Communications on Applied Nonlinear Analysis
ISSN: 1074-133X
Vol. 32 No. 2s (2024)

Pure and Weakly Pure Elements in Lattice Modules

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Article History:

Received: *25-09-2024*

Revised: *02-11-2024*

Accepted: *13-11-2024*

Abstract:

This study concerns with investigation of Pure and Weakly pure elements of lattice modules. An element N of M is called pure, if $aN = N \wedge a1_M$, for each a of L. An element K of M is called weakly pure, if $aN = N \wedge a1_M$, for each idempotent element a of L. Also, this study obtains the relation between pure, idempotent and multiplication elements of lattice modules.

Keywords: Pure element, Weakly Pure element, Idempotent element, Multiplication element.

1. Introduction

A lattice L is called as a multiplicative lattice, if L is complete with commutative, associative and join distributive binary operation called as multiplication. An element 1_L of L act as a identity with respect to multiplication. For $a_1, a_2 \in L$, $(a_1 : a_2) = \vee\{x \in L | a_2 x \leq a_1\}$. Element $p \in L$ such that $p \neq 1_L$ is *prime*, if $p_1.p_2 \leq p$ implies $p_1 \leq p$ or $p_2 \leq p$. The radical of $a \in L$ is denoted by $\sqrt{a}$ and is defined as $\vee\{x \in L | x^k \leq a$, for some $k \in Z^+\} = \wedge\{p \in L | a \leq p$ and p is a prime element$\}$. An element $c \in L$ is called *compact*, if for $t \in I$(I is an index set), $c \leq \vee_t a_t \Rightarrow c \leq \vee_{i=0}^{n} a_{t_i}$, for some $n \in Z^+$. If each element of L is a join of compact elements of L, then L is called a *CG*-lattice. An element $p \in L$ is called meet [join] principal, if $a_1 \wedge a_2 p = ((a_1 : p) \wedge a_2)p$ $[((a_1 p \vee a_2) : p) = a_1 \vee (a_2 : p)]$, $\forall\, a_1, a_2 \in L$. If p $\in L$ is both meet and join principal, then p is called principal element. If every element of L is a join of principal elements of L, then L is called a *PG*-lattice. An element $p \in L$ is said to be weak meet [join] principal, if $a \wedge p = p(a : p)$ $[a \vee (0_L : p) = (pa : p)]$, $\forall\, a \in L$.

An element $a \in L$ is called *semiprime or radical*, if $\sqrt{a} = a$. If $a \in L$ such that $a^2 = a$, then a is called an *idempotent*. Let $c \in L$. If for each $a \in L$ such that $a \leq c$ there exists an element $d \in L$ such that $a = cd$, then c is called *multiplication element*. Note that, $a \in L$ is a multiplication element if and only if it is weak meet principal element in L.

A complete lattice M is called a *lattice module* (L-module), where L is a multiplicative lattice, if the multiplication $aN \in M$, for $a \in L$ and $N \in M$ satisfies,$(ab)N = a(bN)$; for all a,

b in L and for all N in M.

1. $(\vee_\alpha l_\alpha)(\vee_\beta N_\beta) = (\vee_{\alpha\beta} l_\alpha N_\beta)$; for all l_α in L and for all N_β in M.
2. $1_L N = N$; for $1_L \in L$ and $N \in M$.
3. $0_L N = 0_M$; for $0_L \in L$ and $N \in M$.

Note that 0_M is a least and 1_M is a greatest element of M. For $N_1, N_2 \in M$, $(N_1 : N_2) = \vee\{x \in L | xN_2 \leq N_1\}$. For $N \in M$ and $a \in L$, $(N: a) = \vee\{K \in M | aK \leq N\}$. An element $N \in M$ is called *compact*, if for $t \in I$(I is an index set), $N \leq \vee_t B_t \Rightarrow N \leq \vee_{i=0}^{n} B_{t_i}$, for some $n \in Z^+$. If each element of M is a join of compact elements of M, then M is called a *CG*- lattice module. An element $N \in M$ is called meet [join] principal, if $(a \wedge (B : N))N = aN \wedge B[(a \vee (B : N) = ((aN \vee B) : N)]$, $\forall\, a \in L$ and $B \in M$. If $B \in M$ is both meet and join principal, then B is called *principal* element. If each element of M is a join of principal elements of M, then M is called a *PG*-lattice module. An element $N \in M$ is said to be weak meet [join] principal, if $(B : N)N = B \wedge N$ $[(aN : N) = a \vee (0_M : N)]$, $\forall\, a \in L$ and $B \in M$.

