv}))^{\tau}\right)^{\tau}}\right)^{\omega_i}\right)\right)^{\tau}}^{1/\tau}\cdot \\ e^{\sqrt[\tau]{1-\left(1-\left(\bigotimes_{i=1}^{\ell}\left(\sqrt[\tau]{1-\left(1-((\eth\cdot\mathscr{I}_i^{\dashv}))^{\tau}\right)^{\tau}}\right)^{\omega_i}\right)\right)^{\tau}}^{1/\tau}} \end{array} \right].
$$

Proof. To illustrate this, we may first show that,

$$
\bigoplus_{i=1}^{\ell}\omega_i\Theta_i^{\gamma} = \left[\begin{array}{c} \sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-\left(((\eth\cdot\mathscr{R}_i^{\mathsf{T}}))^{\gamma}\right)^{\gamma}\right)^{\omega_i}}\cdot e^{\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-\left(((\eth\cdot\mathscr{I}_i^{\mathsf{T}}))^{\gamma}\right)^{\gamma}\right)^{\omega_i}}} \\ \bigotimes_{i=1}^{\ell}\left(\sqrt[\tau]{1-\left(1-((\eth\cdot\mathscr{R}_i^{\dashv}))^{\tau}\right)^{\tau}}\right)^{\omega_i}\cdot e^{\bigotimes_{i=1}^{\ell}\left(\sqrt[\tau]{1-\left(1-((\eth\cdot\mathscr{I}_i^{\dashv}))^{\tau}\right)^{\tau}}\right)^{\omega_i}} \end{array} \right].
$$

Put $\ell = 2$, $\omega_1\Theta_1 \bigoplus \omega_2\Theta_2$

$$
= \left[\begin{array}{c} \sqrt[\gamma]{\begin{array}{c}\left(\sqrt[\gamma]{1-\left(1-\left(((\eth\cdot\mathscr{R}_1^{\mathsf{T}}))^{\gamma}\right)^{\gamma}\right)^{\omega_1}}\right)^{\gamma}+\left(\sqrt[\gamma]{1-\left(1-\left(((\eth\cdot\mathscr{R}_2^{\mathsf{T}}))^{\gamma}\right)^{\gamma}\right)^{\omega_1}}\right)^{\gamma} \\ -\left(\sqrt[\gamma]{1-\left(1-\left(((\eth\cdot\mathscr{R}_1^{\mathsf{T}}))^{\gamma}\right)^{\gamma}\right)^{\omega_1}}\right)^{\gamma}\cdot\left(\sqrt[\gamma]{1-\left(1-\left(((\eth\cdot\mathscr{R}_2^{\mathsf{T}}))^{\gamma}\right)^{\gamma}\right)^{\omega_1}}\right)^{\gamma}\end{array}} \\ \cdot e^{\sqrt[\gamma]{\begin{array}{c}\left(\sqrt[\gamma]{1-\left(1-\left(((\eth\cdot\mathscr{I}_1^{\mathsf{T}}))^{\gamma}\right)^{\gamma}\right)^{\omega_1}}\right)^{\gamma}+\left(\sqrt[\gamma]{1-\left(1-\left(((\eth\cdot\mathscr{I}_2^{\mathsf{T}}))^{\gamma}\right)^{\gamma}\right)^{\omega_1}}\right)^{\gamma} \\ -\left(\sqrt[\gamma]{1-\left(1-\left(((\eth\cdot\mathscr{I}_1^{\mathsf{T}}))^{\gamma}\right)^{\gamma}\right)^{\omega_1}}\right)^{\gamma}\cdot\left(\sqrt[\gamma]{1-\left(1-\left(((\eth\cdot\mathscr{I}_2^{\mathsf{T}}))^{\gamma}\right)^{\gamma}\right)^{\omega_1}}\right)^{\gamma}\end{array}}}, \\ \left(\sqrt[\tau]{1-\left(1-((\eth\cdot\mathscr{R}_1^{\dashv}))^{\tau}\right)^{\tau}}\right)^{\omega_1}\cdot\left(\sqrt[\tau]{1-\left(1-((\eth\cdot\mathscr{R}_2^{\dashv}))^{\tau}\right)^{\tau}}\right)^{\omega_1} \\ \cdot e^{\left(\sqrt[\tau]{1-\left(1-((\eth\cdot\mathscr{I}_1^{\dashv}))^{\tau}\right)^{\tau}}\right)^{\omega_1}\cdot\left(\sqrt[\tau]{1-\left(1-((\eth\cdot\mathscr{I}_2^{\dashv}))^{\tau}\right)^{\tau}}\right)^{\omega_1}} \end{array} \right]
$$

