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Comparison Between Cauchy's Steepest Descent and Fletcher-Reeves Methods in Solving Unconstrained Neutrosophic Nonlinear Programming Problems

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Abstract:

The main objective of this study is to solve unconstrained single-valued Neutrosophic Nonlinear Programming Problems using Cauchy's Steepest Descent Method(CSDM) and the Fletcher-Reeves Method(FRM). Our approach is based on new arithmetic operations and ranking on the parametric representations of Triangular Neutrosophic Numbers(TNN). We prove some important theorems for Cauchy's Steepest Descent Method and the Fletcher-Reeves Method. Numerical examples are presented to illustrate the theory developed in this article. The outcomes of the proposed methods are compared.

Keywords: Cauchy's Steepest Descent Method; Fletcher Reeves method; Unconstrained optimization; Single-valued neutrosophic number; Arithmetic operations and Ranking

1. Introduction

The notion of fuzzy sets, proposed by Zadeh [27], has played a vital role in addressing real-world challenges. Smarandache [21] introduced neutrosophic sets to shade the problem of fuzzy sets that handles less and inconsistent information. It was invented as an protracted system of fuzzy, intuitionistic and traditional sets. Here indeterminacy is included together with membership and non-membership. Hence each element is associated with three parameters i.e., truth, indeterminacy, and falsity. The neutrosophic sets and logic are the utmost way of handling real-life problems, which obliges a generalization of fuzzy sets. Al-Naemi [1] introduced a novel parameter β_k^{Gh} derived from the memoryless self-scaling DFP quasi-Newton method. The author demonstrated that any line search technique ensuring sufficient descent property can be employed, and further established the validity of the Zoutendijk condition, proving the method's global convergence through a specific step-length approach. Using imprecise parameters to manage independent parameters. According to Andrei [2], an iterative process comprises two distinct components: determining a descent search direction $d_k$followed by a line search to identify an appropriate step size α_k. The search direction in the CSDM is precisely the negative gradient. Antczak [3] discussed an optimization problem involving a fuzzy objective function with inequality and equality constraints, all of which had locally Lipschitz functions. They successfully demonstrated that for this particular non-differentiable fuzzy optimization problem, they could create an associated bi-objective optimization problem. They also established a relationship between the vector optimization problem of Pareto solutions and the non-dominated (weakly) solutions of the original non-differentiable fuzzy optimization problem. Basim

[4] the aim of this work has been to develop new, modified conjugate gradient formulas that perform better than the conventional FR-CG method for picture restoration. The outcomes validate the efficacy of the approach employed in this study to generate variations of the traditional CG procedure. Bellman et al. [5] delved into the field of decision-making processes under conditions of fuzziness. Chakraborty [6] discussed a trapezoidal neutrosophic number is a more useful concept than a triangular one. The removal of area method (i) and mean of interval method (ii) are two important de-neutrosophication methods that are effective in de-neutrosophicating the corresponding number. Hanachi et al. [7] introduced a novel hybrid approach in which the parameter is determined as a convex combination of three parameters $\beta_k^{DY}, \beta_k^{PRP}, \beta_k^{PR}$ the adequacy of the descent and the global convergence has demonstrated. Practical results indicate that the chosen approach is betterÂ and more efficient when compared to the utilized methods. Hassan et al. [8] discussed that the research focus has centered on developing novel, modified conjugate gradient formulations that outperform the traditional FR CG approach in image restoration applications. The findings demonstrate the efficacy of the strategy employed in this research to derive variations of the traditional CG technique. The proposed methods have demonstrated global convergence under the rigorous Wolfe line search conditions. Husin et al. [9] presented a new steepest descent (SD) method, with the primary goal being the modification of the direction that has sufficient descent requirements and global convergence properties. The second strategy involves presenting regression analysis for real-world issues using the suggested modification of the SD method. Hepzibah et al. [10, 11] discussed how unconstrained optimization problems with fuzzy valued functions are taken into consideration as well as how to solve unconstrained optimization problems using Newton's method with SVNTN coefficients. Moreover, single variable and multivariable fuzzy unconstrained optimization problems using Interval Newton's Method are discussed with illustrations. Kaliyaperumal and Das [12] introduced a fuzzy version of the problem, which they addressed using the necessary and sufficient conditions of Lagrangian multipliers with a focus on fuzziness. They illustrated this approach with a numerical example. By solving two numerical examples on using membership functions (MFs) and the other using robust rankings they clarified the model's effectiveness. This model is designed to tackle uncertainties and subjective experiences of decision-makers and can assist in resolving challenges associated with decision-making. Kanaya [13] discussed a method for solving multi-objective nonlinear programming problems involving fuzzy parameters in the objective functions. Utilizing an interactive cutting-plane algorithm, this method relies on the stability set corresponding to α-Pareto optimal solutions obtained through the same method. Comprehensive study on NLNNs and NLN-LPPs was presented by Lachhwani [14]. Based on the proposed modified possibility score function, the author developed a novel solution technique for NLN-LPPs. Nearly the whole range of NN values is covered by the suggested modified possibility score function. Maissam Jdid [15] demonstrated how binary integers can be used to convert certain nonlinear models into linear ones, resulting in integer solutions that are consistent with the nature of the problems being studied. Ming ma et al. [16] defined a new fuzzy arithmetic and applied to fuzzy linear equations. Nagoorgani and Sudha [17] studied the optimality requirements of fuzzy nonlinear unconstrained minimization issues. The cost coefficients are denoted by TFNs and illustrated with several numerical examples. An alternative definition for NLNN was provided by Reig-Mullor [18], which works over some of the limitations mentioned in the literature.

Also, they developed the notions of α, β, γ-cuts, possibilistic variance, possibilistic mean, and possibilistic standard deviation, which are among the primary characteristics of NLNN. Seikh Mijanur Rahaman [19] proposed solution methodologies for solving a unique matrix game, whereby the payoffs are expressed using single-valued neutrosophic numbers (SVNNs). Two non-linear multi-objective programming tasks are defined to determine the ideal values and strategies for the participants. These multi-objective programming issues are converted into bi-objective programming problems by assigning equal significance to the objective functions. Shilpa Ivin Emimal and Irene Hepzibah [20] introduced the optimization of intuitionistic fuzzy valued functions in unconstrained problems. The Interval Newton's Method using an Intuitionistic approach addresses both single and multivariable optimization problems. Fuzzy sets (FS) and neutrosophic sets (NS) were combined by Sujit Das [22] to formulate neutrosophic fuzzy sets (NFS). The primary characteristic of NFS is its ability to handle inconsistent and imprecise data, which is highly useful when managing uncertain and inconsistent real-world applications. Uma Maheswari and Ganesan [23] introduced a fuzzy interpretation of the KKT condition, specifically for FFNLPP, and successfully identified their optimal fuzzy outcomes. Utilizing the Gradient technique, also recognized as CSDM, they transformed it into an unconstrained optimization problem involving multiple fuzzy variables. Vanaja and Ganesan [24,25] developed an interior fuzzy penalty function method to solve Fuzzy Nonlinear Programming Problems (FNLPP). They introduced innovative fuzzy arithmetic and ranking techniques based on the parametric representation of triangular fuzzy numbers. Additionally, they applied the exterior penalty method using fuzzy-valued functions to solve these FNLPPs. Ye [26] presented the notions of NN linear and nonlinear functions, as well as inequalities that include indeterminacy I, along with some basic operations of neutrosophic numbers (NNs). Afterwards, the authors presented general solution techniques for a variety of optimization models for NN nonlinear programming (NN-NP) issues with unconstrained and restricted optimizations.

In this paper, we discuss Single-Valued Neutrosophic Nonlinear Programming Problem (SVNNLPP) involving triangular neutrosophic numbers. We present a neutrosophic version of the Cauchy's Steepest Descent Method (CSDM) and Fletcher- Reeves Method (FRM) approaches based on a new method of neutrosophic arithmetic operations on the parametric representations of neutrosophic numbers and ranking on the neutrosophic numbers of the parametric forms. We prove theorems for the solution of SVNNLPP without converting the given problem to its equivalent crisp form. Numerical examples are presented to illustrate the efficacy of the proposed solution methods.

The rest of this article is organized as follows. Section 2 provides some basic preliminaries and results on neutrosophic numbers. Section 3 provides NNLPP. Theorems on CSDM and FRM are provided in section 4. Section 5 presents the algorithms for CSDM and FRM. Section 6 deals with numerical examples. Result and discussion are given section 7. Finally, a brief conclusion is given in section 8.

2. Preliminaries

We explored Neutrosophic definitions, which capture the core of Neutrosophy by presenting subtle viewpoints on concepts frequently experienced in situations of uncertainty. By employing these definitions, our objective is to shed light the offer valuable perspectives on its impact on theoretical frameworks.

