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A Characterization of Rings in which every Semi-Hopfian Module is Noetherian

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Abstract:

Let R be a ring and M an R-module. It is well known that Hopfian modules are semi-Hopfian, but the converse is not always true. For instance, if p is a prime and M is any direct sum of copies of $\mathbb{Z}_{p^2}$, then M is semi-Hopfian but not Noetherian. In this paper, we introduce and study a new class of rings, called **SHN-rings**, defined as rings for which every semi-Hopfian module is Noetherian. We investigate the structural properties of SHN-rings and explore their relationships with other well-known classes of rings and modules. Over commutative rings, we establish that SHN-rings coincide with Köthe rings, SU-rings, pure-semisimple rings, and Artinian principal ideal rings. Furthermore, we show that over commutative SHN-rings, the notions of multiplication modules, cyclic modules, finitely generated modules, and Noetherian modules are equivalent. Additionally, we prove that in the setting of regular rings, semiprime Artinian rings, PCI-rings, and SHN-rings are equivalent.

Keywords: Semi-Hopfian modules, SHN-rings, SU-rings, Semiprime artinian rings

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1. Introduction

The study of modules over rings plays a fundamental role in modern algebra and ring theory. Among the various classes of modules, Hopfian and semi-Hopfian modules have been of particular interest due to their inherent structural properties. A module is said to be Hopfian if every surjective endomorphism is an isomorphism, while a module is semi-Hopfian if every surjective endomorphism has a kernel that is a direct summand of the module. Recently, some works have been done concerning Hopfian and semi-Hopfian modules see [4, 5, 6, 12].

In this paper, we introduce a new class of rings, referred to as **SHN-rings**, defined as rings for which every semi-Hopfian module is Noetherian. It is known that every Noetherian module is semi-Hopfian, but the converse is not true. For example, Let p be a prime and M be any direct sum of copies of $\mathbb{Z}_{p^2}$. Then we claim that M is a semi-Hopfian $\mathbb{Z}$-module. Let $f: M \to M$ be an epimorphism. Since $p^2M = 0$, f is a $\mathbb{Z}_{p^2}$-epimorphism. Since M is a free $\mathbb{Z}_{p^2}$-module, f splits. This implies that M is a semi-Hopfian $\mathbb{Z}$-module. But it is well known that M is not a Hopfian $\mathbb{Z}$-module (see: Example 3.2 of [9]). Therefore M is not Noetherian. This observation motivates the investigation of rings for which every semi-Hopfian module is Noetherian. These rings, which we term SHN-rings, form the central focus of this paper.

Throughout this paper, all rings are associative with identity and all modules are unitary left R-modules. A module M is called Dedekind finite if $M \cong M \oplus N$ for some module N, $N = 0$. For a ring R, R is Dedekind finite if and only if $ab = 1$ implies that $ba = 1$ for any $a; b \in R$. It is well known that any Hopfian module is Dedekind finite. A module M is called generalized Hopfian if any surjective endomorphism of M has a small kernel. Recall that a Köthe ring is a ring R such that each right, and each left, R-module is a direct sum of cyclique submodules. An R-module M is said to be finitely presented if there is an exact sequence: $0 \to K \to F \to M \to 0$ of R-modules, where F is finitely generated and free, and K is finitely generated. Such a sequence will be called a finite presentation of M. A finitely presented R-module M is called coherent if every finitely generated submodule of M is finitely presented. A ring R is said to be coherent if it is coherent as a R-module. We say that R is a right pure-semisimple ring if every right R-module is a direct sum of indecomposable submodules. R is called a right SU-ring if every right R-module is a direct sum of uniform submodules.

Our main result shows that, for commutative rings, SHN-rings coincide with several important classes of rings, including Köthe rings, SU-rings, pure-semisimple rings, and Artinian principal ideal rings. Additionally, we demonstrate that over a commutative SHN-ring, the classes of multiplication modules, cyclic modules, finitely generated modules, and Noetherian modules are equivalent. This establishes a strong connection between SHN-rings and other prominent classes in module theory.