An element $N \in M$ is said to be proper, if $N < 1_M$. If $N \in M$ such that $N = (N : 1_M)N$, then N is an *idempotent* element of M. Element $N \in M$ is said to be *multiplication*, if for every $K \in M$ with $K \leq N$ there exists an element $a \in L$ such that $K = aN$. It is also noted that, $N \in M$ is a multiplication element if and only if N is weak meet principal in M.

A $L-$lattice module M is called *second*, if for each $a \in L$, $a1_M = 1_M$ or $a1_M = 0_M$. A $L-$lattice module M is called *secondary*, if for each $a \in L$, $a1_M = 1_M$ or $a^n 1_M = 0_M$ for some $n>0$. If $annM = (0_M : 1_M) = 0_L$, then M is called *faithful* $L-$module. A $L-$module M is called *torsion-free*, whenever $aK = 0_M$ implies $K = 0_M$ or $a = 0_L$, for any $a \in L$ and $K \in M$. A $L-$module M is *multiplication*, if for each element $N \in M$ there exists $a \in L$ such that $N = a1_M$. Note that, $L-$module M is a multiplication if and only if $N = (N : 1_M)1_M$ for all $N \in M$ (see [4]).

For $N \in M$, $[N, 1_M]$ is a set of all $K \in M$ such that $N \leq K \leq 1_M$. Note that, $[N, 1_M]$ is a L-lattice module with multiplication $a \circ K = aK \vee N$, where $a \in L$ and $K \in M$ such that $N \leq K$.

This study aims the generalization of some important results studied in [1], [2] for submodules of module over commutative ring to the lattice modules over multiplicative lattices and examine the concepts in multiplicative lattices and multiplication lattice modules.

Remark 1.1. Let M be a multiplication lattice module and N a element of M. If $(N : 1_M)$ is an idempotent, then $N = (N : 1_M)1_M = (N : 1_M)^2 1_M = (N : 1_M)N$, and N is idempotent in M. Conversely, if M is a *CG* and faithful multiplication L-module with N is idempotent in M, then $N = (N : 1_M)1_M = (N : 1_M)N$, and hence $N = (N : 1_M)^2 1_M = (N : 1_M)1_M$, which shows that $(N : 1_M)^2 = (N : 1_M)$ is an idempotent.

Further, for more information on modules, multiplicatice lattices, lattice modules, the reader may refer to [3], [7], [8], [9], [10].

2. Pure Element

We begin this section with the following definitions:

Definition 2.1. Let L be a multiplicative lattice and $c \in L$. c is said to be a *multiplication* element, if for every element a of L such that $a \leq c$ there exists an element $d \in L$ such that $a = cd$. [5].

Definition 2.2. [5] Let L be a multiplicative lattice and M a lattice L-module. $N \in M$ is said to be a *multiplication* element, if for every element K of M such that $K \leq N$ there exists an element $a \in L$ such that $K = aN$.

Definition 2.3. Let L be a multiplicative lattice and M a lattice L-module. $N \in M$ is said to be a *idempotent* element in M, if $N = (N : 1_M)N$.

Proposition 2.4. *Let L be a CG-lattice, M be a nonzero L-lattice module and $0_M \neq N$ is pure element of M. If M is p-secondary lattice module, then $[N, 1_M]$ and $[0_M, N]$ are both p-secondary lattice modules.*

Proof. see [6], Proposition 13.