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Definition 2.1 Let Y be a universe of discourse. A neutrosophic set $\tilde{\mathcal{A}}^{\mathfrak{n}}$ in υ is characterized by a truth-membership function $T_{\tilde{\mathcal{A}}^{\mathfrak{n}}}$ a indeterminacy-membership function $I_{\tilde{\mathcal{A}}^{\mathfrak{n}}}$ and a falsity-membership function $F_{\tilde{\mathcal{A}}^{\mathfrak{n}}}$. $T_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon); I_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon)$ and $F_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon)$ are are real standard elements of [0,1]. It can be written as $\tilde{\mathcal{A}}^{\mathfrak{n}} = \{< \upsilon, (T_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon), I_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon), F_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon)) > | \upsilon \in Y\}$. The function $T_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon); I_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon)$ and $F_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon)$ are real standard or non standard subsets of $]0^{-}, 1^{+}[$ \textcolor{white}{"}

i.e.,$T_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon), I_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon), F_{\tilde{\mathcal{A}}^{\mathfrak{n}}}: Y \to]0^{-}, 1^{+}[$. There is No restriction is applied on the sum of $T_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon); I_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon)$ and $F_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon)$, so $0^{-} \leq \sup T_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon), +\sup I_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon), +\sup F_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon) \leq 3^{+}$. For a fixed $\upsilon = \upsilon \in Y, T_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon), I_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon), F_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon)$, i.e., in simply, $T_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon), I_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon), F_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon)$ is called Neutrosophic Number (NN). \textcolor{white}{``}

Definition 2.2 Let Y be a universe of discourse. A single valued neutrosophic set (SVNS) $\tilde{\mathcal{A}}^{\mathfrak{n}}$ over Y is an object having the form $\tilde{\mathcal{A}}^{\mathfrak{n}} = \{< \upsilon, (T_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon), I_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon), F_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon)) > | \upsilon \in Y\}$ where $T_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon): Y \to]0^{-}, 1^{+}[, I_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon): Y \to]0^{-}, 1^{+}[, F_{\tilde{\mathcal{A}}^{\mathfrak{n}}}: Y \to]0^{-}, 1^{+}[$ are truth, indeterminacy and falsity membership with $0 \leq T_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon), +I_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon), +F_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon) \leq 3$ for all $\upsilon \in Y$. \textcolor{white}{"}

Definition 2.3 A Single Valued Neutrosophic Number (SVNN) $\tilde{\mathcal{A}}^{\mathfrak{n}} = \{< \upsilon, (T_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon), I_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon), F_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon)) > | \upsilon \in Y\}$, subset of a real line, is called generalised neutrosophic number if \textcolor{white}{``}

1. $\tilde{\mathcal{A}}^{\mathfrak{n}}$ is neut-normal.
2. $\tilde{\mathcal{A}}^{\mathfrak{n}}$ is neut-convex.
3. $T_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon)$ is upper semi-continuous, $I_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon)$ is lower semi continuous and $F_{\tilde{\mathcal{A}}^{\mathfrak{n}}}(\upsilon)$ is lower semi continuous.
4. $\tilde{\mathcal{A}}^{\mathfrak{n}}$ is support, i.e. $S(\tilde{\mathcal{A}}^{\mathfrak{n}}) = \upsilon \in Y: T_{\tilde{\mathcal{A}}^{\mathfrak{n}}} > 0, I_{\tilde{\mathcal{A}}^{\mathfrak{n}}} < 1, F_{\tilde{\mathcal{A}}^{\mathfrak{n}}} < 1$ is bounded.

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Definition 2.4 A single valued neutrosophic number $\tilde{\mathcal{A}}^{\mathfrak{n}}$ is Triangular Neutrosophic Number (TNN) and is denoted by $\tilde{\mathcal{A}}^{\mathfrak{n}} = (\langle \mathfrak{a}_{T_1}, \mathfrak{a}_{T_2}, \mathfrak{a}_{T_3} \rangle; \langle \mathfrak{a}_{I_1}, \mathfrak{a}_{I_2}, \mathfrak{a}_{I_3} \rangle; \langle \mathfrak{a}_{F_1}, \mathfrak{a}_{F_2}, \mathfrak{a}_{F_3} \rangle)$ having the membership function, indeterminacy function and non-membership function as follows.a

$$\mu_T(\upsilon) = \begin{cases} \frac{\upsilon - \mathfrak{a}_{T_1}}{\mathfrak{a}_{T_2} - \mathfrak{a}_{T_1}}, & \mathfrak{a}_{T_1} \leq \upsilon \leq \mathfrak{a}_{T_2} \\ \frac{\mathfrak{a}_{T_3} - \upsilon}{\mathfrak{a}_{T_3} - \mathfrak{a}_{T_2}}, & \mathfrak{a}_{T_2} \leq \upsilon \leq \mathfrak{a}_{T_3} \\ 0, & \text{elsewhere} \end{cases}$$

$$\mu_I(\upsilon) = \begin{cases} \frac{\upsilon - \mathfrak{a}_{I_1}}{\mathfrak{a}_{I_2} - \mathfrak{a}_{I_1}}, & \mathfrak{a}_{I_1} \leq \upsilon \leq \mathfrak{a}_{I_2} \\ \frac{\mathfrak{a}_{I_3} - \upsilon}{\mathfrak{a}_{I_3} - \mathfrak{a}_{I_2}}, & \mathfrak{a}_{I_2} \leq \upsilon \leq \mathfrak{a}_{I_3} \\ 0, & \text{elsewhere} \end{cases}$$

$$\mu_F(\upsilon) = \begin{cases} \frac{\upsilon - \mathfrak{a}_{F_1}}{\mathfrak{a}_{F_2} - \mathfrak{a}_{F_1}}, & \mathfrak{a}_{F_1} \le \upsilon \le \mathfrak{a}_{F_2} \\ \frac{\mathfrak{a}_{F_3} - \upsilon}{\mathfrak{a}_{F_3} - \mathfrak{a}_{F_2}}, & \mathfrak{a}_{F_2} \le \upsilon \le \mathfrak{a}_{F_3} \\ 0, & \text{elsewhere} \end{cases}$$

We use $\mathscr{F}(\mathscr{R})$ to represent the set of all triangular neutrosophic numbers defined on $\mathscr{R}$.

Definition 2.5 The (α, β, γ)-cut of neutrosophic set is denoted by $\mathscr{F}(\alpha, \beta, \gamma)$, where $(\alpha, \beta, \gamma) \in [0,1]$ and are fixed numbers, such that $\alpha + \beta + \gamma \le 3$ and is defined as $\mathscr{F}(\alpha, \beta, \gamma) = (\mu_T(\upsilon), \mu_I(\upsilon), \mu_F(\upsilon))$, where $\upsilon \in Y, \mu_T(\upsilon) \ge \alpha, \mu_I(\upsilon) \le \beta, \mu_F(\upsilon) \le \gamma$.

Definition 2.6 A triangular neutrosophic number $\tilde{\mathscr{A}}^{\mathfrak{n}}$ can also be represented as a pair $\tilde{\mathscr{A}}^{\mathfrak{n}}_T = (\underline{\mathfrak{a}}_T;\ \overline{\mathfrak{a}}_T), \tilde{\mathscr{A}}^{\mathfrak{n}}_I = (\underline{\mathfrak{a}}_I;\ \overline{\mathfrak{a}}_I), \tilde{\mathscr{A}}^{\mathfrak{n}}_F = (\underline{\mathfrak{a}}_F;\ \overline{\mathfrak{a}}_F)$ of functions $\underline{\mathfrak{a}}_T(r), \overline{\mathfrak{a}}_T(r), \underline{\mathfrak{a}}_I(r), \overline{\mathfrak{a}}_I(r), \underline{\mathfrak{a}}_F(r), \overline{\mathfrak{a}}_F(r)$ $0 \le r \le 1$ which satisfy the following requirements:

- $\underline{\mathfrak{a}}_T(r)$ is a bounded monotonic increasing left continuous function for membership function.
- $\overline{\mathfrak{a}}_T(r)$ is a bounded monotonic decreasing left continuous function for membership function.
- $\underline{\mathfrak{a}}_I(r)$ is a bounded monotonic increasing left continuous function for indeterminacy function.
- $\overline{\mathfrak{a}}_I(r)$ is a bounded monotonic decreasing left continuous function for indeterminacy function.
- $\underline{\mathfrak{a}}_F(r)$ is a bounded monotonic increasing left continuous function for non-membership function.
- $\overline{\mathfrak{a}}_F(r)$ is a bounded monotonic decreasing left continuous function for non-membership function.
- $\underline{\mathfrak{a}}_T(r) \le \overline{\mathfrak{a}}_T(r), \underline{\mathfrak{a}}_I(r) \le \overline{\mathfrak{a}}_I(r), \underline{\mathfrak{a}}_F(r) \le \overline{\mathfrak{a}}_F(r), 0 \le r \le 1$.