$$
= \left[\begin{array}{c} \sqrt[\gamma]{1-\bigotimes_{i=1}^{2}\left(1-\left(((\eth\cdot\mathscr{R}_1^{\mathsf{T}}))^{\gamma}\right)^{\gamma}\right)^{\omega_i}}\cdot e^{\sqrt[\gamma]{1-\bigotimes_{i=1}^{2}\left(1-\left(((\eth\cdot\mathscr{I}_1^{\mathsf{T}}))^{\gamma}\right)^{\gamma}\right)^{\omega_i}}} \\ \bigotimes_{i=1}^{2}\left(\sqrt[\tau]{1-\left(1-((\eth\cdot\mathscr{R}_i^{\dashv}))^{\tau}\right)^{\tau}}\right)^{\omega_i}\cdot\bigotimes_{i=1}^{2}\left(\sqrt[\tau]{1-\left(1-((\eth\cdot\mathscr{I}_i^{\dashv}))^{\tau}\right)^{\tau}}\right)^{\omega_i} \end{array} \right].
$$

Hence,

$$\bigoplus_{i=1}^{\ell}\omega_i\Theta_i^{\partial} = \begin{bmatrix} \sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-\left(((\partial\cdot\mathscr{R}_1^{\intercal}))^{\gamma}\right)^{\gamma}\right)^{\omega_i}}\cdot e^{\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-\left(((\partial\cdot\mathscr{I}_1^{\intercal}))^{\gamma}\right)^{\gamma}\right)^{\omega_i}}} \\ \bigotimes_{i=1}^{\ell}\left(\sqrt[\tau]{1-\left(1-((\partial\cdot\mathscr{R}_i^{\dashv}))^{\tau}\right)^{\tau}}\right)^{\omega_i}\cdot e^{\bigotimes_{i=1}^{\ell}\left(\sqrt[\tau]{1-\left(1-((\partial\cdot\mathscr{I}_i^{\dashv}))^{\tau}\right)^{\tau}}\right)^{\omega_i}} \end{bmatrix}.$$

If $\ell=\ell+1$, then $\bigoplus_{i=1}^{\ell}\omega_i\Theta_i^{\partial}+\omega_{\ell+1}\Theta_{\ell+1}^{\partial}=\bigoplus_{i=1}^{\ell+1}\omega_i\Theta_i^{\partial}$.
Now,$\bigoplus_{i=1}^{\ell}\omega_i\Theta_i^{\partial}+\omega_{\ell+1}\Theta_{\ell+1}^{\partial}=\omega_1\Theta_1^{\partial}\bigoplus\omega_2\Theta_2^{\partial}\bigoplus...\bigoplus\omega_\ell\Theta_\ell^{\partial}\bigoplus\omega_{\ell+1}\Theta_{\ell+1}^{\partial}$