Proposition 2.5. *Let L be a domain. If M is a multiplication second L-module, then every element in M is pure.*

Proof. Let N be any element of M. Since M is a multiplication second L-module, so M is either divisible or torsion [6]. If M is divisible, then $a1_M = 1_M$, for every $0_L \neq a \in L$. So $aN = N = N \wedge a1_M$, since M is multiplication L-module. If M is torsion, then $a1_M = 0_M$, for every $0_L \neq a \in L$. So, $aN = 0_M = N \wedge a1_M$.

Lemma 2.6. *Let M be a multiplication $L-$module, and $0_M \neq N$ be a pure element of M. Then M is a p-second lattice module if and only if $[0_M, N]$ and $[N, 1_M]$ are both p-second lattice modules.*

Proof. see [6], Proposition 14.

Lemma 2.7. *Let M be a faithful multiplication $L-$module. If N is a pure element of M, then N is multiplication and is idempotent in M.*

Proof. Let K be a element of M. Then $K = (K : 1_M)1_M$. Since N is a pure element of M, we have, $(K : N)N = N \wedge (K : N)1_M \geq N \wedge (K : 1_M)1_M = N \wedge K \geq (K : N)N$, so that $(K : N)N = K \wedge N \Rightarrow$ N a weak meet principal element in M, and N is multiplication. Since N is pure in M, we have that $(N : 1_M)N = N \wedge (N : 1_M)1_M = N$, and hence N is idempotent in M.

Lemma 2.8. *Let M be a multiplication $L-$module. If N is a pure element of M, then $K = (N : 1_M)K$ and $(K : N)N = (K : 1_M)N$, for each K of M.*

Proof. By Lemma 2.7, N is multiplication and is idempotent in M. Let $K \leq N$, then $K = (K : N)N = (K : N)(N : 1_M)N = (N : 1_M)K$. Also, for $K \leq N$, $(K : N)N = (K : N)(N : 1_M)N \leq (K : 1_M)N \leq (K : N)N$, so that $(K : N)N = (K : 1_M)N$.

Lemma 2.9. *Let M be a faithful multiplication $L-$module. If N is a pure element of M, then $a(N : 1_M) = a \wedge (N : 1_M)$, for every a in L.*

Proof. Since N is a pure element of M, so $aN = N \wedge a1_M$. Hence $(aN : 1_M) = ((N \wedge a1_M) : 1_M) = (N : 1_M) \wedge (a1_M : 1_M) = (N : 1_M) \wedge a$. We need to show that $(aN : 1_M) = a(N : 1_M)$. Obviously, $a(N : 1_M) \leq (aN : 1_M)$. Conversely, let $x \leq (aN : 1_M)$. Then $x1_M \leq aN = a(N : 1_M)1_M$. Thus $x \leq a(N : 1_M)$, and hence $(aN : 1_M) \leq a(N : 1_M)$.

Lemma 2.10. *Let M be a faithful multiplication L−module. If $a(N : 1_M) = a \wedge (N : 1_M)$, for every a of L, then N is multiplication and is idempotent in M.*

Proof. Assume $a(N : 1_M) = a \wedge (N : 1_M)$, for all a of L. Take $a = (N : 1_M)$. Then $(N : 1_M)^2 = (N : 1_M)$ and hence $(N : 1_M)$ is an idempotent element of L. Hence $N = (N : 1_M)1_M = (N : 1_M)^2 1_M = (N : 1_M)(N : 1_M)1_M = (N : 1_M)N$, and hence N is idempotent in M. To prove that N is multiplication, let K be any element of M. Let $a = (K : 1_M)$. Then $((K \wedge N) : 1_M) = (K : 1_M) \wedge (N : 1_M) = (K : 1_M)(N : 1_M) \leq (K : N)(N : 1_M)$, and hence $K \wedge N = ((K \wedge N) : 1_M)1_M \leq (K : N)(N : 1_M)1_M \leq (K : N)N \leq K \wedge N$, so that $K \wedge N = (K : N)N$ and N is multiplication. This completes the proof of the theorem.