Definition 2.7 (**Parametric Form**)

Let $\tilde{\mathscr{A}}^{\mathfrak{n}} = (\mathfrak{a}_T, \mathfrak{a}_I, \mathfrak{a}_F)$ be a triangular neutrosophic number and $\overline{\mathfrak{a}}_T(r) = \mathfrak{a}_{T_3} - (\mathfrak{a}_{T_3} - \mathfrak{a}_{T_2})r$, $\underline{\mathfrak{a}}_T(r) = \mathfrak{a}_{T_1} + (\mathfrak{a}_{T_2} - \mathfrak{a}_{T_1})r$, $\overline{\mathfrak{a}}_I(r) = \mathfrak{a}_{I_3} - (\mathfrak{a}_{I_3} - \mathfrak{a}_{I_2})r$, $\underline{\mathfrak{a}}_I(r) = \mathfrak{a}_{I_1} + (\mathfrak{a}_{I_2} - \mathfrak{a}_{I_1})r$, $\overline{\mathfrak{a}}_F(r) = \mathfrak{a}_{F_3} - (\mathfrak{a}_{F_3} - \mathfrak{a}_{F_2})r$, $\underline{\mathfrak{a}}_F(r) = \mathfrak{a}_{F_1} + (\mathfrak{a}_{F_2} - \mathfrak{a}_{F_1})r$, $r \in [0,1]$. The parametric form of the TNN is defined as $\tilde{\mathscr{A}} = (\langle \mathfrak{a}_{T_0}, \mathfrak{a}_{T_*}, \mathfrak{a}_{T^*} \rangle; \langle \mathfrak{a}_{I_0}, \mathfrak{a}_{I_*}, \mathfrak{a}_{I^*} \rangle; \langle \mathfrak{a}_{F_0}, \mathfrak{a}_{F_*}, \mathfrak{a}_{F^*} \rangle)$, where $\mathfrak{a}_{T_*} = \mathfrak{a}_{T_0} - \underline{\mathfrak{a}}_T$ and $\mathfrak{a}_{T^*} = \overline{\mathfrak{a}}_T - \mathfrak{a}_{T_0}$, $\mathfrak{a}_{I_*} = \mathfrak{a}_{I_0} - \underline{\mathfrak{a}}_I$ and $\mathfrak{a}_{I^*} = \overline{\mathfrak{a}}_I - \mathfrak{a}_{I_0}$, $\mathfrak{a}_{F_*} = \mathfrak{a}_{F_0} - \underline{\mathfrak{a}}_F$ and $\mathfrak{a}_{F^*} = \overline{\mathfrak{a}}_F - \mathfrak{a}_{F_0}$ are the left and right fuzziness index functions respectively. The number $\mathfrak{a}_{T_0} = \left(\frac{\underline{\mathfrak{a}}_T(1) + \overline{\mathfrak{a}}_T(1)}{2}\right), \mathfrak{a}_{I_0} = \left(\frac{\underline{\mathfrak{a}}_I(1) + \overline{\mathfrak{a}}_I(1)}{2}\right), \mathfrak{a}_{F_0} = \left(\frac{\underline{\mathfrak{a}}_F(1) + \overline{\mathfrak{a}}_F(1)}{2}\right)$ is called the location index number. When $r = 1$, we get $\mathfrak{a}_{T_0} = \mathfrak{a}_{T_2}, \mathfrak{a}_{I_0} = \mathfrak{a}_{I_2}, \mathfrak{a}_{F_0} = \mathfrak{a}_{F_2}$.

2.1 Arithmetic Operations on Neutrosophic Numbers

We develop a new fuzzy arithmetic using the parametric form of triangular neutrosophic numbers. This involves expressing the numbers in terms of location index and fuzziness index functions for membership, indeterminacy, and non-membership functions. We propose a new neutrosophic arithmetic operation, where the lattice rule, which involves the least upper bound and greatest lower bound in the lattice L, defines the fuzziness index functions and the location index number corresponds to standard arithmetic. That is for $\mathfrak{a}, \mathfrak{b} \in L$, $\mathfrak{a} \vee \mathfrak{b} = \max\{\mathfrak{a}, \mathfrak{b}\}$ and $\mathfrak{a} \wedge \mathfrak{b} = \min\{\mathfrak{a}, \mathfrak{b}\}$. For any two fuzzy numbers $\tilde{\mathscr{A}}^{\mathfrak{n}} = (\langle \mathfrak{a}_{T_0}, \mathfrak{a}_{T_*}, \mathfrak{a}_{T^*} \rangle; \langle \mathfrak{a}_{I_0}, \mathfrak{a}_{I_*}, \mathfrak{a}_{I^*} \rangle; \langle \mathfrak{a}_{F_0}, \mathfrak{a}_{F_*}, \mathfrak{a}_{F^*} \rangle)$, $\tilde{\mathscr{B}}^{\mathfrak{n}} = (\langle \mathfrak{b}_{T_0}, \mathfrak{b}_{T_*}, \mathfrak{b}_{T^*} \rangle; \langle \mathfrak{b}_{I_0}, \mathfrak{b}_{I_*}, \mathfrak{b}_{I^*} \rangle; \langle \mathfrak{b}_{F_0}, \mathfrak{b}_{F_*}, \mathfrak{b}_{F^*} \rangle)$ the arithmetic operations are defined as

$$\begin{aligned} \tilde{\mathscr{A}}^{\mathfrak{n}} * \tilde{\mathscr{B}}^{\mathfrak{n}} = {} & (\langle \mathfrak{a}_{T_0}, \mathfrak{a}_{T_*}, \mathfrak{a}_{T^*} \rangle; \langle \mathfrak{a}_{I_0}, \mathfrak{a}_{I_*}, \mathfrak{a}_{I^*} \rangle; \langle \mathfrak{a}_{F_0}, \mathfrak{a}_{F_*}, \mathfrak{a}_{F^*} \rangle) * (\langle \mathfrak{b}_{T_0}, \mathfrak{b}_{T_*}, \mathfrak{b}_{T^*} \rangle; \\ & \langle \mathfrak{b}_{I_0}, \mathfrak{b}_{I_*}, \mathfrak{b}_{I^*} \rangle; \langle \mathfrak{b}_{F_0}, \mathfrak{b}_{F_*}, \mathfrak{b}_{F^*} \rangle) \\ = {} & (\langle \mathfrak{a}_{T_0} * \mathfrak{b}_{T_0}, \mathfrak{a}_{T_*} \vee \mathfrak{b}_{T_*}, \mathfrak{a}_{T^*} \vee \mathfrak{b}_{T^*} \rangle, \langle \mathfrak{a}_{I_0} * \mathfrak{b}_{I_0}, \mathfrak{a}_{I_*} \vee \mathfrak{b}_{I_*}, \mathfrak{a}_{I^*} \vee \mathfrak{b}_{I^*} \rangle, \\ & \langle \mathfrak{a}_{F_0} * \mathfrak{b}_{F_0}, \mathfrak{a}_{F_*} \vee \mathfrak{b}_{F_*}, \mathfrak{a}_{F^*} \vee \mathfrak{b}_{F^*} \rangle) \end{aligned}$$

In particular for $\tilde{\mathscr{A}}^{\mathfrak{n}} = (\langle \mathfrak{a}_{T_0}, \mathfrak{a}_{T_*}, \mathfrak{a}_{T^*} \rangle; \langle \mathfrak{a}_{I_0}, \mathfrak{a}_{I_*}, \mathfrak{a}_{I^*} \rangle; \langle \mathfrak{a}_{F_0}, \mathfrak{a}_{F_*}, \mathfrak{a}_{F^*} \rangle)$,

$\tilde{\mathscr{B}}^{\mathfrak{n}} = (\langle \mathfrak{b}_{T_0}, \mathfrak{b}_{T_*}, \mathfrak{b}_{T^*} \rangle; \langle \mathfrak{b}_{I_0}, \mathfrak{b}_{I_*}, \mathfrak{b}_{I^*} \rangle; \langle \mathfrak{b}_{F_0}, \mathfrak{b}_{F_*}, \mathfrak{b}_{F^*} \rangle) \in \mathscr{F}(\mathscr{R})$, we have

Addition:

$$\begin{aligned} \tilde{\mathscr{A}}^{\mathfrak{n}} * \tilde{\mathscr{B}}^{\mathfrak{n}} = {} & (\langle \mathfrak{a}_{T_0}, \mathfrak{a}_{T_*}, \mathfrak{a}_{T^*} \rangle; \langle \mathfrak{a}_{I_0}, \mathfrak{a}_{I_*}, \mathfrak{a}_{I^*} \rangle; \langle \mathfrak{a}_{F_0}, \mathfrak{a}_{F_*}, \mathfrak{a}_{F^*} \rangle) + (\langle \mathfrak{b}_{T_0}, \mathfrak{b}_{T_*}, \mathfrak{b}_{T^*} \rangle; \\ & \langle \mathfrak{b}_{I_0}, \mathfrak{b}_{I_*}, \mathfrak{b}_{I^*} \rangle; \langle \mathfrak{b}_{F_0}, \mathfrak{b}_{F_*}, \mathfrak{b}_{F^*} \rangle) \\ = {} & (\langle \mathfrak{a}_{T_0} + \mathfrak{b}_{T_0}, \mathfrak{a}_{T_*} \vee \mathfrak{b}_{T_*}, \mathfrak{a}_{T^*} \vee \mathfrak{b}_{T^*} \rangle, \langle \mathfrak{a}_{I_0} + \mathfrak{b}_{I_0}, \mathfrak{a}_{I_*} \vee \mathfrak{b}_{I_*}, \mathfrak{a}_{I^*} \vee \mathfrak{b}_{I^*} \rangle, \\ & \langle \mathfrak{a}_{F_0} + \mathfrak{b}_{F_0}, \mathfrak{a}_{F_*} \vee \mathfrak{b}_{F_*}, \mathfrak{a}_{F^*} \vee \mathfrak{b}_{F^*} \rangle) \end{aligned}$$

