Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol 32 No. 5s (2025)

Different Weighted Operators such as Generalized Averaging and Generalized Geometric based on Trigonometric $\wp$-rung Interval-Valued Approach

Raed Hatamleh[1], **Abdallah Al-Husban**[2,3], **M.Palanikumar**[4,*], **K. Sundareswari**[5]
[1]Department of Mathematics, Faculty of Science, Jadara University, P.O. Box 733, Irbid 21110, Jordan.
[2]Department of Mathematics, Faculty of Science, Irbid National University, P.O. Box: 2600 Irbid, Jordan.
[3]Jadara Research Center, Jadara University, Irbid 21110, Jordan.
[5]Department of Mathematics, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai-602105, India.
[4]Department of Mathematics, Al- Ameen Engineering College, Erode.
E-mails:[1]raed@jadara.edu.jo, [3]dralhosban@inu.edu.jo, [4]palanimaths86@gmail.com, [5]sundarimaths@gmail.com
*Corresponding author: M.Palanikumar.
Received: 03-10-2024 Revised: 25-11-2024 Accepted: 04-12-2024.

Abstract

A new technique to establish tangent trigonometric $\wp$-rung interval-valued sets is presented in this study. Tangent trigonometric $\wp$-rung interval-valued weighted averaging, geometric, and generalized concepts will all be covered in this article. To obtain the weighted average and geometric, we utilized an aggregating model. Using algebraic approaches, a number of sets with significant properties will be further examined.

Keywords: $\wp$-rung, WA, WG, GWA, GWG.

1 Introduction

To explain uncertainty, a number of theories have been put forth, including fuzzy sets (FS),[1] which have membership grades (MG) that range from 0. Atanassov[2] constructed an intuitionistic FS (IFS) for $\varsigma, \eta \in [0, 1]$ using two MGs: $0 \leq \varsigma + \eta \leq 1$ and positive ς and negative η. Yager[3] developed the Pythagorean FSs (PFS) idea, which is distinguished by its MG and non-MG (NMG) with $\varsigma + \eta \geq 1$ to $\varsigma^2 + \eta^2 \leq 1$. Numerous studies have examined the use of IFSs and PFSs in various fields. Their ability to communicate information is still restricted. Because of this, the experts were still having trouble interpreting the data in these sets and the associated data. Wang et al.[4] investigated the concept of complex IFS with DOMBI prioritized AOs and its application for trustworthy green supplier selection. According to Cuong et al.,[12] the three primary ideas of the picture FS are positive MG (ς), neutral MG ($\wp$), and negative MG (η). Additionally, it offers greater benefits than PFS and IFS. Since $\varsigma, \eta \in [0, 1]$, it has been noted that the picture FS is an upgrade of the IFS that may handle greater inconsistency and $0 \leq \varsigma + \eta \leq 1$. According on the picture FS description, expert comments like "yes," "abstain," "no," and "refusal" will be supplied.

Shahzaib et al.[13] defined the SFS for certain AOs using MADM. SFS requires that $0 \leq \varsigma^2 + \eta^2 \leq 1$ rather than $0 \leq \varsigma + \eta \leq 1$. Hussain et al.[14] first proposed the concept of an intelligent decision support system for SFS. SFSs and their applications in DM were initially presented by Rafiq et al.[15] For instance, $\varsigma^2 + \eta^2 \geq 1$ is a DM problem with a property. Senapati et al.[16] invented Fermatean FS (FFS) in 2019 with the condition that $0 \leq \varsigma^3 + \eta^3 \leq 1$. The concept of generalized orthopair FSs was initially proposed by Yager.[17] In the $\wp$-rung orthogonal pair FS ($\wp$-ROFS), both the MG and the NMG have power $\wp$; however, their sum can never be more than one. Recently many authors discussed the new research and its aggregating operators[19]-.[28] Palanikumar et al.[29–31] investigated a variety of algebraic structures and aggregation methods with applications. Hatamleh[32]-[37] discussed the various real-life applications such as Uryson's operator and linear operators. The rest of this work will be completed in the manner described below. For an introduction, see section 1. In Section 2 deals that basic concepts. Section 3 describes a number of techniques on $\wp$-rung IVNs. Section 4 discusses the AOs based on IVT $\wp$-rung. The conclusion is covered in section 5.

2 Background

This section has several crucial definitions that we should examine for future learning.

Definition 2.1. Let $\mathcal{A}$ be a universal. The PFS $\mho = \Big\{\varkappa, \big\langle \mathcal{Z}^T(\varkappa), \mathcal{Z}^F(\varkappa)\big\rangle \big| \varkappa \in \mathcal{A}\Big\}$, $\mathcal{Z}^T : \mathcal{A} \to (0,1)$ and $\mathcal{Z}^F : \mathcal{A} \to (0,1)$ called the MG and NMG of $\varkappa \in \mathcal{A}$ to $\mho$, respectively and $0 \preceq (\mathcal{Z}^T(\varkappa))^2 + (\mathcal{Z}^F(\varkappa))^2 \preceq 1$. For $\mho = \big\langle \mathcal{Z}^T, \mathcal{Z}^F \big\rangle$ is called a Pythagorean fuzzy number (PFN).

Definition 2.2. Let $\mho_1 = (a_1, b_1) \in N$and $\mho_2 = (a_2, b_2) \in N$. Then the distance between $\mho_1$ and $\mho_2$ is defined as $\aleph(\mho_1, \mho_2) = \sqrt{(a_1 - a_2)^2 + \frac{1}{2}(b_1 - b_2)^2}$, where N is a natural number.

3 Operations for IVT $\wp$ -rung Number

We present the concept of the $\wp$ IV-rung N, which is a tangent trigonometric. As a consequence, the IVT $\wp$ -rung N and its operations were established and $\tan \pi/2 = \varrho$

Definition 3.1. The $\wp$ -rung $\mho = \Big\{\varkappa, \Big\langle \Big([(\varrho \odot \Xi)(\varkappa), (\varrho \odot \Upsilon)(\varkappa)], [(\varrho \odot \Psi)(\varkappa), (\varrho \odot \Omega)(\varkappa)] \Big) \Big\rangle \Big| \varkappa \in \mathcal{A}\Big\}$, where $(\varrho \odot \Xi), (\varrho \odot \Psi) : \mathcal{A} \to (0,1)$ denote the MG, IMG and NMG of $\varkappa \in \mathcal{A}$ to $\mho$, respectively and $0 \leq ((\varrho \odot \Upsilon)(\varkappa))^\wp + ((\varrho \odot \Omega)(\varkappa))^\wp \leq 1$. For convenience, $\mho = \Big\langle \Big([(\varrho \odot \Xi), (\varrho \odot \Upsilon)], [(\varrho \odot \Psi), (\varrho \odot \Omega)] \Big) \Big\rangle$ is represent a IVT $\wp$ -rung N.

Definition 3.2. Let $\mho = \Big\langle \Big([(\varrho \odot \Xi), (\varrho \odot \Upsilon)], [(\varrho \odot \Psi), (\varrho \odot \Omega)] \Big) \Big\rangle, \mho_1 = \langle ([(\varrho \odot \Xi_1), (\varrho \odot \Upsilon_1)], [(\varrho \odot \Psi_1), (\varrho \odot \Omega_1)]) \rangle$,
$\mho_2 = \langle ([(\varrho \odot \Xi_2), (\varrho \odot \Upsilon_2)], [(\varrho \odot \Psi_2), (\varrho \odot \Omega_2)]) \rangle$ be any three IVT $\wp$ -rung Ns, and $\wp > 0$. Then