Theorem 2.11. *Let L be a CG-multiplicative lattice and M be a multiplication L-module. For N, K in M and a in L.*

4. *If a is pure in L and N pure in M, then aN is pure in M. In particular, if a is pure in L, then $a1_M$ is a pure element of M.*

5. *If K is pure in N and N pure in M, then K is pure in M.*

6. *Let $K \vee N$ be a multiplication element. If each of K and N is pure in M, then $K \vee N$ and $K \wedge N$ are pure in M.*

Proof. $1 :\Rightarrow$ Let $b \in L$. We show that, $b(aN) = aN \wedge b1_M$. Assume that, L is local multiplicative lattice. Since a is a pure in L, then $a = 0_L$ or $a = 1_L$. If $a = 0_L$, then we are through. If $a = 1_L$, then the purity of N implies that $b(aN) = bN = N \wedge b1_M = aN \wedge b1_M$.

$2:\Rightarrow$ Let $b \in L$. Then $bK = K \wedge bN$ and $bN = N \wedge b1_M$ and hence, $bK = (K \wedge N) \wedge b1_M = K \wedge b1_M$, since $K \leq N$. So K is pure in M.

$3 :\Rightarrow$ Given K and N are pure in M, $aK = K \wedge a1_M$ and $aN = N \wedge a1_M$. So $aK \wedge aN = (K \wedge N) \wedge a1_M$ and $a(K \vee N) = (K \wedge a1_M) \vee (N \wedge a1_M)$. Since $K \vee N$ is multiplication, so $a(K \wedge N) = aK \wedge aN$ and $(K \vee N) \wedge a1_M = (K \wedge a1_M) \vee (N \wedge a1_M)$ and this shows that $K \wedge N$ and $K \vee N$ are pure elements of M.

In the following theorem we give a relation between pure elements, multiplication elementsand idempotent elements.

Theorem 2.12. *Let L be a CG-multiplicative lattice and M be a faithful multiplication L-modulesuch that 1_M compact. For N in M, the following are equivalent:*

1. *N is a pure element of M.*

2. *N is multiplication and is idempotent in M.*

3. *$(N : 1_M) = a \wedge (N : 1_M)$, for every $a \in L$.*

Proof. $1 \Rightarrow 2$: Assume that N is a pure element of M. Let K be a element of M. We will show that, $N \wedge K = (K : N)N$. Since M is multiplication, $K = (K : 1_M)1_M$. Since N is a pure element of M, so we have, $(K : N)N = N \wedge (K : N)1_M$. Now $(K : N)N = N \wedge (K : N)1_M \geq N \wedge (K : 1_M)1_M = N \wedge K \geq (K : N)N$. Hence, we get $(K : N)N = K \wedge N$. This implies that N is a multiplication in M. Since N is pure element, so we have $(N : 1_M)N = N \wedge (N : 1_M)1_M = N$.

$N = (N : 1_M)N = (N : 1_M)(N : 1_M)1_M = (N : 1_M)^2 1_M$. Hence, we get $(N : 1_M)^2 1_M = (N : 1_M)1_M$. So we have $(N : 1_M)$ is an idempotent element of L. And hence N is idempotent in M.

$2 \Rightarrow 3$: Assume that N is multiplication and idempotent in M. So $(N : 1_M)$ is an idempotent element, then we have $N = (N : 1_M)1_M = (N : 1_M)^2 1_M = (N : 1_M)(N : 1_M)1_M = (N : 1_M)N$. So for any element K of M, we have, $(K : N)N = (K : N)(N : 1_M)N \leq (K : 1_M)N \leq (K : N)N$, that implies $(K : N)N = (K : 1_M)N$. Since N is multiplication element of M, so for every a of L, $a1_M \wedge N = (a1_M : N)N = (a1_M : 1_M)N = aN = a1_M \wedge (N : 1_M)1_M$.

Also $aN = a(N : 1_M)N = a(N : 1_M)1_M$, so $a1_M \wedge (N : 1_M)1_M = a(N : 1_M)1_M$ for any $a \in L$, hence $a1_M \wedge (N : 1_M)1_M = (a \wedge (N : 1_M))1_M$. So we have, $a(N : 1_M) = a \wedge (N : 1_M)$.

