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# Zip Property of Graded and Filtered Affine Schemes

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**Abstract:**

In this paper we study the transfer of zip property between filtered (graded) rings and affine graded (filtered) structure schemes. Under some conditions, the zip property of filtered (graded) rings is preserved under their graded and filtered affine schemes. One may apply these results up to the formal level as in [8].

**Introduction**: Consider a zariskian filtered ring $S$ such that the associated graded ring $G(S) = \bigoplus \frac{F_n S}{F_{n-1} S} \cong \frac{\tilde{S}}{X\tilde{S}}$ is commutative Noetherian domain; [10]. This includes many more geometric applications, i.e. this situation is general in the sense that it allows application of the results to most of the important examples. The topological base space $T$ will be $Spec^g$ of $G(S)$. The canonical element of degree one in $\tilde{S} = \bigoplus F_n S \cong \sum_{n \in Z} F_n SX^n \leq S[X, X^{-1}]$ is the $1 \in F_1 S$ in $S$, we write it as $X$.

For moment let $S$ be a graded ring. For a homogenous element $a \in S$, the annihilator ideal $ann^g(a) = \{s \in S: sa = 0\}$ is a homogenous ideal, as is the ideal annihilator $ann^g(A) = \{s \in S: sA = 0\}$; $A \subseteq S$ a set of homogenous elements and as is the ideal annihilator $ann^g(I) = \{s \in S: sI = 0\}$; $I \trianglelefteq S$ an ideal of homogenous elements.
A graded ring $S$ is said to be zip if $\forall\ A \subseteq S$: $ann^g(A) = 0 \Rightarrow \exists\ A_0 \subseteq A$, finite subset of homogenous elements: $ann^g(A_0) = 0$. In this definition, we can equivalently need to use that $A$ is a graded ideal of $S$. We need only zip expression of commutative case.

For elementary notions, conventions and generalities, which we need here in this paper we refer to the list of references.

**Objectives**: In this paper, we study the transfer of zip property from filtered (graded) rings to the graded and filtered structure affine schemes.

**Results**: According to the work of Leroy and Matczuk ([4] , Theorem 3.2(1) ), who investigated the behavior of the zip property for a localization of a ring , we extend this result for graded and filtered affine schemes.

**Conclusion**: In this research, we investigate the zip property of filtered (graded) rings is preserved under their graded and filtered affine schemes. In the forthcoming work, we hope to come back to introduce the same results on the formal level, one may make this by [8].

**Keywords**: Graded annihilator, Affine schemes, Zip property.

## 1. Introduction

Throughout the paper $S$ will denote a zariskian filtered ring such that the associated graded ring $G(S) = \bigoplus \frac{F_n S}{F_{n-1} S} \cong \frac{\tilde{S}}{X\tilde{S}}$ is commutative Noetherian domain; [10]. This includes many more geometric applications, i.e. this situation is general in the sense that it allows application of the results to most of the important examples. The topological base space $T$ will be $Spec^g$ of $G(S)$. The canonical element of degree one in $\tilde{S} = \bigoplus F_n S \cong \sum_{n \in Z} F_n SX^n \leq S[X, X^{-1}]$ is the $1 \in F_1 S$ in $S$, we write it as $X$.

For moment let $S$ be a graded ring. For a homogenous element $a \in S$, the annihilator ideal $ann^g(a) = \{s \in S: sa = 0\}$ is a homogenous ideal, as is the ideal annihilator $ann^g(A) = \{s \in S: sA = 0\}; A \subseteq S$ a set of homogenous elements and as is the ideal annihilator $ann^g(I) = \{s \in S: sI = 0\}; I \trianglelefteq S$ an ideal of homogenous elements.
A graded ring $S$ is said to be zip if $\forall\ A \subseteq S$: $ann^g(A) = 0 \Rightarrow \exists\ A_0 \subseteq A$, finite subset of homogenous elements: $ann^g(A_0) = 0$. In this definition, we can equivalently need to use that $A$ is a graded ideal of $S$. We need only zip expression of commutative case.