1. $$\mho_1 \triangle \mho_2 = \begin{bmatrix} \sqrt[\wp]{\begin{matrix} ((\varrho \odot \Xi_1))^\wp + ((\varrho \odot \Xi_2))^\wp \\ -((\varrho \odot \Xi_1))^\wp \cdot ((\varrho \odot \Xi_2))^\wp \end{matrix}}, \sqrt[\wp]{\begin{matrix} ((\varrho \odot \Upsilon_1))^\wp + ((\varrho \odot \Upsilon_2))^\wp \\ -((\varrho \odot \Upsilon_1))^\wp \cdot ((\varrho \odot \Upsilon_2))^\wp \end{matrix}}, \\ ((\varrho \odot \Psi_1))^\wp ((\varrho \odot \Psi_2))^\wp, ((\varrho \odot \Omega_1))^\wp ((\varrho \odot \Omega_2))^\wp \end{bmatrix},$$
2. $$\mho_1 \oplus \mho_2 = \begin{bmatrix} ((\varrho \odot \Xi_1))^\wp ((\varrho \odot \Xi_2))^\wp, ((\varrho \odot \Upsilon_1))^\wp ((\varrho \odot \Upsilon_2))^\wp, \\ \sqrt[\wp]{\begin{matrix} ((\varrho \odot \Psi_1))^\wp + ((\varrho \odot \Psi_2))^\wp \\ -((\varrho \odot \Psi_1))^\wp \cdot ((\varrho \odot \Psi_2))^\wp \end{matrix}}, \sqrt[\wp]{\begin{matrix} ((\varrho \odot \Omega_1))^\wp + ((\varrho \odot \Omega_2))^\wp \\ -((\varrho \odot \Omega_1))^\wp \cdot ((\varrho \odot \Omega_2))^\wp \end{matrix}} \end{bmatrix}$$
3. $$\chi \cdot \mho = \begin{bmatrix} \sqrt[\wp]{1 - \big(1 - (\varrho \odot (\Xi)^\wp\big)^\chi}, \sqrt[\wp]{1 - \big(1 - (\varrho \odot (\Upsilon)^\wp\big)^\chi}, \\ ((\varrho \odot (\Psi)^\wp)^\chi, ((\varrho \odot (\Omega)^\wp)^\chi \end{bmatrix}$$
4. $$\mho^\chi = \begin{bmatrix} ((\varrho \odot (\Xi)^\wp)^\chi, ((\varrho \odot (\Upsilon)^\wp)^\chi, \\ \sqrt[\wp]{1 - \big(1 - (\varrho \odot (\Psi)^\wp\big)^\chi}, \sqrt[\wp]{1 - \big(1 - (\varrho \odot (\Omega)^\wp\big)^\chi} \end{bmatrix}.$$

We present ED and HD measures for IVT $\wp$ -rung Ns and investigate their mathematical characteristics.

Definition 3.3. For any two IVT $\wp$ -rung Ns $\mho_1 = \langle ([(\varrho \odot \Xi_1), (\varrho \odot \Upsilon_1)], [(\varrho \odot \Psi_1), (\varrho \odot \Omega_1)]) \rangle, \mho_2 = \langle ([(\varrho \odot \Xi_2), (\varrho \odot \Upsilon_2)], [(\varrho \odot \Psi_2), (\varrho \odot \Omega_2)]) \rangle$. Then

$$\aleph_E(\mho_1, \mho_2) = \sqrt{\frac{1}{2} \left[\begin{bmatrix} 1 + ((\varrho \odot \Xi_1))^2 - ((\varrho \odot \Psi_1))^2 \\ -\big(1 + ((\varrho \odot \Xi_2))^2 - ((\varrho \odot \Psi_2))^2\big) \end{bmatrix}^2 + \begin{bmatrix} ((\varrho \odot \Upsilon_1))^2 - ((\varrho \odot \Omega_1))^2 \\ -\big(((\varrho \odot \Upsilon_2))^2 - ((\varrho \odot \Omega_2))^2\big) \end{bmatrix}^2 \right]}$$

where $\aleph_E(\mho_1, \mho_2)$ is called the ED between $\mho_1$ and $\mho_2$.

$$\aleph_H(\mho_1, \mho_2) = \frac{1}{2} \left[\left| \begin{matrix} 1 + ((\varrho \odot \Xi_1))^2 - ((\varrho \odot \Psi_1))^2 \\ -\big(1 + ((\varrho \odot \Xi_2))^2 - ((\varrho \odot \Psi_2))^2\big) \end{matrix} \right| + \left| \begin{matrix} ((\varrho \odot \Upsilon_1))^2 - ((\varrho \odot \Omega_1))^2 \\ \big(-((\varrho \odot \Upsilon_2))^2 - ((\varrho \odot \Omega_2))^2\big) \end{matrix} \right| \right]$$

where $\aleph_H(\mho_1, \mho_2)$ is called the HD between $\mho_1$ and $\mho_2$.

4 Aggregating operators

We use IVT $\wp$-rung WA, IVT $\wp$-rung WG, GIVT $\wp$-rung WA, and GIVT $\wp$-rung WG to describe the AOs.

4.1 IVT $\wp - rungNWA$

Definition 4 .1. Let $\mho_i = \langle(([(\varrho\odot\Xi_i),(\varrho\odot\Upsilon_i)],[(\varrho\odot\Psi_i),(\varrho\odot\Omega_i)]))\rangle$ be the IVT $\wp$ -rung Ns, $W = (\delta_1,\delta_2,...,\delta_l)$ be the weight of $\mho_i$, $\delta_i \geq 0$ and $\triangle_{i=1}^{l}\delta_i = 1$. Then IVT $\wp$-rung WA $(\mho_1,\mho_2,...,\mho_l) = \triangle_{i=1}^{l}\delta_i\mho_i$.

Theorem 4.2. *Let* $\mho_i = \langle(([(\varrho\odot\Xi_i),(\varrho\odot\Upsilon_i)],[(\varrho\odot\Psi_i),(\varrho\odot\Omega_i)]))\rangle$ *be the IVT* $\wp$ *-rung Ns. Then IVT* $\wp - rungNWA(\mho_1,\mho_2,...,\mho_l)$

$$= \begin{bmatrix} \sqrt[\wp]{1-\nabla_{i=1}^{l}\Big(1-((\varrho\odot\Xi_i))^{\wp}\Big)^{\delta_i}}, \sqrt[\wp]{1-\nabla_{i=1}^{l}\Big(1-((\varrho\odot\Upsilon_i))^{\wp}\Big)^{\delta_i}}, \\ \nabla_{i=1}^{l}(((\varrho\odot\Psi_i))^{\wp})^{\delta_i}, \nabla_{i=1}^{l}(((\varrho\odot\Omega_i))^{\wp})^{\delta_i} \end{bmatrix}.$$

Proof. If $l = 2$, then IVT $\wp$-rung WA$(\mho_1,\mho_2) = \delta_1\mho_1 \triangle \delta_2\mho_2$, where

$$\delta_1\mho_1 = \begin{bmatrix} \sqrt[\wp]{1-\Big(1-((\varrho\odot\Xi_1))^{\wp}\Big)^{\delta_1}}, \sqrt[\wp]{1-\Big(1-((\varrho\odot\Upsilon_1))^{\wp}\Big)^{\delta_1}}, \\ (((\varrho\odot\Psi_1))^{\wp})^{\delta_1}, (((\varrho\odot\Omega_1))^{\wp})^{\delta_1} \end{bmatrix}$$

$$\delta_2\mho_2 = \begin{bmatrix} \sqrt[\wp]{1-\Big(1-((\varrho\odot\Xi_2))^{\wp}\Big)^{\delta_2}}, \sqrt[\wp]{1-\Big(1-((\varrho\odot\Upsilon_2))^{\wp}\Big)^{\delta_2}}, \\ (((\varrho\odot\Psi_2))^{\wp})^{\delta_2}, (((\varrho\odot\Omega_2))^{\wp})^{\delta_2} \end{bmatrix}.$$

Now, $\delta_1\mho_1 \triangle \delta_2\mho_2$

$$= \begin{bmatrix} \sqrt[\wp]{\begin{matrix}\Big(1-\Big(1-((\varrho\odot\Xi_1))^{\wp}\Big)^{\delta_1}\Big)+ \\ \Big(1-\Big(1-((\varrho\odot\Xi_2))^{\wp}\Big)^{\delta_2}\Big) \\ -\Big(1-\Big(1-((\varrho\odot\Xi_1))^{\wp}\Big)^{\delta_1}\Big)\cdot \\ \Big(1-\Big(1-((\varrho\odot\Xi_2))^{\wp}\Big)^{\delta_2}\Big)\end{matrix}}, \sqrt[\wp]{\begin{matrix}\Big(1-\Big(1-((\varrho\odot\Upsilon_1))^{\wp}\Big)^{\delta_1}\Big)+ \\ \Big(1-\Big(1-((\varrho\odot\Upsilon_2))^{\wp}\Big)^{\delta_2}\Big) \\ -\Big(1-\Big(1-((\varrho\odot\Upsilon_1))^{\wp}\Big)^{\delta_1}\Big)\cdot \\ \Big(1-\Big(1-((\varrho\odot\Upsilon_2))^{\wp}\Big)^{\delta_2}\Big)\end{matrix}} \\ (((\varrho\odot\Psi_1))^{\wp})^{\delta_1}, (((\varrho\odot\Psi_2))^{\wp})^{\delta_2}, (((\varrho\odot\Omega_1))^{\wp})^{\delta_1}, (((\varrho\odot\Omega_2))^{\wp})^{\delta_2} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[\wp]{1-\Big(1-((\varrho\odot\Xi_1))^{\wp}\Big)^{\delta_1}\Big(1-((\varrho\odot\Xi_2))^{\wp}\Big)^{\delta_2}}, \\ \sqrt[\wp]{1-\Big(1-((\varrho\odot\Upsilon_1))^{\wp}\Big)^{\delta_1}\Big(1-((\varrho\odot\Upsilon_2))^{\wp}\Big)^{\delta_2}}, \\ (((\varrho\odot\Psi_1))^{\wp})^{\delta_1}\cdot(((\varrho\odot\Psi_2))^{\wp})^{\delta_2}, \\ (((\varrho\odot\Omega_1))^{\wp})^{\delta_1}\cdot(((\varrho\odot\Omega_2))^{\wp})^{\delta_2} \end{bmatrix}$$

