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A Modified Trapezoidal Technique with Improved Accuracy of Nonlinear Algebraic and Transcendental Equations

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Abstract:

For the purpose of evaluating numerical integration, a modified trapezoidal rule is presented in this paper; the proposed method appears to be an effective modification of the trapezoidal rule, which consists of the arithmetic mean of subintervals, while the proposed method uses the arithmetic mean and heronian mean at the subintervals. Compared to the original trapezoidal rule, this rule has a higher accuracy. Utilizing numerical experiments on the basis of local truncation error, the modified and original trapezoidal rules are compared

Keywords: Arithmetic mean, heronian mean, Modified trapezoidal technique, numerical examples, accuracy, and numerical integration.

1. INTRODUCTION

Integration, which is the method for determining the area that a function has drawn on a graph as, I= $\int_a^b f(x)dx$ is the sum or total of f(x)dx on the interval [a,b]. The value of a definite integral is calculated via numerical integration, which uses the integrand's approximations. A quadrature function is a way to represent the area under the curve f(x), which is applied to a single variable during numerical integration. Numerical integration covers all techniques for estimating the numerical values of definite integrals, presuming there are no integrand singularities in the domain. Nowadays, it is essential because computers can connect analytical models to computer processors as well as integrate data in analytical ways. The concept of "numerical integration" was first used in 1915 when David Gibb published A Course in Interpolation & Numeric Integration for Mathematical Laboratory. Applied mathematics, statistics, economics, and engineering are just a few of the numerous fields where numerical integration is used. Numerical integration can be performed by a variety of techniques, such as Euler-Maclaurin formula, Adaptive Quadrature, Gaussian integration, & quadrature methods, which are used to compute difficult-to-integrate functions. Several authors, including R.L. Burden [2], J.H. Mathews [3], S.S. Sastry [1], and others, have written works that recount the various formulas for numerical integrations. J. Oliver [4] used higher-order formula to investigate the various techniques used to assess definite integrals. Also, Gerry Sozio [5] covered a complete description of several numerical integration techniques. Numerical integration is used to estimate likelihoods and posterior distributions using Bayesian techniques [6]. It also includes the value of the definite integral $\int_a^b y\,dx$ which is calculated by modifying the function y using an interpolation formula and then integrating from a to b. We may obtain a quadrature formula by this technique with known numerical values. It is more essential than numerical differentiation in many real-life situations. One of the branches connecting analytical calculations with computer analysis is numerical integration, and it plays a significantly important role in mathematics affiliate today. On the other hand, certain researchers have previously finished significant research projects with a view to modelling and developing the numerous domains of numerical integration for multiple objectives. The analytical formula for the Kramers-Kronig transformation of a Lorentzian function was used by Ohta et al. to assess several numerical integration methods and identify the most efficient approach [7]. The applications of the Maclaurin's formula, the trapezium formula, the Simpson's formula, and consecutive double Fourier transform methods were also contrasted. The electromagnetic fundamental relation's convolution integral was calculated employing a newly developed one-time-step recursion relation and the trapezoidal rule of numerical integration. Also, they have compared various time-domain numerical modelling strategies for material dispersion in their research. Pennestr et al., on the other hand, presented and evaluated eight widely used engineering friction force models, honed in on well-known friction models, and provided a review and comparison based on numerical effectiveness [8]. Uilhoorn et al. examined stiff and nonstiff solvers, specifically integrated explicit Runge-Kutta schemes, to discover a quick and reliable spatial integration solver to gather gas flow transients within the context of particle filtering [9]. Bhonsale et al. looked at the outcomes of three distinct numerical solution methodologies for breaking population balance models the cell average method, the moving pivot method, and the fixed pivot method [10]. Additionally, using these methods, Concepcion Ausin, M. explored more sophisticated numerical integration techniques and compared several numerical integration providers [11]. An integrable polynomial discarding Taylor series has been estimated by Rajesh Kumar Sinha and colleagues [12].

One of the most essential and fundamental mathematical concepts is definite integration. And it has numerous uses in areas like physics and engineering. We must compute integrals in a number of real-world issues. In mathematics, the term "quadrature" refers to the area within the region of the curve f (x) bounded by the ordinates x_0, x_n, and the X-axis. The term "cubature" is occasionally used to describe the numerical integration of a multiple integral.

