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Prime Labeling of Bull Graph

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Abstract:

Let G be a graph. A bijection $f: V \rightarrow \{1,2, \dots..|V|\}$ is called a prime labeling [3] if for each edge $e = uv$ in E, we have $GCD\{ f(u), f(v)\} = 1$. A graph that admits a prime labeling is said to be a prime graph. In this paper we show that bull graph admits Prime labeling in the context of variety graph operations namely duplication of vertex, fusion of vertices and Switching in Bull graph.

Keywords: Prime labeling, Bull graph, Duplication, Fusion and Switching.

1. INTRODUCTION

Graph labeling is one of the stimulating areas with plentiful applications in various fields. In this paper we consider simple and finite graphs only. The notion of prime labeling was introduced by Roger Entringer and was discussed in a paper by A. Tout (1982 P 365-368).

This paper is organized as follows. In section 2 we provide the preliminary definitions. In section 3, we prove the main results of the paper, where we prove the graph obtained by duplicating arbitrary vertex of bull graph is a Prime graph, The graph obtained by Switching of any vertex in a bull graph is a Prime graph and we also prove that in a bull graph fusion of any arbitrary vertex with v_1 produces a Prime graph In section 4, we conclude the paper and also provide the insight for future work. For number theory concept refer [2].

2.PRELIMINARY DEFINITIONS

Definition [7]-2.1. Duplication of a vertex v_i of a graph G produces a new graph G_1 by adding a vertex v_i' with $N(v_i') = N(v_i)$. In other words, a vertex v_i' is said to be a duplication of v_i if all the vertices adjacent to v_i are now adjacent to v_i' also.

Definition [7]-2.2. Let u and v be two distinct vertices of a graph G. A new graph G_1 is constructed by fusing two vertices u and v by a single vertex w such that every edge incident to u and v is now incident with w in G_1.

Definition [7] -2.3. A vertex switching G_u in a graph G is obtained by taking a vertex u of G, removing all the edges incident to u and adding edges joining u to every non-adjacent vertex of u in G.

Definition [5], -2.4. The Bull graph is a graph with 5 vertices and 5 edges consisting of a triangle with two disjoint pendant edges.

3.MAIN RESULTS

Theorem-3.1. The graph obtained by duplicating arbitrary vertex of bull graph is a Prime graph.

Proof:

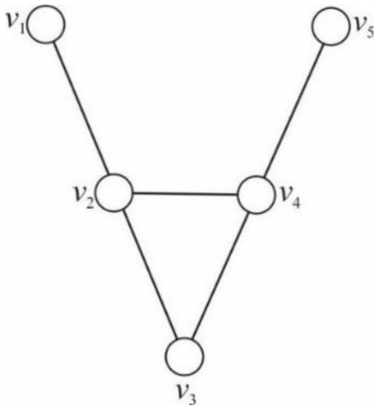

Figure - 1. *Bull graph*

Case-1. Duplication of the vertex v_1

Let G_1 be the graph obtained by duplicating the vertex v_1

Define $\mathscr{B}: V(G_1) \to \{1,2,3, \ldots\ldots, 6\}$ by

$\mathscr{B}(v_i) = i + 1$, $1 \le i \le 5$ and $\mathscr{B}(v_1') = 1$

Evidently all the vertex labels are distinct

For edges in G_1

G.C.D ($\mathscr{B}(v_i)$, $\mathscr{B}(v_{i+1})) = 1, 1 \le i \le 4$

G.C.D ($\mathscr{B}(v_2)$, $\mathscr{B}(v_4)) = 1$

G.C.D ($\mathscr{B}(v_1')$, $\mathscr{B}(v_2)) = 1$

Clearly $\mathscr{B}$ is a prime labeling on G_1.Hence G_1 is a prime graph.