Hence, IVT $\wp - rungNWA(\mho_1,\mho_2)$

$$= \begin{bmatrix} \sqrt[\wp]{1-\nabla_{i=1}^{2}\Big(1-((\varrho\odot\Xi_i))^{\wp}\Big)^{\delta_i}}, \sqrt[\wp]{1-\nabla_{i=1}^{2}\Big(1-((\varrho\odot\Upsilon_i))^{\wp}\Big)^{\delta_i}}, \\ \nabla_{i=1}^{2}(((\varrho\odot\Psi_i))^{\wp})^{\delta_i}, \nabla_{i=1}^{2}(((\varrho\odot\Omega_i))^{\wp})^{\delta_i} \end{bmatrix}.$$

It valid for $l \geq 3$. Thus, IVT $\wp - rungNWA(\mho_1, \mho_2, ..., \mho_l)$

$$= \begin{bmatrix} \sqrt[\wp]{1 - \nabla_{i=1}^{l}\left(1 - ((\varrho \odot \Xi_i))^{\wp}\right)^{\delta_i}}, \sqrt[\wp]{1 - \nabla_{i=1}^{l}\left(1 - ((\varrho \odot \Upsilon_i))^{\wp}\right)^{\delta_i}}, \\ \nabla_{i=1}^{l}(((\varrho \odot \Psi_i))^{\wp})^{\delta_i}, \nabla_{i=1}^{l}(((\varrho \odot \Omega_i))^{\wp})^{\delta_i} \end{bmatrix}.$$

If $l = l + 1$, then IVT $\wp$-rung WA $(\mho_1, \mho_2, ..., \mho_l, \mho_{l+1})$

$$= \begin{bmatrix} \sqrt[\wp]{\begin{array}{c} \triangle_{i=1}^{l}\left(1 - \left(1 - ((\varrho \odot \Xi_i))^{\wp}\right)^{\delta_i}\right) + \left(1 - \left(1 - (\Xi_{l+1})^{\wp}\right)^{\delta_{l+1}}\right) \\ - \nabla_{i=1}^{l}\left(1 - \left(1 - ((\varrho \odot \Xi_i))^{\wp}\right)^{\delta_i}\right) \cdot \left(1 - \left(1 - (\Xi_{l+1})^{\wp}\right)^{\delta_{l+1}}\right) \end{array}}, \\ \sqrt[\wp]{\begin{array}{c} \triangle_{i=1}^{l}\left(1 - \left(1 - ((\varrho \odot \Upsilon_i))^{\wp}\right)^{\delta_i}\right) + \left(1 - \left(1 - (\Upsilon_{l+1})^{\wp}\right)^{\delta_{l+1}}\right) \\ - \nabla_{i=1}^{l}\left(1 - \left(1 - ((\varrho \odot \Upsilon_i))^{\wp}\right)^{\delta_i}\right) \cdot \left(1 - \left(1 - (\Upsilon_{l+1})^{\wp}\right)^{\delta_{l+1}}\right), \end{array}} \\ \nabla_{i=1}^{l}(((\varrho \odot \Psi_i))^{\wp})^{\delta_i}, ((\Psi_{l+1})^{\wp})^{\delta_{l+1}}, \nabla_{i=1}^{l}(((\varrho \odot \Omega_i))^{\wp})^{\delta_i}, ((\Omega_{l+1})^{\wp})^{\delta_{l+1}} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt[\wp]{1 - \nabla_{i=1}^{l+1}\left(1 - ((\varrho \odot \Xi_i))^{\wp}\right)^{\delta_i}}, \sqrt[\wp]{1 - \nabla_{i=1}^{l+1}\left(1 - ((\varrho \odot \Upsilon_i))^{\wp}\right)^{\delta_i}}, \\ \nabla_{i=1}^{l+1}(((\varrho \odot \Psi_i))^{\wp})^{\delta_i}, \nabla_{i=1}^{l+1}(((\varrho \odot \Omega_i))^{\wp})^{\delta_i} \end{bmatrix}.$$

□

Theorem 4.3. *Let $\mho_i = \langle((([(\varrho \odot \Xi_i), (\varrho \odot \Upsilon_i)], [(\varrho \odot \Psi_i), (\varrho \odot \Omega_i)])))\rangle$ be the IVT $\wp$ -rung Ns. Then IVT $\wp$-rung WA $(\mho_1, \mho_2, ..., \mho_l) = \mho$ (idempotency property).*

Proof. Since $(\varrho \odot \Xi_i) = (\varrho \odot \Xi)$, $(\varrho \odot \Psi_i) = (\varrho \odot \Psi)$ and $(\varrho \odot \Upsilon_i) = (\varrho \odot \Upsilon)$, $(\varrho \odot \Omega_i) = (\varrho \odot \Omega)$ and $\triangle_{i=1}^{l}\delta_i = 1$. Now, IVT $\wp - rungNWA(\mho_1, \mho_2, ..., \mho_l)$

$$\begin{aligned} &= \begin{bmatrix} \sqrt[\wp]{1 - \nabla_{i=1}^{l}\left(1 - ((\varrho \odot \Xi_i))^{\wp}\right)^{\delta_i}}, \sqrt[\wp]{1 - \nabla_{i=1}^{l}\left(1 - ((\varrho \odot \Upsilon_i))^{\wp}\right)^{\delta_i}}, \\ \nabla_{i=1}^{l}(((\varrho \odot \Psi_i))^{\wp})^{\delta_i}, \nabla_{i=1}^{l}(((\varrho \odot \Omega_i))^{\wp})^{\delta_i} \end{bmatrix} \\ &= \begin{bmatrix} \sqrt[\wp]{1 - \left(1 - (\varrho \odot (\Xi)^{\wp}\right)^{\triangle_{i=1}^{l}\delta_i}}, \sqrt[\wp]{1 - \left(1 - (\varrho \odot (\Upsilon)^{\wp}\right)^{\triangle_{i=1}^{l}\delta_i}}, \\ ((\varrho \odot (\Psi)^{\wp})^{\triangle_{i=1}^{l}\delta_i}, ((\varrho \odot (\Omega)^{\wp})^{\triangle_{i=1}^{l}\delta_i} \end{bmatrix} \\ &= \begin{bmatrix} \sqrt[\wp]{1 - \left(1 - (\varrho \odot (\Xi)^{\wp}\right)}, \sqrt[\wp]{1 - \left(1 - (\varrho \odot (\Upsilon)^{\wp}\right)}, \\ (\varrho \odot (\Psi)^{\wp}, (\varrho \odot (\Omega)^{\wp} \end{bmatrix} \\ &= \mho. \end{aligned}$$

□

Theorem 4.4. *Let $\mho_i = \langle((([(\varrho \odot \Xi_i), (\varrho \odot \Upsilon_i)], [\underline{(\varrho \odot \Psi_i), (\varrho \odot} \Omega_i)])))\rangle$ be the IVT $\wp$ -rung Ns. Then IVT $\wp$-rung WA$(\mho_1, \mho_2, ..., \mho_l)$, where $\underline{(\varrho \odot \Xi)} = \min(\varrho \odot \Xi_{ij})$, $\overline{(\varrho \odot \Xi)} = \max(\varrho \odot \Xi_{ij})$, $\underline{(\varrho \odot \Psi)} = \min(\varrho \odot \Psi_{ij})$, $\overline{(\varrho \odot \Psi)} = \max(\varrho \odot \Psi_{ij})$ and $\underline{(\varrho \odot \Upsilon)} = \min(\varrho \odot \Upsilon_{ij})$, $\overline{(\varrho \odot \Upsilon)} = \max(\varrho \odot \Upsilon_{ij})$, $\underline{(\varrho \odot \Omega)} = \min(\varrho \odot \Omega_{ij})$, $\overline{(\varrho \odot \Omega)} = \max(\varrho \odot \Omega_{ij})$ and where $1 \leq i \leq l,\ j = 1, 2, ..., i_j$. Then,*

$$\begin{aligned} \left\langle \underline{(\varrho \odot \Xi), (\varrho \odot \Upsilon)}, \overline{(\varrho \odot \Psi), (\varrho \odot \Omega)} \right\rangle &\leq IVT\wp - rungNWA(\mho_1, \mho_2, ..., \mho_l) \\ &\leq \left\langle \overline{(\varrho \odot \Xi), (\varrho \odot \Upsilon)}, \underline{(\varrho \odot \Psi), (\varrho \odot \Omega)} \right\rangle. \end{aligned}$$

(Boundedness property).