$$= \begin{bmatrix} \sqrt[\gamma]{\begin{array}{l}\left(\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-\left(((\partial\cdot\mathscr{R}_i^{\intercal}))^{\gamma}\right)^{\gamma}\right)^{\omega_i}}\right)^{\gamma}+\left(\sqrt[\gamma]{1-\left(1-\left((\mathscr{R}_{\ell+1}^{\intercal})^{\gamma}\right)^{\gamma}\right)^{\omega_1}}\right)^{\gamma}\\ -\left(\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-\left(((\partial\cdot\mathscr{R}_i^{\intercal}))^{\gamma}\right)^{\gamma}\right)^{\omega_i}}\right)^{\gamma}\cdot\left(\sqrt[\gamma]{1-\left(1-\left((\mathscr{R}_{\ell+1}^{\intercal})^{\gamma}\right)^{\gamma}\right)^{\omega_1}}\right)^{\gamma}\end{array}} \\ \cdot e^{\sqrt[\gamma]{\begin{array}{l}\left(\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-\left(((\partial\cdot\mathscr{I}_i^{\intercal}))^{\gamma}\right)^{\gamma}\right)^{\omega_i}}\right)^{\gamma}+\left(\sqrt[\gamma]{1-\left(1-\left((\mathscr{I}_{\ell+1}^{\intercal})^{\gamma}\right)^{\gamma}\right)^{\omega_1}}\right)^{\gamma}\\ -\left(\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell}\left(1-\left(((\partial\cdot\mathscr{I}_i^{\intercal}))^{\gamma}\right)^{\gamma}\right)^{\omega_i}}\right)^{\gamma}\cdot\left(\sqrt[\gamma]{1-\left(1-\left((\mathscr{I}_{\ell+1}^{\intercal})^{\gamma}\right)^{\gamma}\right)^{\omega_1}}\right)^{\gamma}\end{array}}} \\ \bigotimes_{i=1}^{\ell}\left(\sqrt[\tau]{1-\left(1-((\partial\cdot\mathscr{R}_i^{\dashv}))^{\tau}\right)^{\tau}}\right)^{\omega_i}\cdot\left(\sqrt[\tau]{1-\left(1-(\mathscr{R}_{\ell+1}^{\dashv})^{\tau}\right)^{\tau}}\right)^{\omega_1} \\ \cdot e^{\bigotimes_{i=1}^{\ell}\left(\sqrt[\tau]{1-\left(1-((\partial\cdot\mathscr{I}_i^{\dashv}))^{\tau}\right)^{\tau}}\right)^{\omega_i}\cdot\left(\sqrt[\tau]{1-\left(1-(\mathscr{I}_{\ell+1}^{\dashv})^{\tau}\right)^{\tau}}\right)^{\omega_1}} \end{bmatrix}$$

$$\bigoplus_{i=1}^{\ell+1}\omega_i\Theta_i^{\gamma} = \begin{bmatrix} \sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell+1}\left(1-\left(((\partial\cdot\mathscr{R}_1^{\intercal}))^{\gamma}\right)^{\gamma}\right)^{\omega_i}}\cdot e^{\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell+1}\left(1-\left(((\partial\cdot\mathscr{I}_1^{\intercal}))^{\gamma}\right)^{\gamma}\right)^{\omega_i}}} \\ \bigotimes_{i=1}^{\ell+1}\left(\sqrt[\tau]{1-\left(1-((\partial\cdot\mathscr{R}_i^{\dashv}))^{\tau}\right)^{\tau}}\right)^{\omega_i}\cdot e^{\bigotimes_{i=1}^{\ell+1}\left(\sqrt[\tau]{1-\left(1-((\partial\cdot\mathscr{I}_i^{\dashv}))^{\tau}\right)^{\tau}}\right)^{\omega_i}} \end{bmatrix}.$$

$$\left(\bigoplus_{i=1}^{\ell+1}\omega_i\Theta_i^{\partial}\right)^{1/\partial} = \begin{bmatrix} \left(\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell+1}\left(1-\left(((\partial\cdot\mathscr{R}_i^{\intercal}))^{\gamma}\right)^{\gamma}\right)^{\omega_i}}\right)^{1/\gamma}\cdot e^{\left(\sqrt[\gamma]{1-\bigotimes_{i=1}^{\ell+1}\left(1-\left(((\partial\cdot\mathscr{I}_i^{\intercal}))^{\gamma}\right)^{\gamma}\right)^{\omega_i}}\right)^{1/\gamma}} \\ \sqrt[\tau]{1-\left(1-\left(\bigotimes_{i=1}^{\ell+1}\left(\sqrt[\tau]{1-\left(1-((\partial\cdot\mathscr{R}_i^{\dashv}))^{\tau}\right)^{\tau}}\right)^{\omega_i}\right)^{2}\right)^{1/\tau}}\cdot \\ e^{\sqrt[\tau]{1-\left(1-\left(\bigotimes_{i=1}^{\ell+1}\left(\sqrt[\tau]{1-\left(1-((\partial\cdot\mathscr{I}_i^{\dashv}))^{\tau}\right)^{\tau}}\right)^{\omega_i}\right)^{2}\right)^{1/\tau}}} \end{bmatrix}$$

□

Corollary 4.11. *(i) If $(\gamma,\tau)=(1,1)$, then CT (γ,τ)-rung FNWA operator is used instead of the GCT (γ,τ)-rung FNWA operator.*
(ii) If all $\Theta_i=\left\langle\left(((\partial\cdot\mathscr{R}_i^{\intercal})\cdot e^{(\partial\cdot\mathscr{I}_i^{\intercal})},\,(\partial\cdot\mathscr{R}_i^{\dashv})\cdot e^{(\partial\cdot\mathscr{I}_i^{\dashv})})\right)\right\rangle$ and all are equal. Then GCT (γ,τ)-rung FNWA$(\Theta_1,\Theta_2,...,\Theta_\ell)=\Theta$.
(iii) The GCT (γ,τ)-rung FNWA operator meets both boundedness and monotonicity constraints.