Subtraction:

$$\begin{aligned} \tilde{\mathscr{A}}^{\mathfrak{n}} * \tilde{\mathscr{B}}^{\mathfrak{n}} = {} & (\langle \mathfrak{a}_{T_0}, \mathfrak{a}_{T_*}, \mathfrak{a}_{T^*} \rangle; \langle \mathfrak{a}_{I_0}, \mathfrak{a}_{I_*}, \mathfrak{a}_{I^*} \rangle; \langle \mathfrak{a}_{F_0}, \mathfrak{a}_{F_*}, \mathfrak{a}_{F^*} \rangle) - (\langle \mathfrak{b}_{T_0}, \mathfrak{b}_{T_*}, \mathfrak{b}_{T^*} \rangle; \\ & \langle \mathfrak{b}_{I_0}, \mathfrak{b}_{I_*}, \mathfrak{b}_{I^*} \rangle; \langle \mathfrak{b}_{F_0}, \mathfrak{b}_{F_*}, \mathfrak{b}_{F^*} \rangle) \\ = {} & (\langle \mathfrak{a}_{T_0} - \mathfrak{b}_{T_0}, \mathfrak{a}_{T_*} \vee \mathfrak{b}_{T_*}, \mathfrak{a}_{T^*} \vee \mathfrak{b}_{T^*} \rangle, \langle \mathfrak{a}_{I_0} - \mathfrak{b}_{I_0}, \mathfrak{a}_{I_*} \vee \mathfrak{b}_{I_*}, \mathfrak{a}_{I^*} \vee \mathfrak{b}_{I^*} \rangle, \\ & \langle \mathfrak{a}_{F_0} - \mathfrak{b}_{F_0}, \mathfrak{a}_{F_*} \vee \mathfrak{b}_{F_*}, \mathfrak{a}_{F^*} \vee \mathfrak{b}_{F^*} \rangle) \end{aligned}$$

Multiplication:

$$\begin{aligned} \tilde{\mathscr{A}}^{\mathfrak{n}} * \tilde{\mathscr{B}}^{\mathfrak{n}} = {} & (\langle \mathfrak{a}_{T_0}, \mathfrak{a}_{T_*}, \mathfrak{a}_{T^*} \rangle; \langle \mathfrak{a}_{I_0}, \mathfrak{a}_{I_*}, \mathfrak{a}_{I^*} \rangle; \langle \mathfrak{a}_{F_0}, \mathfrak{a}_{F_*}, \mathfrak{a}_{F^*} \rangle) \times (\langle \mathfrak{b}_{T_0}, \mathfrak{b}_{T_*}, \mathfrak{b}_{T^*} \rangle; \\ & \langle \mathfrak{b}_{I_0}, \mathfrak{b}_{I_*}, \mathfrak{b}_{I^*} \rangle; \langle \mathfrak{b}_{F_0}, \mathfrak{b}_{F_*}, \mathfrak{b}_{F^*} \rangle) \\ = {} & (\langle \mathfrak{a}_{T_0} \times \mathfrak{b}_{T_0}, \mathfrak{a}_{T_*} \vee \mathfrak{b}_{T_*}, \mathfrak{a}_{T^*} \vee \mathfrak{b}_{T^*} \rangle, \langle \mathfrak{a}_{I_0} \times \mathfrak{b}_{I_0}, \mathfrak{a}_{I_*} \vee \mathfrak{b}_{I_*}, \mathfrak{a}_{I^*} \vee \mathfrak{b}_{I^*} \rangle, \\ & \langle \mathfrak{a}_{F_0} \times \mathfrak{b}_{F_0}, \mathfrak{a}_{F_*} \vee \mathfrak{b}_{F_*}, \mathfrak{a}_{F^*} \vee \mathfrak{b}_{F^*} \rangle) \end{aligned}$$

Division:

$$\begin{aligned} \tilde{\mathscr{A}}^{\mathfrak{n}} * \tilde{\mathscr{B}}^{\mathfrak{n}} = {} & (\langle \mathfrak{a}_{T_0}, \mathfrak{a}_{T_*}, \mathfrak{a}_{T^*} \rangle; \langle \mathfrak{a}_{I_0}, \mathfrak{a}_{I_*}, \mathfrak{a}_{I^*} \rangle; \langle \mathfrak{a}_{F_0}, \mathfrak{a}_{F_*}, \mathfrak{a}_{F^*} \rangle) \div (\langle \mathfrak{b}_{T_0}, \mathfrak{b}_{T_*}, \mathfrak{b}_{T^*} \rangle; \\ & \langle \mathfrak{b}_{I_0}, \mathfrak{b}_{I_*}, \mathfrak{b}_{I^*} \rangle; \langle \mathfrak{b}_{F_0}, \mathfrak{b}_{F_*}, \mathfrak{b}_{F^*} \rangle) \\ = {} & (\langle \mathfrak{a}_{T_0} \div \mathfrak{b}_{T_0}, \mathfrak{a}_{T_*} \vee \mathfrak{b}_{T_*}, \mathfrak{a}_{T^*} \vee \mathfrak{b}_{T^*} \rangle, \langle \mathfrak{a}_{I_0} \div \mathfrak{b}_{I_0}, \mathfrak{a}_{I_*} \vee \mathfrak{b}_{I_*}, \mathfrak{a}_{I^*} \vee \mathfrak{b}_{I^*} \rangle, \\ & \langle \mathfrak{a}_{F_0} \div \mathfrak{b}_{F_0}, \mathfrak{a}_{F_*} \vee \mathfrak{b}_{F_*}, \mathfrak{a}_{F^*} \vee \mathfrak{b}_{F^*} \rangle) \end{aligned}$$

provided $\mathfrak{b}_{T_0}, \mathfrak{b}_{I_0}, \mathfrak{b}_{F_0} \neq 0$.

2.2 **Ranking of neutrosophic Numbers**

The ranking of neutrosophic numbers plays an essential role in the decision-making process within a fuzzy environment. Various authors in the literature have proposed different types of ranking methods. This article uses a highly effective ranking technique based on the graded mean. For $\tilde{\mathscr{A}}^{\mathfrak{n}} =$

$(\langle \mathfrak{a}_{T_0}, \mathfrak{a}_{T_*}, \mathfrak{a}_{T^*} \rangle; \langle \mathfrak{a}_{I_0}, \mathfrak{a}_{I_*}, \mathfrak{a}_{I^*} \rangle; \langle \mathfrak{a}_{F_0}, \mathfrak{a}_{F_*}, \mathfrak{a}_{F^*} \rangle) \in \mathscr{F}(\mathscr{R})$, define $\Re: \mathscr{F}(\mathscr{R}) \to \mathscr{R}$ by $\Re(\tilde{\mathscr{A}}^{\mathfrak{n}}) = \left(\frac{\langle \mathfrak{a}_{T_*}+4\mathfrak{a}_{T_0}+\mathfrak{a}_{T^*} \rangle, \langle \mathfrak{a}_{I_*}+4\mathfrak{a}_{I_0}+\mathfrak{a}_{I^*} \rangle, \langle \mathfrak{a}_{F_*}+4\mathfrak{a}_{F_0}+\mathfrak{a}_{F^*} \rangle}{6}\right)$. For any two triangular fuzzy numbers $\tilde{\mathscr{A}}^{\mathfrak{n}} = (\langle \mathfrak{a}_{T_0}, \mathfrak{a}_{T_*}, \mathfrak{a}_{T^*} \rangle; \langle \mathfrak{a}_{I_0}, \mathfrak{a}_{I_*}, \mathfrak{a}_{I^*} \rangle; \langle \mathfrak{a}_{F_0}, \mathfrak{a}_{F_*}, \mathfrak{a}_{F^*} \rangle)$ and $\tilde{\mathscr{B}}^{\mathfrak{n}} = (\langle \mathfrak{b}_{T_0}, \mathfrak{b}_{T_*}, \mathfrak{b}_{T^*} \rangle; \langle \mathfrak{b}_{I_0}, \mathfrak{b}_{I_*}, \mathfrak{b}_{I^*} \rangle; \langle \mathfrak{b}_{F_0}, \mathfrak{b}_{F_*}, \mathfrak{b}_{F^*} \rangle) \in \mathscr{F}(\mathscr{R})$, we have the following comparison:

- If $\Re(\tilde{\mathscr{A}}^{\mathfrak{n}}) < \Re(\tilde{\mathscr{B}}^{\mathfrak{n}})$, then $\tilde{\mathscr{A}}^{\mathfrak{n}} \prec \tilde{\mathscr{B}}^{\mathfrak{n}}$
- If $\Re(\tilde{\mathscr{A}}^{\mathfrak{n}}) > \Re(\tilde{\mathscr{B}}^{\mathfrak{n}})$, then $\tilde{\mathscr{A}}^{\mathfrak{n}} \succ \tilde{\mathscr{B}}^{\mathfrak{n}}$
- If $\Re(\tilde{\mathscr{A}}^{\mathfrak{n}}) = \Re(\tilde{\mathscr{B}}^{\mathfrak{n}})$, then $\tilde{\mathscr{A}}^{\mathfrak{n}} \approx \tilde{\mathscr{B}}^{\mathfrak{n}}$.