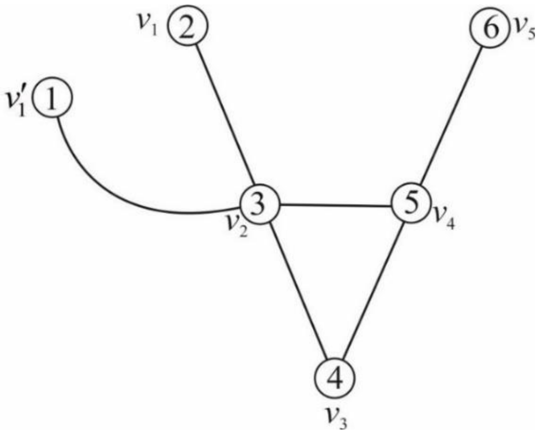

Figure - 2. *Prime labeling of duplication of vertex* v_1 *in Bull graph*

Case-2. Duplication of the vertex v_2

Let G_2 be the graph obtained by duplicating the vertex v_2

Define $\mathscr{B}: V(G_2) \to \{1,2,3, \ldots\ldots, 6\}$ by

$\mathscr{B}(v_i) = i + 1$, $1 \le i \le 5$ and $\mathscr{B}(v_2') = 1$

clearly all the vertex labels are distinct

For edges in G_2

G.C.D ($\mathscr{B}(v_i), \mathscr{B}(v_{i+1})) = 1, 1 \leq i \leq 4$

G.C.D ($\mathscr{B}(v_2), \mathscr{B}(v_4)) = 1$

G.C.D ($\mathscr{B}(v_2'), \mathscr{B}(v_1)) = 1$

G.C.D ($\mathscr{B}(v_2'), \mathscr{B}(v_3)) = 1$

G.C.D ($\mathscr{B}(v_2'), \mathscr{B}(v_4)) = 1$

Therefore $\mathscr{B}$ is a prime labeling on G_2.

Hence G_2 is a prime graph.

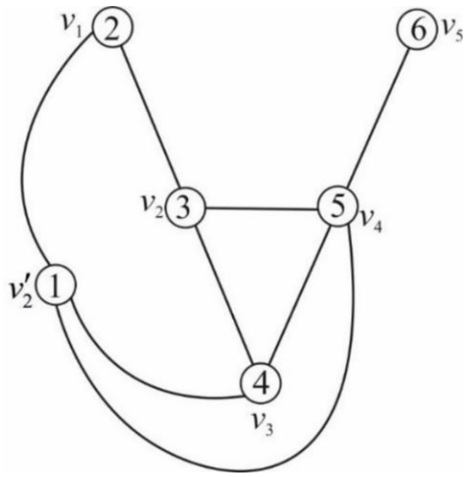

Figure - 3. Prime labeling of duplication of vertex v_2 in Bull graph

Case-3. Duplication of the vertex v_3

Let G_3 be the graph obtained by duplicating the vertex v_3

Define $\mathscr{B}: V(G_3) \rightarrow \{1,2,3, \ldots \ldots ,6\}$ by

$\mathscr{B}(v_i) = i + 1, \ 1 \leq i \leq 5$ and $\mathscr{B}(v_3') = 1$

clearly all the vertex labels are distinct

For edges in G_3

G.C.D ($\mathscr{B}(v_i), \mathscr{B}(v_{i+1})) = 1, 1 \leq i \leq 4$

G.C.D ($\mathscr{B}(v_2), \mathscr{B}(v_4)) = 1$

G.C.D ($\mathscr{B}(v_3'), \mathscr{B}(v_2)) = 1$

G.C.D ($\mathscr{B}(v_3'), \mathscr{B}(v_4)) = 1$

Thus $\mathscr{B}$ is a prime labeling on G_3.

Hence G_3 is a prime graph.