$3 \Rightarrow 1$: Let $a \in L$. we have $(N : 1_M)1_M \wedge a1_M = ((N : 1_M) \wedge a)1_M$. Since $(N : 1_M) \wedge a = a(N : 1_M)$, implies that $N \wedge a1_M = (N : 1_M)1_M \wedge a1_M = ((N : 1_M) \wedge a)1_M = a(N : 1_M)1_M = aN$. Hence N is a pure element in M.

Theorem 2.13. *Let L be a CG-multiplicative lattice and M a faithful multiplication L-module.If N is pure in M, then $(N : 1_M)$ is the smallest element $a \in L$, such that $N = aN$.*

Proof. Let Λ be the collection of all elements a of L with the property that $N = aN$. Then $N = \wedge_{a\in\Lambda} aN = (\wedge_{a\in\Lambda} a)N$. It follows that $(N : 1_M) = ((\wedge_{a\in\Lambda} a)N : 1_M) = (\wedge_{a\in\Lambda} a)(N : 1_M)$, and hence $(N : 1_M) \leq (\wedge_{a\in\Lambda} a)$. But N is pure, and hence an idempotent. Thus $N = (N : 1_M)N$, and this means that $(N : 1_M) \in \Lambda$. So $(N : 1_M)$ is the smallest element of Λ.

Let M be a L-module. A proper element P of M is called a *prime* element of M, if $P \neq 1_M$ *and* whenever $rN \leq P$, for some $N \in M$ and $r \in L$, then $N \leq P$ or $r \leq (P : 1_M)$. The M *-radical*, rad N, of an element N of M is defined as the meet of all prime elements of M containing N. If a is an element of L, then $\sqrt{a}$ is defined as the meet of all prime elements of L containing a. If a is a pure (and hence idempotent) element of L, then $a = a\sqrt{a}$.

Lemma 2.14. *Let N be a element of an L-module M. Then $\sqrt{(N : 1_M)}\, 1_M \leq radN$.*

Proof. If $radN = 1_M$, the result is clear. Otherwise, if P is any prime element of M which contains N, then $(N : 1_M) \leq (P : 1_M)$. As P is a prime element of M, so $(P : 1_M)$ is a prime element of L. Hence $\sqrt{(N : 1_M)} \leq (P : 1_M)$ and thus $\sqrt{(N : 1_M)}1_M \leq (P : 1_M)\, 1_M = P$. Since P is an arbitrary element containing N, we have $\sqrt{(N : 1_M)}1_M \leq radN$.

Proposition 2.15. *[4] Let L be a multiplicative PG-lattice. Let M be a multiplication L-module and $ann(M) \leq b$ for some prime element $b \in L$. If $a1_M \leq b1_M$ for some $a \in L$, then $a \leq b$ or $b1_M = 1_M$.*

Lemma 2.16. *Let L be a multiplicative PG-lattice. Let M be a multiplication L-module such that*

1_M compact and $ann(M) \leq a$ for prime element $a \in L$. Then $a1_M$ is a prime element of M.

Proof. Note that $a1_M \neq 1_M$ and for $b \in L$ and $N \in M$, suppose that $bN \leq a1_M$. As M is a multiplication L-module, we have $N = c1_M$, for $c \in L$, so $bN = b(c1_M) \leq a1_M$. Proposition 2.15 implies that $bc \leq a$, hence $b \leq a$ or $c \leq a = (a1_M : 1_M)$, then $N = c1_M \leq a1_M$ and the proof is complete.

Theorem 2.17. *Let L be a multiplicative PG-lattice. Let M be a multiplication L-module suchthat 1_M compact and let B be a element of M. Then $radB = \sqrt{(B : 1_M)}1_M$.*

Proof. By Lemma 2.14, $\sqrt{(B : 1_M)}1_M \leq radB$. Since M is a multiplication L-module, $radB = (radB : 1_M)1_M$. it suffices then to show that $(radB : 1_M) \leq \sqrt{(B : 1_M)}$. Let a be any prime element such that $(B : 1_M) \leq a$. Since a is a prime element containing $annM$, then $a1_M$ is a prime element of M containing $B = (B : 1_M)1_M$. Hence, $(radB : _M)1_M = radB \leq a1_M$, so that $(radB : 1_M) \leq a$. Consequently, $(radB : 1_M) \leq \sqrt{(B : 1_M)}$.