Note 1. $\Re\left(\tilde{\mathscr{A}}^{\mathfrak{n}} + \tilde{\mathscr{B}}^{\mathfrak{n}}\right) = \Re(\tilde{\mathscr{A}}^{\mathfrak{n}}) + \Re\left(\tilde{\mathscr{B}}^{\mathfrak{n}}\right)$

Note 2. $\Re\left(\tilde{\mathscr{A}}^{\mathfrak{n}}.\tilde{\mathscr{B}}^{\mathfrak{n}}\right) = \Re(\tilde{\mathscr{A}}^{\mathfrak{n}}).\Re\left(\tilde{\mathscr{B}}^{\mathfrak{n}}\right)$

Note 3. $\Re\left(\tilde{\mathscr{A}}^{\mathfrak{n}}/\tilde{\mathscr{B}}^{\mathfrak{n}}\right) = \Re(\tilde{\mathscr{A}}^{\mathfrak{n}})/\Re\left(\tilde{\mathscr{B}}^{\mathfrak{n}}\right)$, provided $\Re\left(\tilde{\mathscr{B}}^{\mathfrak{n}}\right) \neq 0$.

3. Neutrosophic Non Linear Programming Problems (NNLPP)

Let general NNLPP

$$\begin{array}{lll} \text{"} \quad min & \tilde{f}^{\mathcal{N}}(\tilde{v}_i) & \\ sub.\ to & \tilde{h}_i^{\mathcal{N}}(\tilde{v}_i) \approx \tilde{0} \ for \ i = 1,2,\cdots,\ell & \\ & \tilde{g}_j^{\mathcal{N}}(\tilde{v}_i) \preceq \tilde{0} \ for \ j = 1,2,\cdots,m & \\ & \tilde{v}_i \succeq \tilde{0}\text{"} & \end{array} \qquad (1)$$

where $\tilde{f}^{\mathcal{N}}, \tilde{h}_1^{\mathcal{N}}, \cdots, \tilde{h}_l^{\mathcal{N}}, \tilde{g}_1^{\mathcal{N}}, \cdots, \tilde{g}_m^{\mathcal{N}}$ are continuous neutrosophic valued functions defined on R^n.

A vector $\tilde{v}_i = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \cdots, \tilde{v}_n)$ is considered a feasible solution to the NNLPP (1) if it meets all the constraints and adheres to the non-negativity condition of the NNLPP. The collection of all such feasible solutions is known as the feasible region, which is defined by these criteria.

$$\tilde{v} = \{\tilde{\boldsymbol{v}}_{\boldsymbol{i}} \in \mathrm{R}^n / \tilde{h}_i^{\mathcal{N}}(\tilde{v}) \approx \tilde{0} \ for \ i = 1,2,\cdots,\ell, \tilde{g}_j^{\mathcal{N}}(\tilde{v}) for \ j = 1,2,\cdots,m \ and \ \tilde{v}_i \succeq \tilde{0}\}.$$

4. Solution Methods

We investigate two Neutrosophic optimization methods namely Neutrosophic Cauchy's Steepest Descent Method and the Fletcher-Reeves Method (also called the Conjugate Gradient Method). These techniques give strong frameworks for improving objective functions when facing uncertainty, incompleteness, and inconsistency. Our goal is to explain the theoretical basics and real-world impacts of these Neutrosophic optimization methods using a series of theorems and proofs. This will show how effective they are in handling complicated optimization problems.

4.1 Neutrosophic Cauchy's Steepest Descent Method

Consider an unconstrained minimization problem $\min \tilde{f}^{\mathcal{N}}(\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_{\mathrm{n}})$. The goal is to discover a local minimizer, also known as a minimizer. $\tilde{f}^{\mathcal{N}}(\tilde{v})$ represents the objective function for this minimization job. Gradient-based approaches are based on the observation that $\tilde{f}^{\mathcal{N}}$ decreases the fastest at a point p in $\mathscr{R}$ while advancing in the direction of $-\nabla\tilde{f}^{\mathcal{N}}(P)$. As a result, the iterative approach consists of,

$$\tilde{\upsilon}^{\kappa+1} = \tilde{\upsilon}^{\kappa} + (\tilde{\gamma}^{\kappa}\tilde{d}(\tilde{\upsilon}^{\kappa}))^{\mathcal{N}} \qquad (2)$$

where as $\tilde{\upsilon}^{\kappa}$ is the current estimate of $\tilde{\upsilon}^{*}$, $(\tilde{d}^{\kappa})^{\mathcal{N}} = (\tilde{d}(\tilde{\upsilon}^{\kappa}))^{\mathcal{N}}$ is the search direction and $\alpha^{(\kappa)}$ is the step length parameter in the space $\mathscr{R}^{\mathcal{N}}$ of design variables.

If we take $(\tilde{d}^{\kappa})^{\mathcal{N}} = -(\tilde{g}^{\kappa})^{\mathcal{N}} = -\nabla\tilde{f}^{\mathcal{N}}(\tilde{\upsilon}^{\kappa})$, we get the method of steepest descent. We have $\tilde{\upsilon}^{(\kappa+1)} = \tilde{\upsilon}^{\kappa} - (\alpha^{\kappa}(\tilde{g}^{\kappa}))^{\mathcal{N}}, (\tilde{g}^{\kappa})^{\mathcal{N}} = \nabla\tilde{f}^{\mathcal{N}}(\tilde{\upsilon}^{\kappa})$, where $\tilde{\gamma}^{(\kappa)}$ is the minimizer of the function $\phi(\tilde{\gamma}) = \tilde{f}^{\mathcal{N}}(\tilde{\upsilon}^{\kappa} - (\tilde{\gamma}\tilde{g}^{\kappa}))^{\mathcal{N}}$. To get the value of $\alpha^{(\kappa)}$, one can use any of the one-dimensional search techniques. To start the iterative process effectively, the first approximation $\tilde{\upsilon}^{0}$ must be chosen carefully and according to the particular problem at hand.

4.1.1 Analyzing the Convergence of Quadratic Functions using the Neutrosophic steepest Descent Technique

To demonstrate the convergence properties of gradient-based techniques, we will use a quadratic function represented as

$$\tilde{f}^{\mathcal{N}}(\tilde{\upsilon}) \approx \frac{1}{2}\tilde{\upsilon}^{T}\tilde{\mathcal{Q}}\tilde{\upsilon} \approx \frac{1}{2}\langle\tilde{\mathcal{Q}}\tilde{\upsilon},\tilde{\upsilon}\rangle \quad (3)$$

where $\tilde{\mathcal{Q}}$ is positive definite. If $\tilde{f}^{\mathcal{N}}$ has a minimum at $\tilde{\upsilon}^{*} \approx \tilde{0}$ with $\tilde{f}^{\mathcal{N}}(\tilde{\upsilon}^{*}) \approx \tilde{0}$,

$\tilde{g}^{\mathcal{N}} \approx \nabla\tilde{f}^{\mathcal{N}}(\tilde{\upsilon}) \approx \tilde{\mathcal{Q}}\tilde{\upsilon}$ and $\nabla^{2}\tilde{f}^{\mathcal{N}}(\tilde{\upsilon}) \approx \tilde{\mathcal{Q}}$.