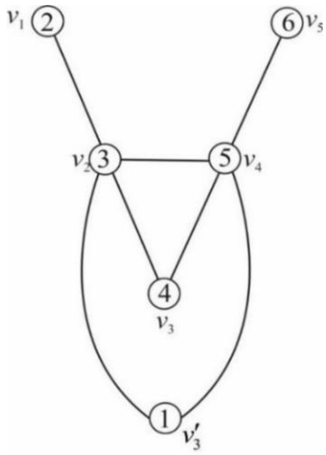

Figure - 4. *Prime labeling of duplication of vertex v_3 in Bull graph*

Case-4. Duplication of the vertex v_4

Let G_4 be the graph obtained by duplicating the vertex v_4

Define $\mathscr{B}: V(G_4) \rightarrow \{1,2,3, \ldots\ldots, 6\}$ by

$\mathscr{B}(v_i) = i + 1,\ 1 \leq i \leq 5$ and $\mathscr{B}(v_4') = 1$

obviously all the vertex labels are distinct

For edges in G_4

G.C.D ($\mathscr{B}(v_i),\ \mathscr{B}(v_{i+1})) = 1, 1 \leq i \leq 4$

G.C.D ($\mathscr{B}(v_2),\ \mathscr{B}(v_4)) = 1$

G.C.D ($\mathscr{B}(v_4'),\ \mathscr{B}(v_2)) = 1$

G.C.D ($\mathscr{B}(v_4'),\ \mathscr{B}(v_3)) = 1$

G.C.D ($\mathscr{B}(v_4'),\ \mathscr{B}(v_5)) = 1$

Clearly $\mathscr{B}$ is a prime labeling on G_4.

Hence G_4 is a prime graph.

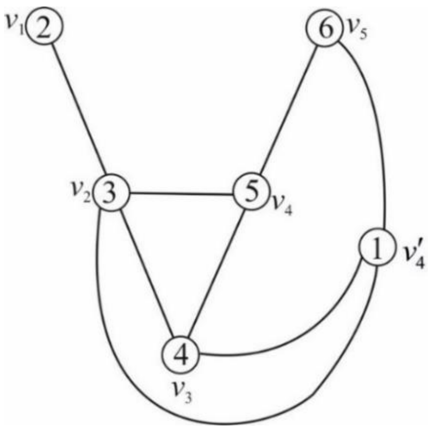

Figure - 5. *Prime labeling of duplication of vertex v_4 in Bull graph*

Case-5. Duplication of the vertex v_5

Let G_5 be the graph obtained by duplicating the vertex v_5

Define $\mathscr{B}: V(\ G_5) \to \{1,2,3, \ldots\ldots ,6\}$ by

$\mathscr{B}(v_i) = i + 1\ ,\ 1 \leq i \leq 5$ and $\mathscr{B}(v_5') = 1$

Evidently all the vertex labels are distinct

For edges in G_5

G.C.D ($\mathscr{B}(v_i\ ),\ \mathscr{B}(v_{i+1})) = 1, 1 \leq i \leq 4$

G.C.D ($\mathscr{B}(v_2),\ \mathscr{B}(v_4)) = 1$

G.C.D ($\mathscr{B}(v_5'),\ \mathscr{B}(v_4)) = 1$

Clearly $\mathscr{B}$ is a prime labeling on G_5.

Hence G_5 is a prime graph.

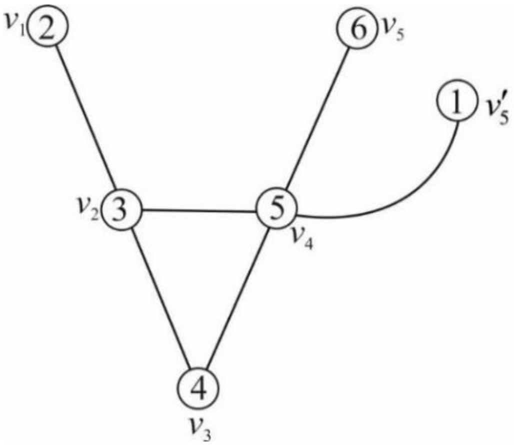

Figure - 6. *Prime labeling of duplication of vertex v_5 of Bull graph*

Thus, in all the cases the graph obtained by duplication of any arbitrary vertex of bull graph is a Prime graph.