Proof. Since, $\underline{(\varrho\odot\Xi)}=\min(\varrho\odot\Xi_{ij})$, $\overline{(\varrho\odot\Xi)}=\max(\varrho\odot\Xi_{ij})$ and $\underline{(\varrho\odot\Xi)}\le(\varrho\odot\Xi_{ij})\le\overline{(\varrho\odot\Xi)}$ and $\underline{(\varrho\odot\Upsilon)}=\min(\varrho\odot\Upsilon_{ij})$, $\overline{(\varrho\odot\Upsilon)}=\max(\varrho\odot\Upsilon_{ij})$ and $\underline{(\varrho\odot\Upsilon)}\le(\varrho\odot\Upsilon_{ij})\le\overline{(\varrho\odot\Upsilon)}$.

Now $\underline{(\varrho\odot\Xi),(\varrho\odot\Upsilon)}$

$$\begin{aligned}
&= \sqrt[\wp]{1-\nabla_{i=1}^{l}\Big(1-(\underline{(\varrho\odot\Xi)})^{\wp}\Big)^{\delta_i}},\ \sqrt[\wp]{1-\nabla_{i=1}^{l}\Big(1-(\underline{(\varrho\odot\Upsilon)})^{\wp}\Big)^{\delta_i}}\\
&\le \sqrt[\wp]{1-\nabla_{i=1}^{l}\Big(1-(((\varrho\odot\Xi_{ij})))^{\wp}\Big)^{\delta_i}},\ \sqrt[\wp]{1-\nabla_{i=1}^{l}\Big(1-(((\varrho\odot\Upsilon_{ij})))^{\wp}\Big)^{\delta_i}}\\
&\le \sqrt[\wp]{1-\nabla_{i=1}^{l}\Big(1-(\overline{(\varrho\odot\Xi)})^{\wp}\Big)^{\delta_i}},\ \sqrt[\wp]{1-\nabla_{i=1}^{l}\Big(1-(\overline{(\varrho\odot\Upsilon)})^{\wp}\Big)^{\delta_i}}\\
&= \overline{(\varrho\odot\Xi)}.
\end{aligned}$$

Since, $\underline{(\varrho\odot(\Psi)^{\wp})}=\min((\varrho\odot\Psi_{ij}))^{\wp}$, $\overline{(\varrho\odot(\Psi)^{\wp})}=\max((\varrho\odot\Psi_{ij}))^{\wp}$ and $\underline{(\varrho\odot(\Psi)^{\wp})}\le((\varrho\odot\Psi_{ij}))^{\wp}\le\overline{(\varrho\odot(\Psi)^{\wp})}$ and $\underline{(\varrho\odot(\Omega)^{\wp})}=\min((\varrho\odot\Omega_{ij}))^{\wp}$, $\overline{(\varrho\odot(\Omega)^{\wp})}=\max((\varrho\odot\Omega_{ij}))^{\wp}$ and $\underline{(\varrho\odot(\Omega)^{\wp})}\le((\varrho\odot\Omega_{ij}))^{\wp}\le\overline{(\varrho\odot(\Omega)^{\wp})}$.

We have,

$$\begin{aligned}
\underline{(\varrho\odot(\Psi)^{\wp}} &= \nabla_{i=1}^{l}\underline{(\varrho\odot(\Psi)^{\wp})}^{\delta_i},\nabla_{i=1}^{l}\underline{(\varrho\odot(\Omega)^{\wp})}^{\delta_i}\\
&\le \nabla_{i=1}^{l}(((\varrho\odot\Psi_{ij}))^{\wp})^{\delta_i},\nabla_{i=1}^{l}(((\varrho\odot\Omega_{ij}))^{\wp})^{\delta_i}\\
&\le \nabla_{i=1}^{l}\overline{(\varrho\odot(\Psi)^{\wp})}^{\delta_i},\nabla_{i=1}^{l}\overline{(\varrho\odot(\Omega)^{\wp})}^{\delta_i}\\
&= \overline{(\varrho\odot(\Psi)^{\wp})},\overline{(\varrho\odot(\Omega)^{\wp})}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}\times\left[\begin{array}{l}\left[\left(\sqrt[\wp]{1-\nabla_{i=1}^{l}\Big(1-(\underline{(\varrho\odot\Xi)})^{\wp}\Big)^{\delta_i}}\right)^2+1-\Big(\nabla_{i=1}^{l}((\overline{(\varrho\odot\Psi)})^{\wp})^{\delta_i}\Big)^2\right]\\+\left[\left(\sqrt[\wp]{1-\nabla_{i=1}^{l}\Big(1-(\underline{(\varrho\odot\Upsilon)})^{\wp}\Big)^{\delta_i}}\right)^2+1-\Big(\nabla_{i=1}^{l}((\overline{(\varrho\odot\Omega)})^{\wp})^{\delta_i}\Big)^2\right]\end{array}\right]\\
\le\ &\frac{1}{2}\times\left[\begin{array}{l}\left[\left(\sqrt[\wp]{1-\nabla_{i=1}^{l}\Big(1-((\varrho\odot(\varrho\odot\Xi_{ij})))^{\wp}\Big)^{\delta_i}}\right)^2+1-\big(\nabla_{i=1}^{l}(((\varrho\odot\Psi_{ij}))^{\wp})^{\delta_i}\big)^2\right]\\+\left[\left(\sqrt[\wp]{1-\nabla_{i=1}^{l}\Big(1-((\varrho\odot(\varrho\odot\Upsilon_{ij})))^{\wp}\Big)^{\delta_i}}\right)^2+1-\big(\nabla_{i=1}^{l}(((\varrho\odot\Omega_{ij}))^{\wp})^{\delta_i}\big)^2\right]\end{array}\right]\\
\le\ &\frac{1}{2}\times\left[\begin{array}{l}\left[\left(\sqrt[\wp]{1-\nabla_{i=1}^{l}\Big(1-(\overline{(\varrho\odot\Xi)})^{\wp}\Big)^{\delta_i}}\right)^2+1-\Big(\nabla_{i=1}^{l}(\underline{(\varrho\odot\Psi)})^{\wp})^{\delta_i}\Big)^2\right]\\+\left[\left(\sqrt[\wp]{1-\nabla_{i=1}^{l}\Big(1-(\overline{(\varrho\odot\Upsilon)})^{\wp}\Big)^{\delta_i}}\right)^2+1-\Big(\nabla_{i=1}^{l}(\underline{(\varrho\odot\Omega)})^{\wp})^{\delta_i}\Big)^2\right]\end{array}\right].
\end{aligned}$$

Hence, $\Big\langle\underline{(\varrho\odot\Xi),(\varrho\odot\Upsilon)},\overline{(\varrho\odot\Psi),(\varrho\odot\Omega)}\Big\rangle\le IVT\wp-rungNWA(\mho_1,\mho_2,...,\mho_l)\le\langle\overline{(\varrho\odot\Xi),(\varrho\odot\Upsilon)},$ $\underline{(\varrho\odot\Psi),(\varrho\odot\Omega)}\rangle$. □