Numerical integration issues have existed at least from the time of the Greeks, when, for instance, the area of a circle was calculated by progressively increasing the number of sides of an inscribed polygon. The discovery of calculus sparked a new branch of study that eventually produced the fundamental principles of numerical integration in the 17th century. Following those centuries of development, the area saw the introduction of computers in more recent times, which allowed for the employment of numerous old and new algorithms to produce speedy and accurate results. Several scientists have already conducted a substantial amount of study in the area of numerical integration [13-17]. The numerical integration $\int_a^b f(x)dx$ represents the area between the ordinates x = a and x = b, the X-axis, and y=f(x). f (x) must be explicitly specified and must be integrable in order for this integration to be possible. When plotting the ordered pairs $(x_i,\ y_i)$, where i = 0, 1, 2…n, and a straight line connects any two successive points.

Numerical integration problem can be defined as follows:

Let y = f(x) be a given function and (x_i, y_i) are the set of n+1 ordered pairs, i = 0,1,2…n, where f(x) is not known explicitly, then it is necessary to determine $\int_{x_0}^{x_n} y\,dx$.

Numerous strategies are used in numerical integration to determine the numerical value of a given integral. Numerical integration is most popular technique for finding a solution to some integration problems since some integration problems cannot be dealt with analytically. Numerous techniques are used to approach and resolve numerical integration problems for unequal data spaces. The process of approximately computing an integral using a numerical technique is referred to as numerical integration. In mathematics, idea of numerical integration is crucial. When definite integrals cannot be solved analytically, engineers and scientists generally use numerical integration to approximate definite integrals. Numerical integrations can be used to resolve boundary value and initial value issues using either ordinary or partial differential equations. The establishment of broad physics laws is typically not possible with numerical solutions, and they frequently fail to show how desired variables depend on various problem-solving parameters. There are numerous numerical techniques for approximating the integral for unequal data points if a smooth function integrand inside a certain interval. Which numerically approach produces a more accurate answer is the question that emerges. To select the optimal numerical strategy, we apply a variety of numerical integration formulas for an unequal data space. Romesh Kumar Muthumalai intended to determine the error of numerical integration and differentiation. Additionally, he created a few new formulas for numerical differentiation utilising split differences, which are very helpful for estimating derivatives if additional information about derivatives becomes accessible. Selina Parvin, M.R. Hossain, and Md. Mamun-Ur-Rashid Khan created a completely novel algorithm for numerical integration techniques for unequal data space. Rajinder Thukral follow Simpson's approach to effectively solve nonlinear equations. To find the zeros of nonlinear equations, he improved the conventional Simpson's third order approach and created a novel formula for second order derivative estimation. In order to solve nonlinear equations with cubic convergence, J Jayakumar generalised Simpson's Newton approach. A novel class of Newton's method is developed to solve a single nonlinear equation for that [18-21]. For solving nonlinear equations, Hamidah Eskandari invented the Simpson's algorithm. He suggested using the integration method to pinpoint the nonlinear equation's root [22]. It is strongly contested that the (composite) trapezoidal rule, the simplest and likely oldest algorithm for numerical quadrature, is adequate to achieve high accuracy quadrature. This is mostly since it integrates standard integrands over a finite interval that has a low order of exactness. However, it has been demonstrated that the discretization error of the trapezoidal method is exponentially negligible in the reciprocal step size for integrals over the real line R of analytical functions in an open strip containing R. For numerical quadrature of analytical functions in particular situations, the trapezoidal rule is thus among the most effective methods. It is desirable to increase the decay rate of the integrand by employing appropriate analytical transformations of the integration variable in order to map common intervals to the real line and decrease the number of terms in infinite trapezoidal sums.

The modified integrand may not have a strip of analyticality since such alterations could result in the emergence of new singularities in the complex plane. Goodwin in 1949, Schwartz in 1969, and Stenger later examined the trapezoidal rule in reference to the quadrature of analytical functions. Takahasi and Mori in 1973 discussed the possibility of adopting transformations of the integration variable.