The next result generalizes the above facts to pure element of multiplication L-module.

Proposition 2.18. *Let L be a CG-multiplicative lattice and M a faithful multiplication L-module. Let N be a pure element of M. Then*

1. $N = \sqrt{(N : 1_M)}N$,

2. $(N : 1_M)radN = N = (radN : 1_M)N$.

Proof. 1 :⇒ Let be the collection of all prime elements a of L contains $(N : 1_M)$. Then $\sqrt{(N : 1_M)} = \wedge_{a\in\Lambda} a$, and so, $\sqrt{(N : 1_M)}\, N = (\wedge_{a\in\Lambda} a)N = \wedge_{a\in\Lambda} aN$. For each $a \in \Lambda$, $N = (N : 1_M)N \leq aN \leq N$ so that $N = aN$, and hence $N = \wedge_{a\in\Lambda} a\, N = \sqrt{(N : 1_M)}N$.

2 :⇒ *I*t follows from (1), and theorem 2.17, that $N = \sqrt{(N : 1_M)}\, N = \sqrt{(N : 1_M)}\, (N : 1_M)\, 1_M = (N : 1_M) radN$. But $radN \leq 1_M$ and M is a multiplication L-module. Thus $radN = (radN : 1_M)\, 1_M$, and hence $(N : 1_M)radN = (N : 1_M)(radN : 1_M)\, 1_M = (radN : 1_M)\, N$.

III. Weakly Pure Element

In this section we give basic definition of weakly pure element of multiplication L-module, and prove some results related to weakly pure element. We begin with following definition.

Definition 3.1. A proper element N of L-module M is called *weakly pure*, if $aN = N \wedge a1_M$, for every idempotent element a of L.

Lemma 3.2. *Let M be a faithful multiplication L−module. If N is a weakly pure element of M, then $a(N : 1_M) = a \wedge (N : 1_M)$, for every idempotent element a of L.*

Proof. Proof follows by Lemma 2.9.

Proposition 3.3. *Let M be a faithful multiplication L−module. If N is a weakly pure element of M, then $(N : 1_M)$ is idempotent.*

Proof. By Lemma 3.2, we have $(N : 1_M)^2 = (N : 1_M) \wedge (N : 1_M) = (N : 1_M)$.

Theorem 3.4. *Let M be a faithful multiplication L−module, and N is a weakly pure element* *of M. Then N is primary element of M if and only if it is weakly primary element of M.*

Proof. It is enough to show that, if N is weakly primary, then N is primary. Assume that $0_M \neq N$ is a weakly primary element of M that is not primary. Then by Proposition 3.3, we have $N = (N : 1_M)1_M = (N : 1_M)^2 1_M = (N : 1_M)N = 0_M$, which is a contradiction. Thus N is primary.

Proposition 3.5. *Let M be a prime multiplication faithful L-module and $0_M \neq N$ be a proper weakly pure element of M. Then $ann(N : 1_M) = 0_L$.*

Proof. For every $a \leq ann(N : 1_M)$, we have $a(N : 1_M) = 0_L$, hence $aN = a(N : 1_M)N = 0_M$,so that $a \leq annN = annM = 0_L$, since M is prime. Hence $a = 0_L$, so $ann(N : 1_M) = 0_L$.

Proposition 3.6. *Let L be a Noetherian multiplicative lattice with Jacobson radical r^*,and M amultiplication L-module and N is a weakly pure element of M. Then there is a maximal element r of L such that $(N : 1_M) \nleq r$.*

Proof. Otherwise, $(N: 1_M) \leq r^*$, so $(N: 1_M) = \wedge_{i=1}^{\infty} (N: 1_M)^i = 0_L$, by Proposition 3.3, hence $N = (N : 1_M)1_M = 0_M$, which is a contradiction. Hence there is a maximal element r of L such that $(N : 1_M) \nleq r$.

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