## 2. Zip graded affine schemes

A graded sheaf $\underline{O}_T^g$ of graded rings, over a topological space $T$, is zip graded sheaf over $T$ if locally is zip sheaf i.e. $\forall\ P \in T \Rightarrow \underline{O}_{T,P}^g$ is zip graded ring.

As in section one, we consider $T = Spec^g(G(S))$, the graded prime spectrum of $G(S)$. Write $\beta$ for the basis of the Zariski topology on $T$ consisting of the basic open sets $T(f) = \{P \in T; f \notin P\}$; $f$ homogenous element. We may define graded structure sheaf on $T$: we may associate to $T(f)$ the graded ring $Q_f^g(G(S)) = S_f^{-1} \cdot G(S)$; just, by inverting the homogenous set $S_f = \{1, f, f^2, \cdots\}$ of $G(S)$ in the classical way and we obtain the graded structure sheaf $Q_T^g$ on $T$ having as the stalk at $P \in T$ the graded local ring $Q_P^g(G(S)) = S_P^{-1} \cdot G(S)$; $S_P = h(G(S) - P)$.

**Proposition 1.** With the same consideration

If $S$ is zip commutative filtered ring, then $\tilde{S}$ is zip commutative graded ring.

**Proof:** Follows from [4], [5] just at the filtered (graded) level.

**Proposition 2.** Under the assumption and notation mentioned above we have:

**i.** $G(S)$ and $\tilde{S}/(1-X)\tilde{S}$ are zip graded rings.

ii. $Q_f^g(G(S))$ is zip graded ring; $f \in G(S)$.

iii. $Q_P^g(G(S))$ is zip graded ring; $P \in T$.

**Proof:** i. By assumption, since $G(S)$ and $\tilde{S}/(1-X)\tilde{S}$ are commutative Noetherian domains.

ii. or iii. This is an adaptation of Theorem 3.2. (1) of [4]; indeed, this theorem is phrased for ring theory but the fact that we work with graded objects. The proof of Theorem 3.2. (1) of [4] then carries over after the common modifications of graded nature.

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Now, we come the first main result of this paper which comes directly from definition and the above proposition.

**Proposition 3.** With conventions notation as before:

$\left(T, \underline{O}_T^g\right)$ is zip graded affine scheme.

Also, we may define the graded structure sheaf $\underline{O}_{n,T}^g$ on $T$: we may associate to $T(f); f \in G(S)$ homogenous element, the graded ring $S_{\tilde{f}(n)}^{-1}\left(\frac{\tilde{S}}{X^n\tilde{S}}\right)$, where $S_{\tilde{f}(n)}$ is the homogenous image of $S_{\tilde{f}}$ in $\tilde{S}$ and we obtain $\underline{O}_{n,T}^g$ on $T$ having as stalk at $P \in T$, the graded local ring $Q_{p(n)}^g\left(\tilde{S}/_{X^n\tilde{S}}\right) = S_{\tilde{p}(n)}^{-1}\left(\tilde{S}/_{X^n\tilde{S}}\right)$.

**Proposition 4.** Under the same assumptions**:**

If $\tilde{S}$ is zip domain, then $\forall\ n \in Z^+$, $\tilde{S}/_{X^n\tilde{S}} = \bar{\tilde{S}}(n)$ zip graded ring.