Although summation and mathematical integration are closely connected, symbol of integration is a stylized capital "S" to emphasize such connection [23]. The development of numerical methods, such as integration, for numerical data or advanced analytics solutions was carried out to solve various real-world engineering challenges. Many numerical approaches have been developed to simplify the process to overcome these critical challenges. To make life easier when facing these challenging situations, numerous numerical techniques have been created. The trapezoidal technique, which is an enormously popular & easy technique because of its relatively high accuracy, is one of the most prominent methods for integration. Unfortunately, this approach cannot be turned error-free without splitting the integration interval into numerous smaller intervals, which may not always be practical. When this isn't performed, the result will be inaccurate and will not be close to the correct response. A improved numerical strategy based on the trapezoidal rule idea is suggested in order to obtain findings that are substantially more accurate even with just one interval period. It involves some additional complexity but produces results that are significantly more precise. For the purpose of solving engineering challenges, numerical methods have drawn the attention of the research community. Its effectiveness and the development of high-speed calculations accomplished on twenty-first-century processors are what have sparked this interest. A case study of the growing significance of numerical approach capabilities in engineering software is Matlab. This book is one of the frequently used textbooks that institutions prefer to utilise when teaching numerical techniques to students. As part of their higher education requirements, all engineers must examine numerical methods [23]. It is recognised as a great resource for engineers and students to study, perform, and engage with engineering difficulties. It includes a wide range of integration techniques, including the Trapezoidal rule, Simpson rules, Romberg, etc. For the purpose of comparison, this book is used while discussing the equations and Trapezoidal rule illustrations.

In [24], the authors tried to find a solution to the Runge phenomenon in order to obtain perfection of a higher order compared to that of the usual quadrature methods, which are only capable of achieving accuracy of the fourth order. Utilising the trapezoidal rule frequency accuracy over the interior of the intervals, they provide method. The trapezoidal rule has also been critically examined from both a mathematical and historical perspective by Trefethen and Weideman, who also demonstrated how it is connected to computational techniques used in many areas of scientific computing [25]. It was found to be associated with approaches for computing the eigenvalues of matrices and operators, as well as with complex ideas, special functions, integral equations, and rational approximation. Similar research was done by the authors of [26] on techniques for extremely precise numerical integration. They looked at numerous techniques from 1980 to the present and highlighted a few that, when used in accordance with effective techniques for integer detection, required to allow for analytical interpretation of numerous

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"impossible" integrands before these methods were explored.

As far as we know, none of authors used the approaches mentioned in the paper.

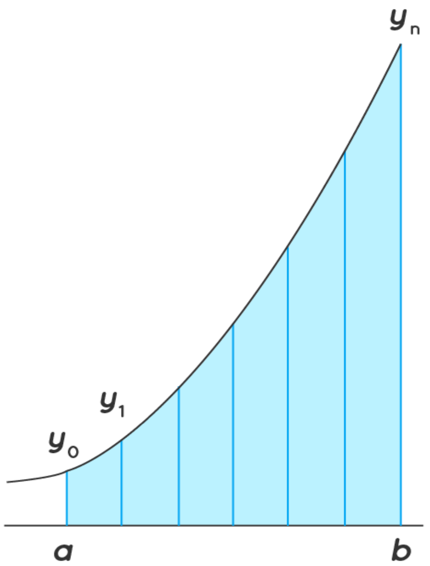
The following is how the paper is set up: The problem formulation is presented in Section 2, followed by numerical examples in Section 3, discussion of the findings in Section 4, and the study's conclusion in Section 4.

2. MATERIAL AND METHODS

The numerical techniques are used to determine the actual and approximation solution of nonlinear algebraic, transcendental equation $\int_1^2 \frac{1}{1+x^2}\,dx, \int_2^4 \frac{1}{1+x}\,dx, \int_1^3 e^x\,dx,\ \int_3^5 \log x\,dx,$ $\int_1^2 2^x\,dx,\ \int_0^1 \sec x \tan x\,dx,$ and $\int_0^1 e^x + \sin x\,dx$

Trapezoidal Method and Modified Trapezoidal Method: The technique of calculating the value of definite integrals is known as numerical integration in numerical analysis. when an analytical technique failed or when an integrand function was provided in a data table. It has several uses in the fields of engineering and physics. In physics, numerical integration problems sometimes require first subdividing the integrand into an equidistant mesh and then applying a basic rule of integration to finally attain the needed precision.