Theorem-3.2. The graph obtained by Switching of any vertex in a bull graph is a Prime graph.

Proof.

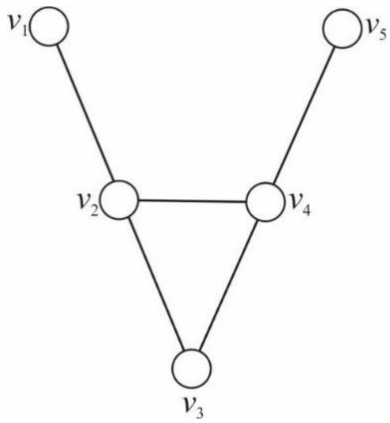

Figure - 7. *Bull graph*

Case-1. switching the vertex v_1

Let G_1 be the graph obtained by switching the vertex v_1

Define $\wp: V(G_1) \rightarrow \{1,2,3,\ldots\ldots,5\}$ by

$\wp(v_1) = 1$, $\wp(v_2) = 5$, $\wp(v_3) = 4$, $\wp(v_4) = 3$, $\wp(v_5) = 2$

Evidently all the vertex labels are distinct

For edges in G_1

G.C.D ($\wp(v_i), \wp(v_{i+1})) = 1, 2 \leq i \leq 4$

G.C.D ($\wp(v_2), \wp(v_4)) = 1$

G.C.D ($\wp(v_1), \wp(v_3)) = 1$

G.C.D ($\wp(v_1), \wp(v_4)) = 1$

G.C.D ($\wp(v_1), \wp(v_5)) = 1$

Thus $\wp$ is a prime labeling on G_1. Hence G_1 is a prime graph.

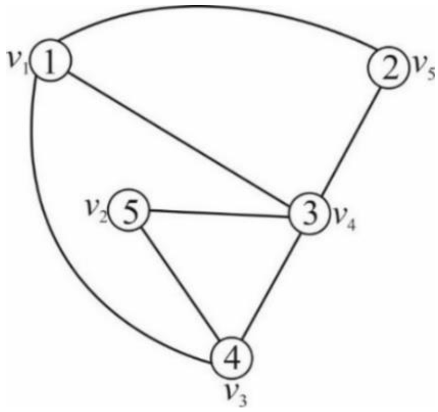

Figure - 8. *Prime labeling of switching of vertex v_1 in Bull graph*

Case-2. switching the vertex v_2

Let G_2 be the graph obtained by switching the vertex v_2

Define $\wp: V(G_2) \rightarrow \{1,2,3,\ldots\ldots,5\}$ by

$\wp(v_1) = 1$, $\wp(v_2) = 5$, $\wp(v_3) = 4$, $\wp(v_4) = 3$, $\wp(v_5) = 2$

clearly all the vertex labels are distinct

For edges in G_2

G.C.D ($\wp(v_i), \wp(v_{i+1})) = 1, 3 \leq i \leq 4$

G.C.D ($\wp(v_2), \wp(v_5)) = 1$

Hence $\wp$ is a prime labeling on G_2.

Thus G_2 is a prime graph.