4.4 Generalized CT (γ,τ)-rung FNWG (GCT (γ,τ)-rung FNWG)

Definition 4.12. Let $\Theta_i = \left\langle \left(((\eth \cdot \mathscr{R}_i^{\mathsf{T}}) \cdot e^{(\eth\cdot \mathscr{I}_i^{\mathsf{T}})}, (\eth \cdot \mathscr{R}_i^{\dashv}) \cdot e^{(\eth\cdot \mathscr{I}_i^{\dashv})}) \right) \right\rangle$ be the CT (γ,τ)-rung FNs. Then GCT(γ,τ)-rung FNWG $(\Theta_1,\Theta_2,...,\Theta_\ell) = \frac{1}{\partial}\left(\bigotimes_{i=1}^{\ell} (\partial\Theta_i)^{\omega_i} \right)$.

Corollary 4.13. *Let* $\Theta_i = \left\langle \left(((\eth \cdot \mathscr{R}_i^{\mathsf{T}}) \cdot e^{(\eth\cdot \mathscr{I}_i^{\mathsf{T}})}, (\eth \cdot \mathscr{R}_i^{\dashv}) \cdot e^{(\eth\cdot \mathscr{I}_i^{\dashv})}) \right) \right\rangle$ *be the CT* (γ,τ)*-rung FNs. Then GCT*(γ,τ)*-rung FNWG*$(\Theta_1,\Theta_2,...,\Theta_\ell)$

$$= \left[\begin{array}{c} \sqrt[\gamma]{1-\left(1-\left(\bigotimes_{i=1}^{\ell}\left(\sqrt[\gamma]{1-\left(1-((\eth\cdot\mathscr{R}_i^{\mathsf{T}}))^{\gamma}\right)^{\gamma}}\right)^{\omega_i}\right)^{\gamma}\right)^{1/\gamma}} \cdot e^{\sqrt[\gamma]{1-\left(1-\left(\bigotimes_{i=1}^{\ell}\left(\sqrt[\gamma]{1-\left(1-((\eth\cdot\mathscr{I}_i^{\mathsf{T}}))^{\gamma}\right)^{\gamma}}\right)^{\omega_i}\right)^{\gamma}\right)^{1/\gamma}}} \\ \left(\sqrt[\tau]{1-\bigotimes_{i=1}^{\ell}\left(1-\left(((\eth\cdot\mathscr{R}_i^{\dashv}))^{\tau}\right)^{\tau}\right)^{\omega_i}}\right)^{1/\tau} \cdot e^{\left(\sqrt[\tau]{1-\bigotimes_{i=1}^{\ell}\left(1-\left(((\eth\cdot\mathscr{I}_i^{\dashv}))^{\tau}\right)^{\tau}\right)^{\omega_i}}\right)^{1/\tau}} \end{array} \right].$$

Corollary 4.14. *(i) When* $\partial = 1$*, the GCT* (γ,τ)*-rung FNWG is converted to the* (γ,τ)*-rung FNWG.*
(ii) GCT(γ,τ)*-rung FNWG operators satisfy the boundedness and monotonicity characteristics.*
(iii) If all $\Theta_i = \left\langle \left(((\eth \cdot \mathscr{R}_i^{\mathsf{T}}) \cdot e^{(\eth\cdot \mathscr{I}_i^{\mathsf{T}})}, (\eth \cdot \mathscr{R}_i^{\dashv}) \cdot e^{(\eth\cdot \mathscr{I}_i^{\dashv})}) \right) \right\rangle$ *are equal.*
Then GCT(γ,τ)*-rung FNWG*$(\Theta_1,\Theta_2,...,\Theta_\ell) = \Theta$.

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