Then

$$\mathscr{R}(\Phi(\tilde{\gamma}^{\mathcal{N}})) = \mathscr{R}\left(\tilde{f}^{\mathcal{N}}(\tilde{\upsilon} - \tilde{\gamma}\tilde{g}^{\mathcal{N}})\right) = \frac{1}{2}\mathscr{R}(\langle[\tilde{\mathcal{Q}} - 2\tilde{\gamma}\tilde{\mathcal{Q}}^{2} + \tilde{\gamma}^{2}\tilde{\mathcal{Q}}^{3}]\tilde{\upsilon},\tilde{\upsilon}\rangle)$$

This gives $\mathscr{R}(\tilde{\gamma}^{\mathcal{N}}) = \frac{\mathscr{R}(\langle\tilde{\mathcal{Q}}^{2}\tilde{\upsilon},\tilde{\upsilon}\rangle))}{\mathscr{R}(\langle\tilde{\mathcal{Q}}^{3}\tilde{\upsilon},\tilde{\upsilon}\rangle))} = \frac{\mathscr{R}(\langle\tilde{\mathcal{Q}}\tilde{\upsilon},\tilde{\mathcal{Q}}\tilde{\upsilon}\rangle)}{\mathscr{R}(\langle\tilde{\mathcal{Q}}(\tilde{\mathcal{Q}}\tilde{\upsilon}),\tilde{\mathcal{Q}}\tilde{\upsilon}\rangle)} = \frac{\mathscr{R}(\langle\tilde{g}^{\mathcal{N}},\tilde{g}^{\mathcal{N}}\rangle)}{\mathscr{R}(\langle\tilde{\mathcal{Q}}\tilde{g}^{\mathcal{N}},\tilde{g}^{\mathcal{N}}\rangle)}$

Therefore, the iteration process of steepest descent for the quadratic function is expressed as

$$\mathscr{R}(\tilde{\upsilon}^{\kappa+1}) = \mathscr{R}\left(\tilde{\upsilon}^{\kappa} - \frac{\langle(\tilde{g}^{\kappa})^{\mathcal{N}},(\tilde{g}^{\kappa})^{\mathcal{N}}\rangle}{\langle\tilde{\mathcal{Q}}(\tilde{g}^{\kappa})^{\mathcal{N}},(\tilde{g}^{\kappa})^{\mathcal{N}}\rangle}(\tilde{g}^{\kappa})^{\mathcal{N}}\right) \qquad (4)$$

Using the above formulation and the fact that

$$\mathscr{R}\left(\tilde{f}^{\mathcal{N}}(\tilde{\upsilon}^{\kappa+1})\right) = \mathscr{R}\left(\tilde{f}^{\mathcal{N}}(\tilde{\upsilon}^{\kappa}) - \frac{1}{2}\frac{[\langle(\tilde{g}^{\kappa})^{\mathcal{N}},(\tilde{g}^{\kappa})^{\mathcal{N}}\rangle]^{2}}{\langle\tilde{\mathcal{Q}}(\tilde{g}^{\kappa})^{\mathcal{N}},(\tilde{g}^{\kappa})^{\mathcal{N}}\rangle}\right)$$

The gives

$$\mathscr{R}\left(\frac{\tilde{f}^{\mathcal{N}}(\tilde{\upsilon}^{\kappa}) - \tilde{f}^{\mathcal{N}}(\tilde{\upsilon}^{\kappa+1})}{\tilde{f}^{\mathcal{N}}(\tilde{\upsilon}^{\kappa})}\right) = \mathscr{R}\left(\frac{[\langle(\tilde{g}^{\kappa})^{\mathcal{N}},(\tilde{g}^{\kappa})^{\mathcal{N}}\rangle]^{2}}{\langle\tilde{\mathcal{Q}}(\tilde{g}^{\kappa})^{\mathcal{N}},(\tilde{g}^{\kappa})^{\mathcal{N}}\rangle\langle(\tilde{g}^{\kappa})^{\mathcal{N}},\tilde{\mathcal{Q}}^{-1}(\tilde{g}^{\kappa})^{\mathcal{N}}\rangle}\right)$$

and hence

$$\mathscr{R}(\tilde{\mathrm{f}}^{\mathcal{N}}(\tilde{\mathrm{\upsilon}}^{\kappa+1})) = \mathscr{R}((1-\mu_{\kappa})\tilde{\mathrm{f}}^{\mathcal{N}}(\tilde{\mathrm{\upsilon}}^{\kappa}))$$

(4a)

where $\mu_{\kappa} = \frac{[\langle(\tilde{g}^{\kappa})^{\mathcal{N}},(\tilde{g}^{\kappa})^{\mathcal{N}}\rangle]^2}{\langle\tilde{\mathcal{Q}}(\tilde{g}^{\kappa})^{\mathcal{N}},(\tilde{g}^{\kappa})^{\mathcal{N}}\rangle\langle(\tilde{g}^{\kappa})^{\mathcal{N}},\tilde{\mathcal{Q}}^{-1}(\tilde{g}^{\kappa})^{\mathcal{N}}\rangle}$

Theorem 4.1 Let the neutrosophic quadratic function denoted as $\tilde{\mathrm{f}}^{\mathcal{N}}$, defined by the expressions in (3), $\tilde{\mathrm{f}}^{\mathcal{N}}(\tilde{\upsilon}) \approx \frac{1}{2}\tilde{\upsilon}^{\mathrm{T}}\tilde{\mathcal{Q}}\tilde{\upsilon} \approx \frac{1}{2}\langle\tilde{\mathcal{Q}}\tilde{\upsilon},\tilde{\upsilon}\rangle$ and $(\tilde{\upsilon}^{\mathrm{K}})$ represent the iterates generated by the process

$$\tilde{\upsilon}^{\kappa+1} = \tilde{\upsilon}^{\kappa} - \frac{\langle(\tilde{g}^{\kappa})^{\mathcal{N}},(\tilde{g}^{\kappa})^{\mathcal{N}}\rangle}{\langle\tilde{\mathcal{Q}}(\tilde{g}^{\kappa})^{\mathcal{N}},(\tilde{g}^{\kappa})^{\mathcal{N}}\rangle}(\tilde{g}^{\kappa})^{\mathcal{N}}$$

The sequence $\tilde{\upsilon}^{\mathrm{K}}$ converges linearly towards the minimizer $\tilde{\upsilon}^{*}$, which is approximately identical to $\tilde{0}$, starting from any initial approximation $\tilde{\upsilon}^{0}$.

Proof: To prove $\tilde{\upsilon}^{\kappa+1} = \tilde{\upsilon}^{\kappa} - \frac{\langle(\tilde{g}^{\kappa})^{\mathcal{N}},(\tilde{g}^{\kappa})^{\mathcal{N}}\rangle}{\langle\tilde{\mathcal{Q}}(\tilde{g}^{\kappa})^{\mathcal{N}},(\tilde{g}^{\kappa})^{\mathcal{N}}\rangle}(\tilde{g}^{\kappa})^{\mathcal{N}}$, it is enough to prove

$$\mathscr{R}(\tilde{\upsilon}^{\kappa+1}) = \mathscr{R}\left(\tilde{\upsilon}^{\kappa} - \frac{\langle(\tilde{g}^{\kappa})^{\mathcal{N}},(\tilde{g}^{\kappa})^{\mathcal{N}}\rangle}{\langle\tilde{\mathcal{Q}}(\tilde{g}^{\kappa})^{\mathcal{N}},(\tilde{g}^{\kappa})^{\mathcal{N}}\rangle}(\tilde{g}^{\kappa})^{\mathcal{N}}\right)$$

outlined in Equation (4) . That is it is enough to prove

$$\mathscr{R}\left(\tilde{\mathrm{f}}^{\mathcal{N}}(\tilde{\upsilon}^{\kappa+1})\right) = \mathscr{R}((1-\tilde{\mu}_{\kappa})\tilde{\mathrm{f}}^{\mathcal{N}}(\tilde{\upsilon}^{\kappa})) \qquad (5)$$

This gives $\mathscr{R}\left(\tilde{\mathrm{f}}^{\mathcal{N}}(\tilde{\upsilon}^{\kappa})\right) = \prod_{i=0}^{\kappa-1}\mathscr{R}((1-\tilde{\mu}_{\kappa})\tilde{\mathrm{f}}^{\mathcal{N}}(\tilde{\upsilon}^{(0)}))$, $(\tilde{\upsilon}^{\kappa})$ converges to $\mathscr{R}(\tilde{\upsilon}^{*}) = 0$ if and only if $\mathscr{R}(\tilde{\mathrm{f}}^{\mathcal{N}}(\tilde{\upsilon}^{\kappa})) \to 0$ and in the view the above equation, this is possible if and only if

$\prod_{i=0}^{\infty}\mathscr{R}(1-\tilde{\mu}_{\kappa}) = 0$, which is true, because $\mathscr{R}(1-\tilde{\mu}_{\kappa}) \leq \mathscr{R}\left(\frac{\tilde{\gamma}_{\max}-\tilde{\gamma}_{\min}}{\tilde{\gamma}_{\max}}\right) \leq 1$.