Moreover, we investigate the equivalence between semiprime Artinian rings, PCI-rings, and SHN-rings, particularly in the context of regular rings, providing new insights into the structure of these rings and their modules. Our findings extend the current understanding of semi-Hopfian modules and open avenues for further exploration in module theory and ring theory.

The contributions of this paper enhance the theoretical framework surrounding SHN-rings, providing a comprehensive classification of these rings and their modules, while offering a deeper understanding of the role that semi-Hopfian modules play in ring theory.

2. Preliminary results

Lemma 2.1 (Theorem 09 of [9]). For a commutative ring R, the following are equivalent:

1. R is an artinian pricipal ideal ring;
2. R is I-ring;
3. R is S-ring

Lemma 2.2 (Theorem 3.1 and Theorem 3.4 of [8]). For a commutative ring R, the following are equivalent:

1. R is an artinian pricipal ideal ring;
2. R is Köthe-ring;

Lemma 2.3. Every SHN-ring is S-ring.

Proof. Let R be a SHN-ring and M an R-module. If M is Hopfian then, given that every Hopfian module is semi-Hopfian then M is Noetherian. Hence R is S-ring.

Lemma 2.4. (Proposition 3.6 of [12]) The following are equivalent for a module M.

1. M is Hopfian;
2. M is Dedekind finite;
3. M is generalized Hopfian and semi-Hopfian.

Lemma 2.5 (Theorem 1 of [10]). Let R be a ring.

1. Let M be a co-Hopfian (Hopfian) module. If M decomposes as a direct sum of a family M_i of non trivial R-modules, then each M_i is co-Hopfian (Hopfian).

2. Let M_i be a family of non trivial R-module for which $\mathrm{Hom}(M_i, M_j) = \mathrm{Hom}(M_j, M_i) = 0$ whenever $i \neq j$. If M_i is co-Hopfian (Hopfian) then so is $\oplus M_i$

Recall that an R-module M is called locally noetherian if all its finitely generated submodules are noetherian.

Proposition 2.6. Every module over commutative SHN-ring is locally noetherian.

Proof. Let R be a commutative SHN-ring and let M be a R-module and N a finitely generated submodule of M. Since over commutative ring, every finitely generated module is Hopfian then N is Hopfian and so semi-Hopfian. As R is SHN-ring then M is Noetherian.

Recall that an R-module is called locally coherent if every finitely generated ideal of R is coherent.

Proposition 2.7. Let R be a SHN-ring, then R is locally coherent.

Proof. Let R be a SHN-ringh and let I be a finitely generated ideal of R. As I is finitely generated then it is Hopfian R-module and so semi Hopfian. Since R is a SHN-ring then I is Noetherian. Therefore I is coherent because every Noetherian module is coherent. Hence R is a coherent ring.

Proposition 2.8. Let R be a commutative SHN-ring and M a R-module. The following are equivalent:

1. M is Noetherian;
2. M is Strongly Hopfian;
3. M is Hopfian;
4. M is Semi-Hopfian.

Proof. $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ Result from définitions
$(4) \Rightarrow (1)$ Over commutative SHN-ring, every semi-Hopfien module is noetherian.

3. Rings whose semi-Hopfian modules are Noetherian

Theorem 3.1 .Let R be a commutative ring. We suppose for all family of no trivial R-modules M_i, $Hom(M_i, M_j) = Hom(M_j, M_i) = 0$. The following are equivalent:

1. R is SHN-ring.
2. R is Köthe ring
3. R is artinian principal ideal ring.

Proof.$(1) \Rightarrow (2)$ Results from Lemma 2.3 and Lemma 2.1.

$(2) \Rightarrow (1)$ Let M be a semi-Hopfian R-module. R is pricipal ideal ring implies every R-module is a direct sum of cyclic modules. Let $M = \oplus M_i$. By [Proposition 3.7 of [12]], for all $i \in I; M_i$ is semi-Hopfian. Every M_i is cyclic and therefore finitely generated. All M_i are also finitely cogenerated because over artinian ring, finitely generated and finitely cogenerated coincide. As result all M_i is Dedekind finite. By Lemma 2.4, M_i is Hopfian. As result from Lemma 2.5 that $M = \oplus M_i$ is Hopfian. Referring to Lemma 2.1 and Lemma 2.2, R is a S-ring. Therefore M is Noetherian.
$2) \Leftrightarrow 3)$ Comes from Lemma 2.2.