Theorem 4.5. *Let* $\mho_i=\langle([(\varrho\odot\Xi_{t_{ij}}),(\varrho\odot\Upsilon_{t_{ij}})],[(\varrho\odot\Psi_{t_{ij}}),(\varrho\odot\Omega_{t_{ij}})])\rangle$ *and* $W_i=\langle([(\varrho\odot\Xi_{h_{ij}}),(\varrho\odot\Upsilon_{h_{ij}})],[(\varrho\odot\Psi_{h_{ij}}),(\varrho\odot\Omega_{h_{ij}})])\rangle$*, be the IVT* $\wp$*-rung WAs. For any* i*, if there is* $(\varrho\odot\Xi_{t_{ij}})^2\le(\varrho\odot\Xi_{h_{ij}})^2$ *and* $(\varrho\odot\Psi_{t_{ij}})^2\ge(\varrho\odot\Psi_{h_{ij}})^2$ *and* $(\varrho\odot\Upsilon_{t_{ij}})^2\le(\varrho\odot\Upsilon_{h_{ij}})^2$ *and* $(\varrho\odot\Omega_{t_{ij}})^2\ge(\varrho\odot\Omega_{h_{ij}})^2$ *or* $\mho_i\le W_i$*. Prove that IVT* $\wp-rungNWA(\mho_1,\mho_2,...,\mho_l)\le IVT\wp-rungNWA(W_1,W_2,...,W_l)$*, where* $(i=1,2,...,l)$; $(j=1,2,...,i_j)$ *(monotonicity property).*

Proof. For any i, $(\varrho \odot \Xi_{t_{ij}})^2 \leq (\varrho \odot \Xi_{h_{ij}})^2$.
Therefore, $1 - ((\varrho \odot \Xi_{t_i}))^2 \geq 1 - ((\varrho \odot \Xi_{h_i}))^2$.
Hence, $\bigtriangledown_{i=1}^{l} \left(1 - ((\varrho \odot \Xi_{t_i}))^2\right)^{\delta_i} \geq \bigtriangledown_{i=1}^{l} \left(1 - ((\varrho \odot \Xi_{h_i}))^2\right)^{\delta_i}$
and $\sqrt[\wp]{1 - \bigtriangledown_{i=1}^{l} (1 - ((\varrho \odot \Xi_{t_i}))^{\wp})^{\delta_i}} \leq \sqrt[\wp]{1 - \bigtriangledown_{i=1}^{l} (1 - ((\varrho \odot \Xi_{h_i}))^{\wp})^{\delta_i}}$.
Similarly, $(\varrho \odot \Upsilon_{t_{ij}})^2 \leq (\varrho \odot \Upsilon_{h_{ij}})^2$.
Therefore, $1 - ((\varrho \odot \Upsilon_{t_i}))^2 \geq 1 - ((\varrho \odot \Upsilon_{h_i}))^2$.
Hence, $\bigtriangledown_{i=1}^{l} \left(1 - ((\varrho \odot \Upsilon_{t_i}))^2\right)^{\delta_i} \geq \bigtriangledown_{i=1}^{l} \left(1 - ((\varrho \odot \Upsilon_{h_i}))^2\right)^{\delta_i}$
and $\sqrt[\wp]{1 - \bigtriangledown_{i=1}^{l} (1 - ((\varrho \odot \Upsilon_{t_i}))^{\wp})^{\delta_i}} \leq \sqrt[\wp]{1 - \bigtriangledown_{i=1}^{l} (1 - ((\varrho \odot \Upsilon_{h_i}))^{\wp})^{\delta_i}}$.

For any i, $\left((\varrho \odot \Psi_{t_{ij}})\right)^2 \geq \left((\varrho \odot \Psi_{h_{ij}})\right)^2$ and $\left((\varrho \odot \Psi_{t_{ij}})\right)^{\wp} \geq \left((\varrho \odot \Psi_{h_{ij}})\right)^{\wp}$.
Therefore, $1 - \left(\bigtriangledown_{i=1}^{l}(\varrho \odot \Psi_{t_{ij}})\right)^{\wp} \leq 1 - \left(\bigtriangledown_{i=1}^{l}(\varrho \odot \Psi_{h_{ij}})\right)^{\wp}$.
Similarly, for any i,
$\left((\varrho \odot \Omega_{t_{ij}})\right)^2 \geq \left((\varrho \odot \Omega_{h_{ij}})\right)^2$ and $\left((\varrho \odot \Omega_{t_{ij}})\right)^{\wp} \geq \left((\varrho \odot \Omega_{h_{ij}})\right)^{\wp}$.
Therefore, $-\left(\bigtriangledown_{i=1}^{l}(\varrho \odot \Omega_{t_{ij}})\right)^{\wp} \leq -\left(\bigtriangledown_{i=1}^{l}(\varrho \odot \Omega_{h_{ij}})\right)^{\wp}$.

Hence,

$$\frac{1}{2} \times \left[\begin{array}{l} \left[\left(\sqrt[\wp]{1 - \bigtriangledown_{i=1}^{l}\left(1 - ((\varrho \odot \Xi_{ti}))^{\wp}\right)^{\delta_i}}\right)^2 + 1 - \left(\bigtriangledown_{i=1}^{l}((\varrho \odot \Psi_{t_i}))^{\wp}\right)^2\right] \\ + \left[\left(\sqrt[\wp]{1 - \bigtriangledown_{i=1}^{l}\left(1 - ((\varrho \odot \Upsilon_{ti}))^{\wp}\right)^{\delta_i}}\right)^2 + 1 - \left(\bigtriangledown_{i=1}^{l}((\varrho \odot \Omega_{t_i}))^{\wp}\right)^2\right] \end{array}\right]$$

$$\leq \frac{1}{2} \times \left[\begin{array}{l} \left[\left(\sqrt[\wp]{1 - \bigtriangledown_{i=1}^{l}\left(1 - ((\varrho \odot \Xi_{hi}))^{\wp}\right)^{\delta_i}}\right)^2 + 1 - \left(\bigtriangledown_{i=1}^{l}((\varrho \odot \Psi_{h_i}))^{\wp}\right)^2\right] \\ + \left[\left(\sqrt[\wp]{1 - \bigtriangledown_{i=1}^{l}\left(1 - ((\varrho \odot \Upsilon_{hi}))^{\wp}\right)^{\delta_i}}\right)^2 + 1 - \left(\bigtriangledown_{i=1}^{l}((\varrho \odot \Omega_{h_i}))^{\wp}\right)^2\right] \end{array}\right].$$

Hence, IVT $\wp - rungNWA\,(\mho_1, \mho_2, ..., \mho_l) \leq IVT\wp - rungNWA\,(W_1, W_2, ..., W_l)$. □

4.2 IVT $\wp$-rung WG

Definition 4.6. Let $\mho_i = \langle(([(\varrho \odot \Xi_i), (\varrho \odot \Upsilon_i)], [(\varrho \odot \Psi_i), (\varrho \odot \Omega_i)]))\rangle$ be the IVT $\wp$ -rung Ns. Then $\wp$NWG $(\mho_1, \mho_2, ..., \mho_l) = \bigtriangledown_{i=1}^{l} \mho_i^{\delta_i}$.

Corollary 4.7. *Let* $\mho_i = \langle(([(\varrho \odot \Xi_i), (\varrho \odot \Upsilon_i)], [(\varrho \odot \Psi_i), (\varrho \odot \Omega_i)]))\rangle$ *be the IVT* $\wp$ *-rung Ns. Then IVT* $\wp$*-rung WG* $(\mho_1, \mho_2, ..., \mho_l)$

$$= \left[\frac{\bigtriangledown_{i=1}^{l}(((\varrho \odot \Xi_i))^{\wp})^{\delta_i}, \bigtriangledown_{i=1}^{l}(((\varrho \odot \Upsilon_i))^{\wp})^{\delta_i}}{\sqrt[\wp]{1 - \bigtriangledown_{i=1}^{l}\left(1 - ((\varrho \odot \Psi_i))^{\wp}\right)^{\delta_i}}, \sqrt[\wp]{1 - \bigtriangledown_{i=1}^{l}\left(1 - ((\varrho \odot \Omega_i))^{\wp}\right)^{\delta_i}}}\right].$$

Corollary 4.8. *(i) Let* $\mho_i = \langle(([(\varrho \odot \Xi_i), (\varrho \odot \Upsilon_i)], [(\varrho \odot \Psi_i), (\varrho \odot \Omega_i)]))\rangle$ *be the IVT* $\wp$ *-rung Ns and all are equal. Then* $\wp$*NWG*$(\mho_1, \mho_2, ..., \mho_l) = \mho$.
(ii) It has other properties, including boundedness and monotonicity, as well as having $\wp$*NWG.*

4.3 Generalized IVT $\wp$-rung WA (GIVT$\wp$NWA)

Definition 4.9. Let $\mho_i = \langle(([(\varrho \odot \Xi_i), (\varrho \odot \Upsilon_i)], [(\varrho \odot \Psi_i), (\varrho \odot \Omega_i)]))\rangle$ be the IVT $\wp$ -rung N. Then GIVT $\wp$-rung WA $(\mho_1, \mho_2, ..., \mho_l) = \left(\triangle_{i=1}^{l} \delta_i \mho_i^{\chi} \right)^{1/\chi}$.