Assume that a continuous function y = f(x) defined on [a,b]. The interval [a,b] divided into n equal sub – intervals each of width h = s.t. a = $x_0 < x_1 < x_2 < \cdots < x_n = b$



Let f(x) = pn(x) = $\sum_{k=0}^{n} L_k(x) f(x_k)$ ...(1)

Where pn(x) is polynomial of degree ≤ n &

Lk(x) = $\frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{k-1})(x-x_{k+1})\ldots\ldots(x-x_n)}{(x_k-x_0)(x_k-x_1)\ldots\ldots(x_k-x_{k-1})(x_k-x_{k+1})\ldots\ldots(x_k-x_n)}$

Integrating equation (1) w.r.t. x over limit a to b

$\int_a^b f(x)dx$ = $\int_a^b p_n$(x)dx = $\int_a^b [\sum_{k=0}^{n} L_k(x) f(x_k)]dx$

$\int_a^b f(x)dx = \int_a^b p_n(\text{x})\text{dx} = \sum_{k=0}^{n}[\int_a^b L_k(x)dx]\, f(x_k)$

$\int_a^b f(x)dx = \int_a^b p_n(\text{x})\text{dx} = \sum_{k=0}^{n} A_k f(x_k) \ldots(2)$

Where $A_k = \int_a^b L_k(x)dx$

Equation (2) is known as Newton – Cotes formula.

Put n = 1 in equation (2), we get

$\int_a^b f(x)dx = \int_a^b p_1(\text{x})\text{dx} = \sum_{k=0}^{1} A_k f(x_k)$

$\int_a^b f(x)dx = \int_a^b p_1(\text{x})\text{dx} = A_0\text{f}(x_0) + A_1\text{f}(x_1)$

$\int_a^b f(x)dx = \int_a^b p_1(\text{x})\text{dx} = A_0 y_0 + A_1 y_1 \ldots(3)$

Now we find A_0 & A_1

$A_0 = \int_a^b \frac{x-b}{a-b}\text{dx} = \int_{x_0}^{x_1} \frac{x-x_1}{x_0-x_1}\text{dx} = \int_{t_0}^{x_0+h} \frac{x-x_0-h}{x_0-x_0-h}\text{dx}$

After solve above we get $A_0 = \frac{h}{2}$

$A_1 = \int_a^b \frac{x-a}{b-a}\text{dx} = \int_{x_0}^{x_1} \frac{x-x_0}{x_1-x_0}\text{dx} = \int_{t_0}^{x_0+h} \frac{x-x_0-h}{-h}\text{dx}$

After solving we get $A_1 = \frac{h}{2}$

Put values of A_0 & A_1 in (3)

$\int_a^b f(x)dx = \int_a^b p_1(\text{x})\text{dx} = \frac{h}{2}[y_0 + y_1]$

Repeat the above process n times & adding, we get

$\int_a^b f(x)dx = \frac{h}{2}[y_0 + 2(y_1 + y_2 + \ldots + y_{n-1}) + y_n]$

This is known as Trapezoidal rule.

Modified Trapezoidal rule is based on Arithmetic & Heronian mean and it is the two Arithmetic & Heronian mean respectively.

Firstly, we tabulate the function f by taking h = $\frac{b-a}{n}$

x	y
x0	y0

x1	y1
x2	y2
.	.
.	.
.	.
xn-1	yn-1
xn	yn

$I_1 = \int_{x_0}^{x_1} f(x)dx = \frac{h}{2}(y_0 + y_1)$

$I_2 = \int_{x_1}^{x_2} f(x)dx = \frac{h}{2}(y_1 + y_2)$

$I_3 = \int_{x_2}^{x_3} f(x)dx = \frac{h}{3}(y_2 + \sqrt{y_2 y_3} + y_3)$

$I_4 = \int_{x_3}^{x_4} f(x)dx = \frac{h}{3}(y_3 + \sqrt{y_3 y_4} + y_4)$

$I_5 = \int_{x_4}^{x_5} f(x)dx = \frac{h}{2}(y_4 + y_5)$

Continuing like this and adding, we get

$$\int_a^b f(x)dx = h[\left(\frac{y_0 + y_1}{2}\right) + \left(\frac{y_1 + y_2}{2}\right) + \left(\frac{y_2 + \sqrt{y_2 y_3} + y_3}{3}\right) + \left(\frac{y_3 + \sqrt{y_3 y_4} + y_4}{3}\right) + \cdots$$

This is called modified Trapezoidal rule.