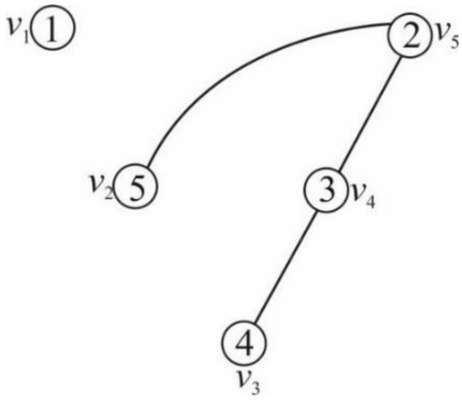

Figure - 9. *Prime labeling of switching of vertex v_2 in Bull graph*

Case-3. switching the vertex v_3

Let G_3 be the graph obtained by switching the vertex v_3

Define $\wp: V(G_3) \rightarrow \{1,2,3,\ldots\ldots,5\}$ by

$\wp(v_1) = 5, \wp(v_2) = 4, \wp(v_3) = 1, \wp(v_4) = 3, \wp(v_5) = 2$

Visibly all the vertex labels are distinct

For edges in G_3

G.C.D ($\wp(v_1), \ \wp(v_2)) = 1$

G.C.D ($\wp(v_1), \ \wp(v_3)) = 1$

G.C.D ($\wp(v_2), \ \wp(v_4)) = 1$

G.C.D ($\wp(v_4), \ \wp(v_5)) = 1$

G.C.D ($\wp(v_3), \ \wp(v_5)) = 1$

Therefore $\wp$ is a prime labeling on G_3. Hence G_3 is a prime graph.

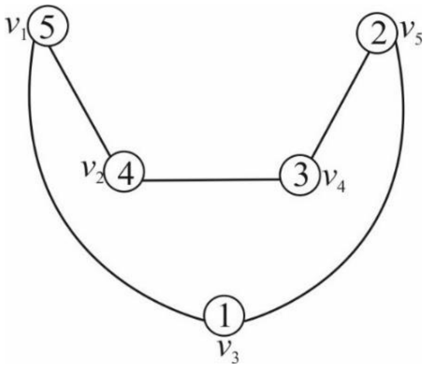

Figure - 10. *Prime labeling of switching of vertex v_3 in Bull graph*

Case-4. switching the vertex v_4

Let G_4 be the graph obtained by switching the vertex v_4

Define $\wp: V(G_4) \rightarrow \{1,2,3,\ldots\ldots,5\}$ by

$\wp(v_1) = 1, \wp(v_2) = 5, \wp(v_3) = 4, \wp(v_4) = 3, \wp(v_5) = 2$

Clearly all the vertex labels are distinct

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For edges in G_4

G.C.D ($\wp(v_i)$, $\wp(v_{i+1})) = 1, 1 \leq i \leq 2$

G.C.D ($\wp(v_1)$, $\wp(v_4)) = 1$

Hence $\wp$ is a prime labeling on G_4.

Therefore G_4 is a prime graph

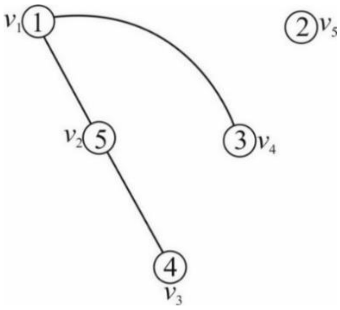

Figure - 11. *Prime labeling of switching of vertex v_4 in Bull graph*

Case-5. switching the vertex v_5

Let G_5 be the graph obtained by switching the vertex v_5

Define $\wp: V(G_5) \rightarrow \{1,2,3,\ldots\ldots,5\}$ by

$\wp(v_1) = 2, \wp(v_2) = 3, \wp(v_3) = 4, \wp(v_4) = 5, \wp(v_5) = 1$

Visibly all the vertex labels are distinct

For edges in G_5

G.C.D ($\wp(v_i)$, $\wp(v_{i+1})) = 1, 1 \leq i \leq 3$

G.C.D ($\wp(v_2)$, $\wp(v_4)) = 1$

G.C.D ($\wp(v_1)$, $\wp(v_5)) = 1$

G.C.D ($\wp(v_2)$, $\wp(v_5)) = 1$

G.C.D ($\wp(v_3)$, $\wp(v_5)) = 1$

Hence $\wp$ is a prime labeling on G_5.

So G_5 is a prime graph

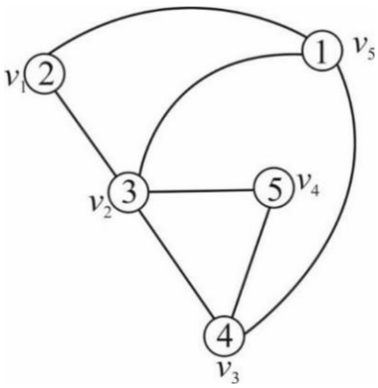

Figure - 12. *Prime labeling of switching of vertex v_5 in Bull graph*

Thus, in all the cases the graph obtained by Switching of any arbitrary vertex of bull graph is a Prime graph.