Theorem 4.10. *Let $\mho_i = \langle(([(\varrho \odot \Xi_i), (\varrho \odot \Upsilon_i)], [(\varrho \odot \Psi_i), (\varrho \odot \Omega_i)]))\rangle$ be the IVT $\wp$ -rung Ns. Then GIVT $\wp$-rung WA $(\mho_1, \mho_2, ..., \mho_l)$*

$$= \left[\begin{array}{c} \left(\sqrt[\wp]{1-\triangledown_{i=1}^{l}\left(1-\left(((\varrho\odot\Xi_i))^{\wp}\right)^{\wp}\right)^{\delta_i}}\right)^{1/\wp}, \left(\sqrt[\wp]{1-\triangledown_{i=1}^{l}\left(1-\left(((\varrho\odot\Upsilon_i))^{\wp}\right)^{\wp}\right)^{\delta_i}}\right)^{1/\wp}, \\ \sqrt[\wp]{1-\left(1-\left(\triangledown_{i=1}^{l}\left(\sqrt[\wp]{1-\left(1-((\varrho\odot\Psi_i))^{\wp}\right)^{\wp}}\right)^{\delta_i}\right)^{\wp}\right)^{1/\wp}}, \\ \sqrt[\wp]{1-\left(1-\left(\triangledown_{i=1}^{l}\left(\sqrt[\wp]{1-\left(1-((\varrho\odot\Omega_i))^{\wp}\right)^{\wp}}\right)^{\delta_i}\right)^{\wp}\right)^{1/\wp}} \end{array}\right].$$

Proof. To illustrate this, we may first show that,

$$\triangle_{i=1}^{l}\delta_i\mho_i^{\wp} = \left[\begin{array}{c} \sqrt[\wp]{1-\triangledown_{i=1}^{l}\left(1-\left(((\varrho\odot\Xi_i))^{\wp}\right)^{\wp}\right)^{\delta_i}}, \sqrt[\wp]{1-\triangledown_{i=1}^{l}\left(1-\left(((\varrho\odot\Upsilon_i))^{\wp}\right)^{\wp}\right)^{\delta_i}} \\ \triangledown_{i=1}^{l}\left(\sqrt[\wp]{1-\left(1-((\varrho\odot\Psi_i))^{\wp}\right)^{\wp}}\right)^{\delta_i}, \triangledown_{i=1}^{l}\left(\sqrt[\wp]{1-\left(1-((\varrho\odot\Omega_i))^{\wp}\right)^{\wp}}\right)^{\delta_i} \end{array}\right].$$

Put $l = 2$, $\delta_1\mho_1 \triangle \delta_2\mho_2$

$$= \left[\begin{array}{c} \sqrt[\wp]{\begin{array}{c} \left(\sqrt[\wp]{1-\left(1-\left(((\varrho\odot\Xi_1))^{\wp}\right)^{\wp}\right)^{\delta_1}}\right)^{\wp} + \left(\sqrt[\wp]{1-\left(1-\left(((\varrho\odot\Xi_2))^{\wp}\right)^{\wp}\right)^{\delta_1}}\right)^{\wp} \\ -\left(\sqrt[\wp]{1-\left(1-\left(((\varrho\odot\Xi_1))^{\wp}\right)^{\wp}\right)^{\delta_1}}\right)^{\wp}\cdot\left(\sqrt[\wp]{1-\left(1-\left(((\varrho\odot\Xi_2))^{\wp}\right)^{\wp}\right)^{\delta_1}}\right)^{\wp} \end{array}} \\ , \sqrt[\wp]{\begin{array}{c} \left(\sqrt[\wp]{1-\left(1-\left(((\varrho\odot\Upsilon_1))^{\wp}\right)^{\wp}\right)^{\delta_1}}\right)^{\wp} + \left(\sqrt[\wp]{1-\left(1-\left(((\varrho\odot\Upsilon_2))^{\wp}\right)^{\wp}\right)^{\delta_1}}\right)^{\wp} \\ -\left(\sqrt[\wp]{1-\left(1-\left(((\varrho\odot\Upsilon_1))^{\wp}\right)^{\wp}\right)^{\delta_1}}\right)^{\wp}\cdot\left(\sqrt[\wp]{1-\left(1-\left(((\varrho\odot\Upsilon_2))^{\wp}\right)^{\wp}\right)^{\delta_1}}\right)^{\wp}, \end{array}} \\ \left(\sqrt[\wp]{1-\left(1-((\varrho\odot\Psi_1))^{\wp}\right)^{\wp}}\right)^{\delta_1}, \left(\sqrt[\wp]{1-\left(1-((\varrho\odot\Psi_2))^{\wp}\right)^{\wp}}\right)^{\delta_1} \\ , \left(\sqrt[\wp]{1-\left(1-((\varrho\odot\Omega_1))^{\wp}\right)^{\wp}}\right)^{\delta_1}, \left(\sqrt[\wp]{1-\left(1-((\varrho\odot\Omega_2))^{\wp}\right)^{\wp}}\right)^{\delta_1} \end{array}\right]$$

$$= \left[\begin{array}{c} \sqrt[\wp]{1-\triangledown_{i=1}^{2}\left(1-\left(((\varrho\odot\Xi_1))^{\wp}\right)^{\wp}\right)^{\delta_i}}, \sqrt[\wp]{1-\triangledown_{i=1}^{2}\left(1-\left(((\varrho\odot\Upsilon_1))^{\wp}\right)^{\wp}\right)^{\delta_i}} \\ \triangledown_{i=1}^{2}\left(\sqrt[\wp]{1-\left(1-((\varrho\odot\Psi_i))^{\wp}\right)^{\wp}}\right)^{\delta_i}, \triangledown_{i=1}^{2}\left(\sqrt[\wp]{1-\left(1-((\varrho\odot\Omega_i))^{\wp}\right)^{\wp}}\right)^{\delta_i} \end{array}\right].$$

Hence,

$$\triangle_{i=1}^{l}\delta_i\mho_i^{\chi} = \left[\begin{array}{c} \sqrt[\wp]{1-\triangledown_{i=1}^{l}\left(1-\left(((\varrho\odot\Xi_1))^{\wp}\right)^{\wp}\right)^{\delta_i}}, \sqrt[\wp]{1-\triangledown_{i=1}^{l}\left(1-\left(((\varrho\odot\Upsilon_1))^{\wp}\right)^{\wp}\right)^{\delta_i}} \\ \triangledown_{i=1}^{l}\left(\sqrt[\wp]{1-\left(1-((\varrho\odot\Psi_i))^{\wp}\right)^{\wp}}\right)^{\delta_i}, \triangledown_{i=1}^{l}\left(\sqrt[\wp]{1-\left(1-((\varrho\odot\Omega_i))^{\wp}\right)^{\wp}}\right)^{\delta_i} \end{array}\right].$$

If $l = l+1$, then $\triangle_{i=1}^{l}\delta_i\mho_i^{\chi} + \delta_{l+1}\mho_{l+1}^{\chi} = \triangle_{i=1}^{l+1}\delta_i\mho_i^{\chi}$.
Now,$\triangle_{i=1}^{l}\delta_i\mho_i^{\chi} + \delta_{l+1}\mho_{l+1}^{\chi} = \delta_1\mho_1^{\chi} \triangle \delta_2\mho_2^{\chi} \triangle ... \triangle \delta_l\mho_l^{\chi} \triangle \delta_{l+1}\mho_{l+1}^{\chi}$