As $\mathscr{R}(\tilde{\upsilon}^{*}) = 0$ and $\mathscr{R}\left(\tilde{\mathrm{f}}^{\mathcal{N}}(\tilde{\upsilon})\right) = \frac{1}{2}\mathscr{R}(\tilde{\mathcal{Q}}\tilde{\upsilon},\tilde{\upsilon})$, Rayleigh inequality gives

$$\begin{aligned}\mathscr{R}\left(\frac{\tilde{\gamma}_{\min}}{2}\parallel\tilde{\upsilon}^{\kappa+1}-\tilde{\upsilon}^{\kappa}\parallel^2\right) &\leq \mathscr{R}(\tilde{\mathrm{f}}^{\mathcal{N}}(\tilde{\upsilon}^{\kappa+1}-\tilde{\upsilon}^{(*)})) \\ &= \mathscr{R}(\tilde{\mathrm{f}}^{\mathcal{N}}(\tilde{\upsilon}^{\kappa+1})).\end{aligned} \qquad (6)$$

Similarly,

$$\begin{aligned}\mathscr{R}\left(\frac{\tilde{\gamma}_{\max}}{2}\parallel\tilde{\upsilon}^{\kappa}-\tilde{\upsilon}^{(*)}\parallel^2\right) &\geq \mathscr{R}(\tilde{\mathrm{f}}^{\mathcal{N}}(\tilde{\upsilon}^{\kappa})-\tilde{\upsilon}^{(*)}) \\ &= \mathscr{R}(\tilde{\mathrm{f}}^{\mathcal{N}}(\tilde{\upsilon}^{\kappa}))\end{aligned} \qquad (7)$$

Consequently (6) and (7), we get

$$\begin{aligned}\mathscr{R}\left(\frac{\tilde{\gamma}_{\min}}{2}\parallel\tilde{\upsilon}^{\kappa+1}-\tilde{\upsilon}^{\kappa}\parallel^2\right) &\leq \mathscr{R}(\tilde{\mathrm{f}}^{\mathcal{N}}(\tilde{\upsilon}^{\kappa+1})) \\ &= \mathscr{R}((1-\tilde{\mu}_{\kappa})\tilde{\mathrm{f}}^{\mathcal{N}}(\tilde{\upsilon}^{\kappa})) \\ &\leq \mathscr{R}\left(\frac{\tilde{\gamma}_{\max}-\tilde{\gamma}_{\min}}{2}\parallel\tilde{\upsilon}^{\kappa}-\tilde{\upsilon}^{*}\parallel^2\right)\end{aligned}$$

Which implies $\mathscr{R}\left(\frac{\|\tilde{\upsilon}^{\kappa+1}-\tilde{\upsilon}^*\|}{\|\tilde{\upsilon}^{\kappa}-\tilde{\upsilon}^*\|}\right) \leq \mathscr{R}\left(\sqrt{\frac{\tilde{\gamma}_{\max}-\tilde{\gamma}_{\min}}{\tilde{\gamma}_{\max}}}\right)$ as $\mathscr{R}\left(\sqrt{\frac{\tilde{\gamma}_{\max}-\tilde{\gamma}_{\min}}{\tilde{\gamma}_{\max}}}\right) > 0$

Hence $\frac{\|\tilde{\upsilon}^{\kappa+1}-\tilde{\upsilon}^*\|}{\|\tilde{\upsilon}^{\kappa}-\tilde{\upsilon}^*\|} \leq \sqrt{\frac{\tilde{\gamma}_{\max}-\tilde{\gamma}_{\min}}{\tilde{\gamma}_{\max}}}$.

It implies that the convergence of $\tilde{\upsilon}^{\kappa}$ to $\tilde{\upsilon}^*$ is linear.

4.2 Exploring the Neutrosophic Fletcher-Reeves Technique (Neutrosophic Conjugate Gradient Method)

In the Cauchy's Steepest Descent Method, the determination of $(\tilde{d}^{\kappa})^{\mathcal{N}}$ as $-(\tilde{g}^{\kappa})^{\mathcal{N}}$ leads the iterative process of $\tilde{\upsilon}^{\kappa}$ towards the minimizer $\tilde{\upsilon}^*$ in a zigzagging manner. Therefore, there is a necessity to generate a new search direction function $(\tilde{d}^{\kappa})^{\mathcal{N}}$ that will accelerate the convergence of the iterates $\{\tilde{\upsilon}^{\kappa}\}$ towards $\tilde{\upsilon}^*$.

Consider the quadratic function given in (3), $\tilde{f}^{\mathcal{N}}(\tilde{\upsilon}) \approx \frac{1}{2}\tilde{\upsilon}^{T}\tilde{\mathcal{Q}}\tilde{\upsilon} \approx \frac{1}{2}\langle\tilde{\mathcal{Q}}\tilde{\upsilon},\tilde{\upsilon}\rangle$ with $\tilde{\mathcal{Q}}$ positive definite. We shall generate $\mathscr{R}((\tilde{d}^{\kappa})^{\mathcal{N}})$ as manually conjugate direction with respect to $\mathscr{R}(\tilde{\mathcal{Q}}), \mathscr{R}(\langle\tilde{\mathcal{Q}}\left(\tilde{d}^{i}\right)^{\mathcal{N}}, \left(\tilde{d}^{j}\right)^{\mathcal{N}}\rangle) = 0, i \neq j$.

The procedure for conjugate direction generation is

$\mathscr{R}\left((\tilde{d}^{0})^{\mathcal{N}}\right) = -\mathscr{R}((\tilde{g}^{0})^{\mathcal{N}}) = -\mathscr{R}\left(\nabla\tilde{f}^{\mathcal{N}}\left(\tilde{\upsilon}^{0}\right)\right) = -\mathscr{R}\left(\tilde{\mathcal{Q}}^{\mathcal{N}}(\tilde{\upsilon}^{0})\right)$ with $\mathscr{R}(\tilde{\upsilon}^{0})$ being initial guess. Then

$\mathscr{R}(\tilde{\upsilon}^{\kappa+1}) = \mathscr{R}\left(\tilde{\upsilon}^{\kappa} - \tilde{\alpha}^{\kappa}(\tilde{d}^{\kappa})^{\mathcal{N}}\right)$, we get $\mathscr{R}(\tilde{\alpha}^{\kappa}) = -\mathscr{R}\left(\frac{\langle(\tilde{d}^{\kappa})^{\mathcal{N}},(\tilde{d}^{\kappa})^{\mathcal{N}}\rangle}{\langle(\tilde{\mathcal{Q}}(\tilde{d}^{\kappa}))^{\mathcal{N}},(\tilde{d}^{\kappa})^{\mathcal{N}}\rangle}\right)$. This $\mathscr{R}(\tilde{\alpha}^{\kappa})$ minimize the function $\mathscr{R}(\tilde{\Phi}(\tilde{\alpha})) = \mathscr{R}\left(\tilde{f}\left(\tilde{\upsilon} - \tilde{\alpha}\tilde{d}^{\kappa}\right)\right)$ and hence $\mathscr{R}(\tilde{g}^{\kappa+1})$ is orthognal to $\mathscr{R}(\tilde{d}^{\kappa})$, for $\mathscr{R}(\tilde{\alpha})$ that minimizes the equation is given by $\mathscr{R}(\tilde{\phi}'(\tilde{\alpha})) = \mathscr{R}\left(\langle\nabla\tilde{f}^{\mathcal{N}}(\tilde{\upsilon} - \tilde{\alpha}(\tilde{d}^{\kappa}))^{\mathcal{N}}, (\tilde{d}^{\kappa})^{\mathcal{N}}\rangle\right) = 0$ which is same as $\mathscr{R}\left(\langle(\tilde{g}^{\kappa+1})^{\mathcal{N}}, (\tilde{d}^{\kappa})^{\mathcal{N}}\rangle\right) = 0$. The nextconjugate direction $\mathscr{R}(\tilde{d}^{\kappa+1})^{\mathcal{N}}$ is given by

$\mathscr{R}(\tilde{d}^{\kappa+1})^{\mathcal{N}} + \mathscr{R}\left(\tilde{\beta}^{\kappa}(\tilde{d}^{\kappa})^{\mathcal{N}}\right)$ where $\mathscr{R}\left((\tilde{d}^{\kappa})^{\mathcal{N}}\right)$ is so chosen that $\mathscr{R}\left((\tilde{d}^{\kappa})^{\mathcal{N}}\right)$ is conjugate to $\mathscr{R}\left((\tilde{d}^{\kappa+1})^{\mathcal{N}}\right)$ with respect to $\mathscr{R}(\tilde{\mathcal{Q}})$. This gives $\mathscr{R}(\tilde{\beta}^{\kappa}) = \mathscr{R}\left(\frac{(\tilde{g}^{\kappa+1})^{\mathcal{N}},\tilde{\mathcal{Q}}(\tilde{d}^{\kappa})^{\mathcal{N}}}{(\tilde{d}^{\kappa})^{\mathcal{N}},\tilde{\mathcal{Q}}(\tilde{d}^{\kappa})^{\mathcal{N}}}\right)$.