Theorem-3.3. In a bull graph fusion of any arbitrary vertex with v_1 produces a Prime graph.

Proof.

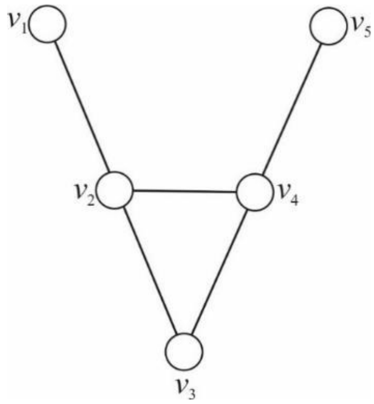

Figure - 13. *Bull graph*

Case-1. Fusion of v_2 with v_1

Let G_1 be the graph obtained by fusion of v_2 with v_1

Define $\mathcal{U}: V(G_1) \rightarrow \{1,2,3,4\}$ by

$\mathcal{U}(v_1 = v_2) = 1, \mathcal{U}(v_3) = 2, \mathcal{U}(v_4) = 3, \mathcal{U}(v_5) = 4$

Evidently all the vertex labels are distinct

For edges in G_1

G.C.D ($\mathcal{U}(v_i), \mathcal{U}(v_{i+1})) = 1, 3 \leq i \leq 4$

G.C.D ($\mathcal{U}(v_1 = v_2), \mathcal{U}(v_3)) = 1$

G.C.D ($\mathcal{U}(v_1 = v_2), \mathcal{U}(v_4)) = 1$

Hence $\mathcal{U}$ is a prime labeling on G_1.

So G_1 is a prime graph

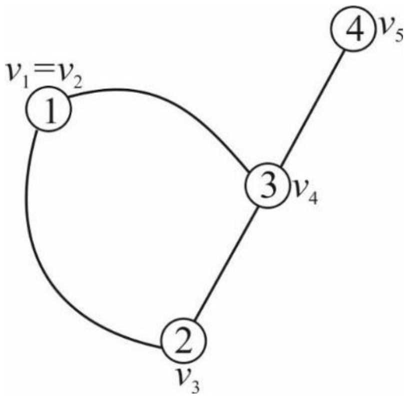

Figure - 14 . *Prime labeling of fusion of vertices v_2 with v_1 in Bull graph*

Case-2. Fusion of v_3 with v_1

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Let G_2 be the graph obtained by fusion of v_3 with v_1

Define $\mathcal{U}: V(G_1) \rightarrow \{1,2,3,4\}$ by

$\mathcal{U}(v_1 = v_3) = 1, \mathcal{U}(v_2) = 2, \mathcal{U}(v_4) = 3, \mathcal{U}(v_5) = 4$