$$= \left[\begin{array}{c} \sqrt[\wp]{\begin{array}{c}\left(\sqrt[\wp]{1-\triangledown_{i=1}^{l}\left(1-\left(((\varrho\odot\Xi_i))^{\wp}\right)^{\wp}\right)^{\delta_i}}\right)^{\wp} + \left(\sqrt[\wp]{1-\left(1-\left((\Xi_{l+1})^{\wp}\right)^{\wp}\right)^{\delta_1}}\right)^{\wp} \\ -\left(\sqrt[\wp]{1-\triangledown_{i=1}^{l}\left(1-\left(((\varrho\odot\Xi_i))^{\wp}\right)^{\wp}\right)^{\delta_i}}\right)^{\wp}\cdot\left(\sqrt[\wp]{1-\left(1-\left((\Xi_{l+1})^{\wp}\right)^{\wp}\right)^{\delta_1}}\right)^{\wp}\end{array}} \\ , \sqrt[\wp]{\begin{array}{c}\left(\sqrt[\wp]{1-\triangledown_{i=1}^{l}\left(1-\left(((\varrho\odot\Upsilon_i))^{\wp}\right)^{\wp}\right)^{\delta_i}}\right)^{\wp} + \left(\sqrt[\wp]{1-\left(1-\left((\Upsilon_{l+1})^{\wp}\right)^{\wp}\right)^{\delta_1}}\right)^{\wp} \\ -\left(\sqrt[\wp]{1-\triangledown_{i=1}^{l}\left(1-\left(((\varrho\odot\Upsilon_i))^{\wp}\right)^{\wp}\right)^{\delta_i}}\right)^{\wp}\cdot\left(\sqrt[\wp]{1-\left(1-\left((\Upsilon_{l+1})^{\wp}\right)^{\wp}\right)^{\delta_1}}\right)^{\wp}\end{array}} \\ \triangledown_{i=1}^{l}\left(\sqrt[\wp]{1-\left(1-((\varrho\odot\Psi_i))^{\wp}\right)^{\wp}}\right)^{\delta_i}, \left(\sqrt[\wp]{1-\left(1-(\Psi_{l+1})^{\wp}\right)^{\wp}}\right)^{\delta_1} \\ , \triangledown_{i=1}^{l}\left(\sqrt[\wp]{1-\left(1-((\varrho\odot\Omega_i))^{\wp}\right)^{\wp}}\right)^{\delta_i}, \left(\sqrt[\wp]{1-\left(1-(\Omega_{l+1})^{\wp}\right)^{\wp}}\right)^{\delta_1} \end{array}\right]$$

$$\triangle_{i=1}^{l+1}\delta_i\mho_i^{\wp} = \left[\begin{array}{c} \sqrt[\wp]{1-\triangledown_{i=1}^{l+1}\left(1-\left(((\varrho\odot\Xi_1))^{\wp}\right)^{\wp}\right)^{\delta_i}}, \sqrt[\wp]{1-\triangledown_{i=1}^{l+1}\left(1-\left(((\varrho\odot\Upsilon_1))^{\wp}\right)^{\wp}\right)^{\delta_i}} \\ \triangledown_{i=1}^{l+1}\left(\sqrt[\wp]{1-\left(1-((\varrho\odot\Psi_i))^{\wp}\right)^{\wp}}\right)^{\delta_i}, \triangledown_{i=1}^{l+1}\left(\sqrt[\wp]{1-\left(1-((\varrho\odot\Omega_i))^{\wp}\right)^{\wp}}\right)^{\delta_i} \end{array}\right].$$

$$\left(\triangle_{i=1}^{l+1}\delta_i\mho_i^{\chi}\right)^{1/\chi} = \left[\begin{array}{c} \left(\sqrt[\wp]{1-\triangledown_{i=1}^{l+1}\left(1-\left(((\varrho\odot\Xi_i))^{\wp}\right)^{\wp}\right)^{\delta_i}}\right)^{1/\wp}, \left(\sqrt[\wp]{1-\triangledown_{i=1}^{l+1}\left(1-\left(((\varrho\odot\Upsilon_i))^{\wp}\right)^{\wp}\right)^{\delta_i}}\right)^{1/\wp} \\ \sqrt[\wp]{1-\left(1-\left(\triangledown_{i=1}^{l+1}\left(\sqrt[\wp]{1-\left(1-((\varrho\odot\Psi_i))^{\wp}\right)^{\wp}}\right)^{\delta_i}\right)^2\right)^{1/\wp}}, \\ \sqrt[\wp]{1-\left(1-\left(\triangledown_{i=1}^{l+1}\left(\sqrt[\wp]{1-\left(1-((\varrho\odot\Omega_i))^{\wp}\right)^{\wp}}\right)^{\delta_i}\right)^2\right)^{1/\wp}} \end{array}\right]$$

□

Corollary 4.11. *(i) If $\wp = 1$, then IVT $\wp$-rung WA operator is used instead of the GIVT $\wp$-rung WA operator.*
(ii) If all $\mho_i = \langle((([(\varrho\odot\Xi_i),(\varrho\odot\Upsilon_i)],[(\varrho\odot\Psi_i),(\varrho\odot\Omega_i)])))\rangle$ and all are equal.
Then GIVT $\wp$-rung WA$(\mho_1,\mho_2,...,\mho_l) = \mho$.
(iii) The GIVT $\wp$-rung WA operator meets both boundedness and monotonicity constraints.

4.4 Generalized IVT $\wp$-rung NWG (GIVT $\wp$-rung NWG)

Definition 4 .12. Let $\mho_i = \langle(([(\varrho \odot \Xi_i), (\varrho \odot \Upsilon_i)], [(\varrho \odot \Psi_i), (\varrho \odot \Omega_i)]))\rangle$ be the IVT $\wp$-rung Ns. Then GIVT$\wp$-rung NWG $(\mho_1, \mho_2, ..., \mho_l) = \frac{1}{\chi}\left(\nabla_{i=1}^{l} (\chi\mho_i)^{\delta_i}\right)$.

Corollary 4.13. *Let $\mho_i = \langle(([(\varrho \odot \Xi_i), (\varrho \odot \Upsilon_i)], [(\varrho \odot \Psi_i), (\varrho \odot \Omega_i)]))\rangle$ be the IVT $\wp$-rung Ns. Then GIVT$\wp$-rung NWG$(\mho_1, \mho_2, ..., \mho_l)$*

$$= \begin{bmatrix} \sqrt[\wp]{1 - \left(1 - \left(\nabla_{i=1}^{l}\left(\sqrt[\wp]{1 - \left(1 - ((\varrho \odot \Xi_i))^{\wp}\right)^{\wp}}\right)^{\delta_i}\right)^{\wp}\right)^{1/\wp}}, \\ \sqrt[\wp]{1 - \left(1 - \left(\nabla_{i=1}^{l}\left(\sqrt[\wp]{1 - \left(1 - ((\varrho \odot \Upsilon_i))^{\wp}\right)^{\wp}}\right)^{\delta_i}\right)^{\wp}\right)^{1/\wp}} \\ \left(\sqrt[\wp]{1 - \nabla_{i=1}^{l}\left(1 - \left(((\varrho \odot \Psi_i))^{\wp}\right)^{\wp}\right)^{\delta_i}}\right)^{1/\wp}, \left(\sqrt[\wp]{1 - \nabla_{i=1}^{l}\left(1 - \left(((\varrho \odot \Omega_i))^{\wp}\right)^{\wp}\right)^{\delta_i}}\right)^{1/\wp} \end{bmatrix}.$$

Corollary 4.14. *(i) When $\chi = 1$, the GIVT $\wp$-rung WG is converted to the $\wp$-rung NWG.*
(ii) GIVT$\wp$-rung NWG operators satisfy the boundedness and monotonicity characteristics.
(iii) If all $\mho_i = \langle(([(\varrho \odot \Xi_i), (\varrho \odot \Upsilon_i)], [(\varrho \odot \Psi_i), (\varrho \odot \Omega_i)]))\rangle$ are equal. Then GIVT$\wp$-rung NWG$(\mho_1, \mho_2, ..., \mho_l) = \mho$.

5 Conclusion:

This work presents novel weighted operators, such as geometric and averaging operators. Boundedness, idempotency, commutativity, associativity, and monotonicity are some of the characteristics of these operators. To describe the weighted vector, we looked at a number of common metrics. Many aggregation operator criteria have been studied. Some findings have been made after a few aggregating techniques for these IVT $\wp$ -rung Ns have been examined.

References

[1] L. A. Zadeh, Fuzzy sets, Information and control, 8(3), (1965), 338-353.

[2] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets and Systems, 20(1), (1986), 87–96.

[3] R. R. Yager, Pythagorean membership grades in multi criteria decision-making, IEEE Trans. Fuzzy Systems, 22, (2014), 958–965.

[4] Wang, P., Zhu, B., Yu, Y., Ali, Z., & Almohsen, B. Complex intuitionistic fuzzy DOMBI prioritized aggregation operators and their application for resilient green supplier selection. Facta Universitatis, Series: Mechanical Engineering, 21(3), (2023), 339-357.