Corollary 3.2. Let R be a ring in which all idempotents are central. We suppose for all family of non trivial R-module M_i, $Hom(M_i, M_j) = Hom(M_j, M_i) = 0$, for $i \neq j$. The following are equivalent:

1. R is an SHN-ring.
2. R is an artinian serial ring.

Proof.$(1) \Leftrightarrow (2)$ Follows from Theorem 3.2 and Corollary 3.3 of [2] and Theorem 3.1.

Recall that a ring R is called Krull-Schmidt if every finitely presented left R-module is a direct sum of modules with local endomorphism rings.

Lemma 3.3 (Proposition 2.6 of [7]). If every finitely generated left R-module is pure-injective, then R is a Krull-Schmidt left perfect ring.

Corollary 3.4. Every SHN-ring is Krull-Schimidt left perfect ring.

Proof. Let R a SHN-ring. By Theorem 3.1, R is Artinian principal ideal ring. It is well known, every Artinian ring is Noetherian then R is coherent. Referring to Theorem 2.8 of [7] every finitely generated R-module is pure-injective. Therefore R is Krull-Schimidt left perfect ring.

Theorem 3.5. Let R be a commutative ring. The following are equivalent:

1. R is an SHN-ring;

2. R is an SU-ring;

3. R is a pure-semisimple ring.

Proof. $(1) \Rightarrow (2)$ Let R be a SHN-ring, then by Theorem 3.1 R is a Köthe ring. It's result from Corollary 3.3 of [16] that every R-module is serial. Thus every R-module is direct sum of uniserial modules. Since uniserial modules are uniform then every R-module is direct sum of uniform modules and therefore R is SU-modules.
$(2) \Rightarrow (3)$ It is from to Theorem 4.1.13 of [14].

$(3) \Rightarrow (1)$ See [3].

Recall that if R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R: rM \subseteq N\}$ will be denoted by $[N: M]$. Then $[0: M]$ is the annihilator of M, $\mathrm{Ann}(M)$.

An R-module M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. In this case we can take $I = [N: M]$. Clearly, M is a multiplication module if and only if for each $m \in M, Rm = [Rm: M]M$.
The ring R is said to be semilocal if $R/J(R)$ is semisimple artinian, with $J(R)$ for the Jacobson radical of R.

Lemma 3.6 (Proposition 4, of [1]). Let R be a semilocal ring. Then an R-module is a multiplication module if, and only if, it is cyclic.

Theorem 3.7. Let R be a commutative SHN-ring and M an R-module. Then the following are equivalent:

1. M is a multiplication module;

2. M is cyclic;

3. M is finitely generated;

4. M is Noetherian.

Proof.(1) ⇔ (2). Since R be a SHN-ring. Then by Theorem 3.1, R is artinian and so semilocal. It results from Lemma 3.6 that the implications 1) and 2) are equivalent. It's easy to see that (2) ⇒ (3).

(3) ⇒ (4) Let M be a finitely generated module. It is well know, over commutative ring, every finitely generated module is Hopfian, it follows that M is Hopfian and as Hopfian module is semi-Hopfian then M is semi-Hopfian. Since R is a SHN-ring then M is Noetherian.

(4) ⇒ (1) Let M be a Noetherian module then M is a finitely generated module. Since by hypothesis R is a SHN-ring then it result for Theorem 3.1 that R is an artinian principal ideal ring and therefore M is direct sum of cyclic modules. Each cyclic module being a multiplication module, we deduce that M is the direct sum of the multiplication module. As M is of finite generate and R is a SHN-ring so R is Artinian and thus M is of finite length. It follows that M is a finite direct sum of multiplication module and consequently M is a multiplication module.

Lemma 3.8. If R is a commutative coherent ring, then R is an S-ring.