**Proof**. It is easily checked that, if $\bar{\tilde{I}}(n) \trianglelefteq \bar{\tilde{I}}\ \bar{\tilde{S}}(n)$: $ann^g\left(\bar{\tilde{I}}(n)\right) = 0; \bar{\tilde{I}}(n) = \tilde{I}/X^n\tilde{S}; \tilde{I} \trianglelefteq \bar{\tilde{I}}\ \tilde{S}$ .Then $ann^g(\tilde{I}) = 0$;. in $S$ . Then there exists $\tilde{I}_0 \trianglelefteq \tilde{I}(finite) : ann^g(\tilde{I}_0) = 0$. Then there exists $\tilde{I}_0/X^n\tilde{S} = \bar{\tilde{I}}_0 \trianglelefteq \bar{\tilde{I}}: ann^g\left(\bar{\tilde{I}}_0\right) = 0$ and we have $\bar{\tilde{S}}(n), \forall n$**,** is zip.

**Proposition 5. (Graded version of Theorem 3.2.(1) in [4])**

Under the same consideration: $Q_{\bar{\tilde{S}}}^g\left(\bar{\tilde{S}}\right)$ is zip graded ring.

Again, we can mention the second main important result of this paper which comes directly from definition and the above proposition.

**Proposition 6.** With conventions and notations as before:

If $\tilde{S}$ is zip domain then the graded structure affine scheme $\left(T, \underline{O}_{n,T}^g\right)$, $\forall\ n \in Z^+$, is zip graded affine scheme.

A similar result holds in case of the Rees graded micro-localization rings $\tilde{Q}_{\tilde{f}}^\mu(\tilde{S})$; $S_f = \{f, f^2, \cdots\}$ in $G(S)$. That is

$\tilde{Q}_{\tilde{f}}^\mu(\tilde{S}) = \lim\limits_{\overleftarrow{n}}{}^g\ Q_{\bar{\tilde{f}}(n)}^g\left(\bar{\tilde{S}}(n)\right) = \lim\limits_{\overleftarrow{n}}{}^g\ \left(\bar{\tilde{f}}(n)\right)^{-1}\left(\bar{\tilde{S}}(n)\right); \bar{\tilde{S}}(n) = \tilde{S}/_{X^n\tilde{S}}$ , see [1] and [9].

Now, if $\tilde{I}_{\tilde{f}}^\mu \trianglelefteq \tilde{Q}_{\tilde{f}}^\mu(\tilde{S})$ such that $ann^g\left(\tilde{I}_{\tilde{f}}^\mu\right) = 0$, $\tilde{I}_{\tilde{f}}^\mu = Q_{\tilde{f}(n)}^g\left(\tilde{I}_{\tilde{f}}^\mu\right)$ and $\tilde{I}_{\tilde{f}}^\mu \trianglelefteq \bar{\tilde{S}}(n) = Q_{\tilde{f}(n)}^g(\bar{\tilde{S}})$, representing $\tilde{I}_{\tilde{f}}^\mu$ at a level $n$ in the inverse limit. Hence $ann^g\left(Q_{\tilde{f}(n)}^g\left(\tilde{I}_{\tilde{f}}^\mu\right)\right) = 0$ and there exists $\bar{\tilde{I}}_0^\mu(n) \trianglelefteq \bar{\tilde{S}}(n)$ (as above) finite such that $ann^g\left(Q_{\tilde{f}(n)}^g\left(\bar{\tilde{I}}_0(n)\right)\right) = 0$. Then there exists $\tilde{I}_{0\tilde{f}}^\mu \trianglelefteq \tilde{I}_{\tilde{f}}^\mu$ (finite) such that $ann^g\left(\tilde{I}_{0\tilde{f}}^\mu\right) = 0$ and $\tilde{Q}_{\tilde{f}}^\mu(\tilde{S})$ is zip ring. It is noted that one may start the argument at any $m$ larger than $n$.