Clearly all the vertex labels are distinct

For edges in G_2

G.C.D ($\mathcal{U}(v_1 = v_3), \mathcal{U}(v_2)) = 1$

G.C.D ($\mathcal{U}(v_1 = v_3), \mathcal{U}(v_4)) = 1$

G.C.D ($\mathcal{U}(v_2), \mathcal{U}(v_4)) = 1$

G.C.D ($\mathcal{U}(v_4), \mathcal{U}(v_5)) = 1$

Hence $\mathcal{U}$ is a prime labeling on G_2. Therefore G_2 is a prime graph

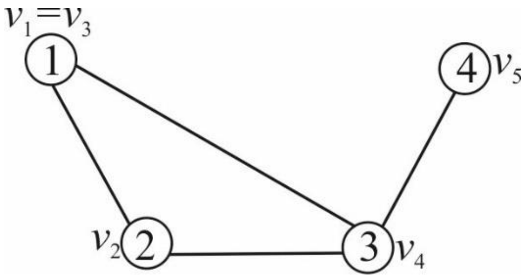

Figure - 15 . *Prime labeling of fusion of vertices v_3 with v_1 in Bull graph*

Case-3. Fusion of v_4 with v_1

Let G_3 be the graph obtained by fusion of v_4 with v_1

Define $\mathcal{U}: V(G_1) \rightarrow \{1,2,3,4\}$ by

$\mathcal{U}(v_1 = v_4) = 1, \mathcal{U}(v_2) = 2, \mathcal{U}(v_3) = 3, \mathcal{U}(v_5) = 4$

Evidently all the vertex labels are distinct

For edges in G_3

G.C.D ($\mathcal{U}(v_1 = v_4), \mathcal{U}(v_2)) = 1$

G.C.D ($\mathcal{U}(v_1 = v_4), \mathcal{U}(v_3)) = 1$

G.C.D ($\mathcal{U}(v_1 = v_4), \mathcal{U}(v_5)) = 1$

G.C.D ($\mathcal{U}(v_2), \mathcal{U}(v_3)) = 1$

Hence $\mathcal{U}$ is a prime labeling on G_3. So G_3 is a prime graph

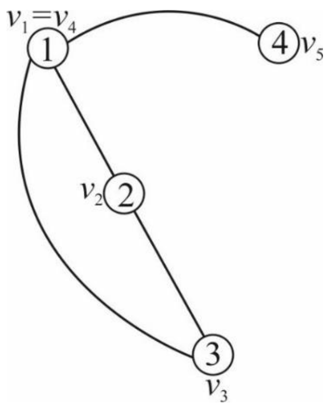

Figure - 16 . *Prime labeling of fusion of vertices v_4 with v_1 in Bull graph*

Case-4. Fusion of v_5 with v_1

Let G_4 be the graph obtained by fusion of v_5 with v_1

Define $\mathcal{U}: V(G_1) \rightarrow \{1,2,3,4\}$ by

$\mathcal{U}(v_1 = v_5) = 4,\ \mathcal{U}(v_2) = 3,\ \mathcal{U}(v_3) = 2,\ \mathcal{U}(v_4) = 1$

Obviously all the vertex labels are distinct

For edges in G_4

G.C.D ($\mathcal{U}(v_2),\ \mathcal{U}(v_3)) = 1$

G.C.D ($\mathcal{U}(v_2),\ \mathcal{U}(v_4)) = 1$

G.C.D ($\mathcal{U}(v_3),\ \mathcal{U}(v_4)) = 1$

G.C.D ($\mathcal{U}(v_1 = v_5),\ \mathcal{U}(v_2)) = 1$

G.C.D ($\mathcal{U}(v_1 = v_5),\ \mathcal{U}(v_4)) = 1$

Thus $\mathcal{U}$ is a prime labeling on G_4. Hence G_4 is a prime graph.

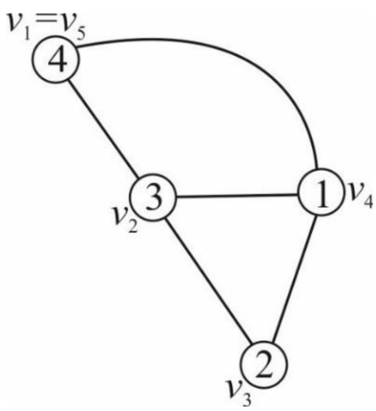

Figure - 17. *Prime labeling of fusion of vertices v_5 with v_1 in Bull graph*

Thus, in all the cases the graph obtained by fusion of any arbitrary vertex to v_1 of bull graph is a Prime graph.

4.CONCLUSION AND FUTURE WORK

In this paper we have proved that bull graph admits Prime labeling in the context of graph operations namely duplication, fusion and switching. There exist many such graphs that admit Prime labeling. An investigation to identify such graphs can be considered as future work.

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