Proof. Let R be a commutative coherent ring and let M be a Hopfian R-module. If M is not Noetherian then There exists a submodule N of M that is not finitely generated. Hence there exists a sequence $N_1 \subset N_2 \subset N_3 \subset \cdots$ of submodules of M, where each N_i is strictly contained in N_{i+1} and $N = \bigcup_{i\geq 1} N_i$. Let f be an endomorphism of M defined by:

$$f(m) = \begin{cases} m & \text{if } m \notin N, \\ \text{the image of } m \text{ in } N_{i+1} & \text{if } m \in N_i. \end{cases}$$

So f is surjective, because every element of M is either fixed or moved to the next submodule and is not injective, because Kerf contains at least the elements of N_1. This contradicts the fact that M is a Hopfian module and therefore M is Noetherian.

Theorem 3.9. For a commutative ring R the following assertions are equivalent:

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1. R is a SHN-ring;

2. Every finitely presented R-module is coherent.

3. Every free R-module is locally coherent;

4. R is left coherent.

Proof.$(1) \Rightarrow (2)$ Let R be a SHN-ring and let M a finitely presented R-module. As finitely presented module are finitely generated then M is finitely generated and so M is Hopfian because over commutative ring, every finitely generated module is Hopfian. Since every Hopfian module is semi-Hopfian, then M is semi-Hopfian and so M is Noetherian because R is SHN-ring. As every Noetherian R-module over Noetherian ring is coherent then M is coherent because by Theorem 3.1, we can deduce that R is Noetherian. Hence every finitely presented R-module is coherent.

$(2) \Rightarrow (3) \Rightarrow (4)$ follows from 26.6 of [15]

$(4) \Rightarrow (1)$ Let R be a commutative coherent ring. It results from Lemma 3.8 that R is a S-ring and according to Lemma 2.1 and Theorem 3.1, R is SHN-ring.

Recall that a ring R is called a PCI-ring if and only if every proper cyclic R-module is injective. A fully invariant submodule X of a R-module M is called a semiprime submodule if it is an intersection of prime submodules of M. A R-module M is called a prime module if 0 is a prime submodule of M. A ring R is a prime ring if $_RR$ is a prime module. A R-module M is called a semiprime module if 0 is a semiprime submodule of M. Consequently, the ring R is semiprime ring if $_RR$ is semiprime.

Theorem 3.10. Let R be a commutative regular ring. Then the following assertions are equivalent:

1. R is semiprime Artinian;

2. R is PCI-ring;

3. R is a SHN-ring.

Proof. For $(1) \Rightarrow (2)$ we use this result from Osofsky; see [11] ”A ring R is semiprime Artinian if and only if every cyclic R-module is injective.” It follows from this result that every semiprime ring is a PCI-ring.

$(2) \Rightarrow (3)$ Let R be a PCI-ring. Since R is regular by hypothesis, then according to Proposition 6.11 of [17], R is semisimple. On a semisimple ring semi-Hopfian module and Noetherian module coincide and therefore R is a SHN-ring.

$(3) \Rightarrow (1)$ Let R be a SHN-ring, then it’s result from Theorem 3.1. that R is artinian principal ideal ring. Since R is regular then by Wisbauer 4.5 of [15] , R is semisimple and by Theorem 4.4 of [15],R is semiprime. Hence R is Artinian semiprime ring.

Recall that an ring R is called SI-ring if every singular R-module is injective. The rings of which each of the right (left) cyclic modules is a direct sum of a projective module and an injective module are called CDPI-rings.

Corollary 3.11. Let R be a commutative regular ring. The following assertions are equivalent:

1. R is an SHN-ring;
2. R is a CDPI-ring.

Proof. Let R be a commutative regular ring. By Theorem 3.10, R is SHN-ring if and only if R is semiprime Artinian ring. Hence it results from the Theorem 4.6 of [13] that R is a SHN-ring if and only if R is CDPI-ring.

Corollary 3.12. The following statements are equivalent for a commutative ring R with socle S.

1. R is an SI-ring;
2. R is Von Neumann Regular ring and R/S is a SHN-ring.

Proof. According to Theorem 3.9 of [13] R is an SI-ring if and only R is a Von Neumann Regular ring and R/S is a semiprime Artinian ring and consequently according to Theorem 3.10 we have the equivalence.

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Compliance with ethical standards

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