Now, associating to an open set $X(f)$ the micro-localizations $\tilde{Q}^{\mu}_{\tilde{f}}(\tilde{S})$, res. $Q^{\mu}_{f}(S)$ we obtain sheaf $\underline{\tilde{O}}^{\mu}_{X}$ , res. $\underline{O}^{\mu}_{X}$ having as the completed stalks at $P \in Spec^{g}(G(S))$ (or $T = Proj^{g}\ G(S)$ the ring $\tilde{Q}^{\mu}_{p}(\tilde{S})$,res. $Q^{\mu}_{P}(S)$, see [9]. Note that $\underline{O}^{\mu}_{X}$ is a sheaf of zariski rings and the $X-$adic completion

$$\tilde{Q}^{\mu}_{T,p}{}^{\wedge X} = \varprojlim_{\overleftarrow{n}}{}^{g} \frac{\varinjlim_{\vec{f}}{}^{g}\ \tilde{Q}^{\mu}_{\tilde{f}}(\tilde{S})}{X^{n} \varinjlim_{\vec{f}}{}^{g}\ \tilde{Q}^{\mu}_{\tilde{f}}(\tilde{S})} = \varprojlim_{\overleftarrow{n}}{}^{g}\ \varinjlim_{\vec{f}}{}^{g}\ \frac{\tilde{Q}^{\mu}_{\tilde{f}}(\tilde{S})}{X^{n}\,\tilde{Q}^{\mu}_{\tilde{f}}(\tilde{S})}$$

$$= \varprojlim_{\overleftarrow{n}}{}^{g}\ \varinjlim_{\vec{f}}{}^{g}\ \bar{\tilde{Q}}^{\mu}_{\tilde{f}(n)}(\bar{\tilde{S}}) = \tilde{Q}^{\mu}_{p}(\tilde{S})$$

, the micro-localization at $h(G(S) - P)$.

As above, at a level $n$, obtaining $Q^{\mu}_{P}(S)$ from $\tilde{Q}^{\mu}_{p}(\tilde{S})$ and one may easily prove that $\tilde{Q}^{\mu}_{p}(\tilde{S})$, res. $Q^{\mu}_{P}(S)$, are zip rings.

Hence the following result holds:

**Proposition 7.**

i. $\left(T,\ \underline{\tilde{O}}^{\mu}_{T}\right)$ is zip graded affine scheme.

ii. $\left(T,\ \underline{O}^{\mu}_{T}\right)$ is zip filtered affine scheme.

## References

[1] Asensio M. J., Van den Bergh M. and Van Oystaeyen F., A new algebraic approach to micro-localization of filtered rings, Trans. Amer. Math. Soc. 316 (1989), 15-25.

[2] Hortshorne R., Algebraic geometry, G.T.M. 52, Springer Verlag, New York, 1977.

[3] Huishi L. and Van Oystaeyen F., Zariskian filterations, Comm. In algebra, 17(12) (1989).

[4] Leroy A. and Matczuk J., Zip property of certain extensions, Journal of Pure and Applied Algebra, Vol. 220, No. 1(2016), P. 335-345.

[5] Lunqun O., Jinwang L. and Yuemin X., Extension of zip modules, J. of Advances in Mathematics (China), 43(5) (2014), 683-694.

[6] Nastasescu C. and Van Oystaen F., Graded and filtered rings and modulus, L.N. in Mathematics, Springer-verlag, Berlin, Heidelberg, New York, 1977.

[7] Nastasescu C. and Van Oystaen F., Graded ring theory, M. Library 28, North Holland, Amesterdam, 1981.

[8] Nawal M. NourEldeen, Radwan A. E., and Ahmed Aboubakr, On micro-localization of graded and filtered formal modules, accepted and to appear in Applied Mathematics and Information Sciences (2024).

[9] Radwan A. E. and Van Oystaeyen F., Micro-structure sheaves, formal schemes and quantum sections over projective schemes, In P of contact, France-Belgium, 1992.

[10] Radwan A. E., Filtered and graded micro-affine schemes, J. Inst. Math. And Comp. Sci., 5(1994), 73-81.

[11] Sharp R. Y., Graded annihilators and uniformly F-compatible ideals, Acta Math. (Vietnamica), 40 (2015), 179-195.