[5] Abdallah Shihadeh, Khaled Ahmad Mohammad Matarneh, Raed Hatamleh, Randa Bashir Yousef Hijazeen, Mowafaq Omar Al-Qadri, Abdallah Al-Husban, An Example of Two-Fold Fuzzy Algebras Based On Neutrosophic Real Numbers, Neutrosophic Sets and Systems, 67, (2024), 169-178.

[6] A. Rajalakshmi, Raed Hatamleh, Abdallah Al-Husban, K. Lenin Muthu Kumaran, M. S. Malchijah raj, Various neutrosophic ideals of an ordered ternary semigroups. 32 (3), (2025), 400-417.

[7] Abdallah Al-Husban & Abdul Razak Salleh, Complex fuzzy ring. Proceedings of 2nd International Conference on Computing, Mathematics and Statistics, (2015), 241-245.

[8] . Abdallah Al-Husban & Abdul Razak Salleh 2015. Complex Fuzzy Hyperring Based on Complex Fuzzy Spaces. Proceedings of 2nd Innovation and Analytics Conference & Exhibition (IACE), 1691, AIP Publishing 2015, 040009-040017.

[9] Al-Husban, A., & Salleh, A. R. Complex fuzzy hypergroups based on complex fuzzy spaces. International Journal of Pure and Applied Mathematics, 107(4), (2016), 949-958.

[10] Al-Husban, A., Amourah, A., & Jaber, J. J. Bipolar complex fuzzy sets and their properties. Italian Journal of Pure and Applied Mathematics, 43, 2020, 754761.

[11] Mishra, A. R., Rani, P., Cavallaro, F., & Alrasheedi, A. F. Assessment of sustainable wastewater treatment technologies using interval-valued intuitionistic fuzzy distance measure-based MAIRCA method. Facta Universitatis, Series: Mechanical Engineering, 21(3), (2023), 359-386.

[12] B.C. Cuong and V. Kreinovich, Picture fuzzy sets a new concept for computational intelligence problems, in Proceedings of 2013 Third World Congress on Information and Communication Technologies (WIIV 2013), IEEE, (2013), 1–6.

[13] S. Ashraf, S. Abdullah, T. Mahmood, F. Ghani and T. Mahmood, Spherical fuzzy sets and their applications in multi-attribute decision making problems, Journal of Intelligent and Fuzzy Systems, 36, (2019), 2829–284.

[14] Hussain, A., & Ullah, K. An Intelligent Decision Support System for Spherical Fuzzy Sugeno-Weber Aggregation Operators and Real-Life Applications. Spectrum of Mechanical Engineering and Operational Research, 1(1), (2024), 177-188.

[15] M. Rafiq, S. Ashraf, S. Abdullah, T. Mahmood, S. Muhammad, The cosine similarity measures of spherical fuzzy sets and their applications in decision making. Journal of Intelligent & Fuzzy Systems, 36(6), 2019, 6059-6073.

[16] T. Senapati, R.R. Yager, Fermatean, fuzzy sets. J. Ambient Intell. Humaniz. Comput. 11, (2020), 663-674.

[17] R.R. Yager, Generalized Orthopair Fuzzy Sets. IEEE Trans Fuzzy Syst 25(5), 2016, 1222-1230.

[18] R.N. Xu and C.L. Li, Regression prediction for fuzzy time series, Appl. Math. J. Chinese Univ., 16, (2001), 451–461.

[19] C. Sivakumar, Mowafaq Omar Al-Qadri, Abdallah shihadeh, Ahmed Atallah Alsaraireh, Abdallah Al-Husban, P. Maragatha Meenakshi, N. Rajesh, M. Palanikumar. (2024). q-rung square root interval-valued sets with respect to aggregated operators using multiple attribute decision making. Journal of, 23(3), 154-174.

[20] Abu, Ibraheem.,T., T.,Al-Husban, Abdallah.,Alahmade, Ayman.,Azhaguvelavan, S.. , Palanikumar, Murugan. (2024) Computer purchasing using new type neutrosophic sets and its extension based on aggregation operators. International Journal of Neutrosophic Science, 24(1), 171-185.

[21] Raja, K., Meenakshi, P. M., Al-Husban, A., Al-Qadri, M. O., Rajesh, N., & Palanikumar, M. (2024). Multi-criteria group decision making approach based on a new type of neutrosophic vague approach is used to select the shares of the companies for purchase. Full Length Article, 23(3), 296-96.?

[22] J., Lejo. , Damrah, Sadeq. , M., Mutaz. , Al-Husban, Abdallah. , Palanikumar, M.. (2024). Type-I extension Diophantine neutrosophic interval valued soft set in real life applications for a decision making. International Journal of Neutrosophic Science, 24(4), 151-164.

[23] Abu, Ibraheem. , Al-Husban, Abdallah. , J., Lejo. , J., Jamil. , Palanikumar, M.. , Balaji, G.. (2024). Selection process real-life application for new type complex neutrosophic sets using various aggregation operators. International Journal of Neutrosophic Science, 23(4), 136-153.

[24] Shihadeh, A., Matarneh, K. A. M., Hatamleh, R., Al-Qadri, M. O., & Al-Husban, A. (2024). On The Two-Fold Fuzzy n-Refined Neutrosophic Rings For 2= 3. Neutrosophic Sets and Systems, 68, 8-25.

[25] Abdallah Shihadeh, Khaled Ahmad Mohammad Matarneh, Raed Hatamleh, Randa Bashir Yousef Hijazeen, Mowafaq Omar Al-Qadri, Abdallah Al-Husban.(2024). An Example of Two-Fold Fuzzy Algebras Based On Neutrosophic Real Numbers, Neutrosophic Sets and Systems, 67,169-178.

[26] Raed Hatamleh, Abdallah Al-Husban, N. Sundarakannan, M. S. Malchijah Raj. (2025). Complex cubic intuitionistic fuzzy set applied to subbisemirings of bisemirings using homomorphism. 32 (3), PP: 418-435.

[27] Abubaker, Ahmad A, Hatamleh, Raed, Matarneh, Khaled, Al-Husban, Abdallah.(2024). On the Numerica Solutions for Some Neutrosophic Singular Boundary Value Problems by Using (LPM) Polynomials, International Journal of Neutrosophic Science,25(2),197-205.

[28] A., Ahmad. , Hatamleh, Raed. , Matarneh, Khaled. , Al-Husban, Abdallah. On the Irreversible k-Threshold Conversion Number for Some Graph Products and Neutrosophic Graphs. (2025). International Journal of Neutrosophic Science, 25(2), pp. 183-196

[29] M. Palanikumar, K. Arulmozhi, A Iampan, Multi criteria group decision making based on VIKORand TOPSIS methods for Fermatean fuzzy soft with aggregation operators, ICIC Express Letters 16 (10), (2022), 1129–1138.

[30] M. Palanikumar, K. Arulmozhi, MCGDM based on TOPSIS and VIKORusing Pythagorean neutrosophic soft with aggregation operators, Neutrosophic Sets and Systems, (2022), 538–555.

[31] M. Palanikumar, N. Kausar, H. Garg, A. Iampan, S. Kadry, M. Sharaf, Medical robotic engineering selection based on square root neutrosophic normal interval-valued sets and their aggregated operators, AIMS Mathematics, 8(8), (2023), 17402–17432.

[32] R. Hatamleh, On the Compactness and Continuity of Uryson's Operator in Orlicz Space, International Journal of Neutrosophic Science, 24 (3), (2024), 233-239.

[33] R. Hatamleh, On the Numerical Solutions of the Neutrosophic One-Dimensional Sine-Gordon System, International Journal of Neutrosophic Science, 25 (3), (2024), 25-36.

[34] R. Hatamleh, On The Continuous and Differentiable Two-Fold Neutrosophic and Fuzzy Real Functions, Neutrosophic Sets and Systems, 75,(2025), 196-209.

[35] R. Hatamleh, V.A. Zolotarev, On the Universal Models of Commutative Systems of Linear Operators.Journal of Mathematical Physics, Analysis, Geometry, 8(3), (2012), 248-259.

[36] R. Hatamleh, V.A. Zolotarev, On the Abstract Inverse Scattering Problem for Traces Class Pertubations. Journal of Mathematical Physics, Analysis, Geometry, 13(1), (2017), 1-32.

[37] R. Hatamleh, On a Novel Topological Space Based on Partially Ordered Ring of Weak Fuzzy Complex Numbers and its Relation with the Partially ordered Neutrosophic Ring of Real Numbers, Neutrosophic Sets and Systems, 78, (2025), 578-590.