3. NUMERICAL EXAMPLES

The efficiency of original Trapezoidal rule, the Modified Trapezoidal rule-based on Arithmetic, & heronian mean are contrasted in this section. The integral $\int_1^2 \frac{1}{1+x^2}\, dx, \int_2^4 \frac{1}{1+x}\, dx, \int_1^3 e^x\, dx$, $\int_3^5 \log x\, dx$, $\int_1^2 2^x\, dx$, $\int_0^1 \sec x \tan x\, dx$, and $\int_0^1 (e^x + \sin x)\, dx$ are evaluated and results are compared.

Problem 1. $\int_1^2 \frac{1}{1+x^2}\, dx$

Actual solution = 0.3216018366

By using Trapezoidal rule, we obtained the solution = 0.34852251407

By using Modified Trapezoidal rule, we obtained the solution = 0.30315893439

Problem 2. $\int_2^4 \frac{1}{1+x}\, dx$

Actual solution = 0.91629073187

By using Trapezoidal rule, we obtained the solution = 0.5123015873

By using Modified Trapezoidal rule, we obtained the solution = 0.88718467879

Problem 3. $\int_1^3 e^x \; dx$

Actual solution = 17.3672550947

By using Trapezoidal rule, we obtained the solution = 17.7275742529

By using Modified Trapezoidal rule, we obtained the solution = 17.5960037893

Problem 4. $\int_3^5 \log x \; dx$

Actual solution = 2.75135269617

By using Trapezoidal rule, we obtained the solution = 4.12286987021

By using Modified Trapezoidal rule, we obtained the solution = 4.12234686636

Problem 5. $\int_1^2 2^x \; dx$

Actual solution = 2.88539008178

By using Trapezoidal rule, we obtained the solution = 2.89260675394

By using Modified Trapezoidal rule, we obtained the solution = 2.8904933039

Problem 6. $\int_0^1 \sec x \tan x \; dx$

Actual solution = 0.85081571768

By using Trapezoidal rule, we obtained the solution = 0.90012355497

By using Modified Trapezoidal rule, we obtained the solution = 0.88181471573

Problem7. $\int_0^1 (e^x + \sin x) \; dx$

Actual solution = 2.17797952259

By using Trapezoidal rule, we obtained the solution = 2.18452284213

By using Modified Trapezoidal rule, we obtained the solution = 2.1807080442

Table 1

Comparison results of Trapezoidal, modified Trapezoidal methods are demonstrated in the table given below

Function	Exact Value	Trapezoidal Rule	Error	Modified Trapezoidal Rule	Error
$\int_1^2 \frac{1}{1+x^2} \; dx$	0.3216018366	0.34852251407	0.0269206775	0.30315893439	0.01844290221

$\int_2^4 \frac{1}{1+x}\, dx$	0.91629073187	0.5123015873	0.40398914457	0.88718467879	0.02910605308
$\int_1^3 e^x\, dx$	17.3672550947	17.7275742529	0.3603191582	17.5960037893	0.2287486943
$\int_3^5 \log x\, dx$	2.75135269617	4.12286987021	1.371517174	4.12234686636	1.3709941702
$\int_1^2 2^x\, dx$	2.88539008178	2.89260675394	0.0072166722	2.8904933039	0.0051032221
$\int_0^1 \sec x \tan x\, dx$	0.85081571768	0.90012355497	0.0493078373	0.88181471573	0.0309989981
$\int_0^1 (e^x + \sin x)\, dx$	2.17797952259	2.18452284213	0.0065433195	2.1807080442	0.0027285216

4. RESULT DISCUSSION

By concentrating on the preceding results, we moved forward with the interpretation of our numerical approaches for the approximation solutions of non-linear equations. Problems are solved by using Trapezoidal and Modified Trapezoidal methods. The results table indicates that compared to the trapezoidal approaches, the modified trapezoidal methodology converges to the exact answer more frequently. The modified trapezoidal technique has lowest error & highest error for the trapezoidal technique. In conclusion, it has been observed that the modified trapezoidal approach is the most effective way for solving the nonlinear equations since it converges more precisely and fast than the trapezoidal methods.

5. CONCLUSION

We are capable of demonstrating numerical integration with the methods of trapezoidal and modified trapezoidal that we described from the approaches covered in our research in order to get the minimum error value. In our written assignments, we have made an effort to illustrate certain examples and highlight the circumstances in which the modified trapezoidal rule works well. As a result, we can see that the modified trapezoidal rule provides the lowest error value and is, formally, the most practical and appropriate method.

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