To evaluate $\mathscr{R}\left(\langle(\tilde{d}^{\kappa})^{\mathcal{N}}, (\tilde{g}^{\kappa})^{\mathcal{N}}\rangle\right)$

$$\mathscr{R}\left(\langle(\tilde{d}^{\kappa})^{\mathcal{N}}, (\tilde{g}^{\kappa})^{\mathcal{N}}\rangle\right) = -\mathscr{R}(\langle(\tilde{g}^{\kappa})^{\mathcal{N}}, (\tilde{g}^{\kappa})^{\mathcal{N}}\rangle) + \mathscr{R}\left(\tilde{\beta}^{\kappa-1}\langle(\tilde{d}^{\kappa-1})^{\mathcal{N}}, (\tilde{g}^{\kappa})^{\mathcal{N}}\rangle\right)$$

$= -\mathscr{R}(\langle(\tilde{g}^{\kappa})^{\mathcal{N}}, (\tilde{g}^{\kappa})^{\mathcal{N}}\rangle)$ as $\mathscr{R}((\tilde{g}^{\kappa})^{\mathcal{N}}) \perp \mathscr{R}((\tilde{d}^{\kappa-1})^{\mathcal{N}})$ and hence $\mathscr{R}(\tilde{\alpha}^{\kappa}) = -\mathscr{R}\left(\frac{\langle(\tilde{d}^{\kappa})^{\mathcal{N}},(\tilde{g}^{\kappa})^{\mathcal{N}}\rangle}{\langle\tilde{\mathcal{Q}}(\tilde{d}^{\kappa})^{\mathcal{N}},(\tilde{d}^{\kappa})^{\mathcal{N}}\rangle}\right)$.

Theorem 4.2 The neutrosophic gradient vectors $(\tilde{g}^{\kappa})^{\mathcal{N}}$ are mutually orthogonal and the direction search neutrosophic vectors $(\tilde{d}^{\kappa})^{\mathcal{N}}$ are mutually $\mathcal{Q}$-Conjugate.

Proof: To prove the neutrosophic gradient vectors $(\tilde{g}^{\kappa})^{\mathcal{N}}$ are mutually orthogonal and the direction search neutrosophic vectors $(\tilde{d}^{\kappa})^{\mathcal{N}}$ are mutually $\mathcal{Q}$-Conjugate, it is enough to prove the neutrosophic gradient vectors $\mathscr{R}((\tilde{g}^{\kappa})^{\mathcal{N}})$ are mutually orthogonal and the direction search neutrosophic vectors $\mathscr{R}\big((\tilde{d}^{\kappa})^{\mathcal{N}}\big)$ are mutually $\mathcal{Q}$-Conjugate.

Let as prove the theorem by mathematical induction. The result is true for $\kappa = 1$, since $\mathscr{R}((\tilde{d}^{0})^{\mathcal{N}})$ and $\mathscr{R}((\tilde{d}^{1})^{\mathcal{N}})$ are $\mathcal{Q}$-conjugate by the choice of $\mathscr{R}(\tilde{\beta}^{0})$. Also $\mathscr{R}((\tilde{g}^{0})^{\mathcal{N}}) = \mathscr{R}((-\tilde{d}^{0})^{\mathcal{N}})$ is orthogonal to $\mathscr{R}((\tilde{g}^{1})^{\mathcal{N}})$. Assume that $\mathscr{R}((\tilde{d}^{0})^{\mathcal{N}}), \mathscr{R}((\tilde{d}^{1})^{\mathcal{N}}), \ldots, \mathscr{R}((\tilde{d}^{m})^{\mathcal{N}})$ are mutually $\mathcal{Q}$-conjugate and $\mathscr{R}((\tilde{g}^{0})^{\mathcal{N}}), \mathscr{R}((\tilde{g}^{1})^{\mathcal{N}}), \ldots, \mathscr{R}((\tilde{g}^{m})^{\mathcal{N}})$ are mutually orthogonal. That is the result is true for $\kappa = m$. We shall prove the result for $\kappa = m + 1$.

Then $\mathscr{R}((\tilde{g}^{m+1})^{\mathcal{N}}) = \mathscr{R}((\tilde{g}^{m})^{\mathcal{N}}) + \mathscr{R}\big(\tilde{\alpha}^{m}\tilde{\mathcal{Q}}(\tilde{d}^{m})^{\mathcal{N}}\big)$, where $\mathscr{R}(\tilde{\alpha}^{m}) = -\mathscr{R}\left(\frac{\langle(\tilde{d}^{m})^{\mathcal{N}},(\tilde{g}^{m})^{\mathcal{N}}\rangle}{\langle(\tilde{\mathcal{Q}}(\tilde{d}^{m}))^{\mathcal{N}},(\tilde{d}^{m})^{\mathcal{N}}\rangle}\right)$ And hence

$$\begin{aligned}\mathscr{R}(\langle(\tilde{g}^{(m+1)})^{\mathcal{N}},(\tilde{g}^{i})^{\mathcal{N}}\rangle) = \;& \mathscr{R}(\langle(\tilde{g}^{m})^{\mathcal{N}},(\tilde{g}^{i})^{\mathcal{N}}\rangle + \langle\tilde{\alpha}^{m}\rangle\langle\tilde{\mathcal{Q}}(\tilde{g}^{m})^{\mathcal{N}},(\tilde{g}^{i})^{\mathcal{N}}\rangle)\\ &= \mathscr{R}(\langle(\tilde{g}^{m})^{\mathcal{N}},(\tilde{g}^{i})^{\mathcal{N}}\rangle) + \mathscr{R}(\langle\tilde{\alpha}^{m}\rangle\langle\tilde{\mathcal{Q}}(\tilde{g}^{m})^{\mathcal{N}},(\tilde{g}^{i})^{\mathcal{N}}\rangle)\\ &= \mathscr{R}(\langle(\tilde{g}^{m})^{\mathcal{N}},(\tilde{g}^{i})^{\mathcal{N}}\rangle) - \mathscr{R}(\langle(\tilde{g}^{m})^{\mathcal{N}},(\tilde{g}^{i})^{\mathcal{N}}\rangle)\\ &= 0\end{aligned}$$

Also $\mathscr{R}((\tilde{d}^{m+1})^{\mathcal{N}}) = -\mathscr{R}((\tilde{g}^{m+1})^{\mathcal{N}}) + \mathscr{R}\big(\tilde{\beta}^{m}(\tilde{d}^{m})^{\mathcal{N}}\big)$, where

$$\mathscr{R}(\tilde{\beta}^{m}) = \mathscr{R}\left(\frac{\langle(\tilde{g}^{m+1})^{\mathcal{N}},\tilde{\mathcal{Q}}(\tilde{d}^{m})^{\mathcal{N}}\rangle}{\langle(\tilde{d}^{m})^{\mathcal{N}},\tilde{\mathcal{Q}}(\tilde{d}^{m})^{\mathcal{N}}\rangle}\right)$$

As $\mathscr{R}(\tilde{d}^{0})^{\mathcal{N}}, \mathscr{R}(\tilde{d}^{1})^{\mathcal{N}}, \ldots, \mathscr{R}(\tilde{d}^{m})^{\mathcal{N}}$ are $\mathcal{Q}$-conjugate. Taking inner product with $\mathscr{R}\big(\tilde{\mathcal{Q}}(\tilde{d}^{m})^{\mathcal{N}}\big)$, we have

$$\begin{aligned}\langle\mathscr{R}(\tilde{d}^{m+1})^{\mathcal{N}},\mathscr{R}\big(\tilde{\mathcal{Q}}(\tilde{d}^{m})^{\mathcal{N}}\big)\rangle \;&= -\mathscr{R}\big(\langle(\tilde{g}^{m+1})^{\mathcal{N}},\tilde{\mathcal{Q}}(\tilde{d}^{m})^{\mathcal{N}}\rangle\big) + \mathscr{R}\big(\langle\tilde{\beta}^{m}\rangle\langle(\tilde{d}^{m})^{\mathcal{N}},\tilde{\mathcal{Q}}(\tilde{d}^{m})^{\mathcal{N}}\rangle\big)\\ &= -\mathscr{R}\big(\langle(\tilde{g}^{m+1})^{\mathcal{N}},\tilde{\mathcal{Q}}(\tilde{d}^{m})^{\mathcal{N}}\rangle\big) + \mathscr{R}\big(\langle(\tilde{g}^{m+1})^{\mathcal{N}},\tilde{\mathcal{Q}}(\tilde{d}^{m})^{\mathcal{N}}\rangle\big)\\ &= 0\end{aligned}$$

Then, $\mathscr{R}\big(\tilde{\mathcal{Q}}(\tilde{d}^{m})^{\mathcal{N}}\big) = \mathscr{R}\left(\frac{(\tilde{g}^{m+1})^{\mathcal{N}}-(\tilde{g}^{m})^{\mathcal{N}}}{\tilde{\alpha}^{m}}\right)$.

and hence we have

$$\mathscr{R}\left(\langle(\tilde{d}^{\kappa})^{\mathcal{N}},\tilde{\mathcal{Q}}(\tilde{d}^{i})^{\mathcal{N}}\rangle\right) = \mathscr{R}\left(\frac{\langle(\tilde{g}^{\kappa})^{\mathcal{N}},(\tilde{g}^{i+1})^{\mathcal{N}}-(\tilde{g}^{i})^{\mathcal{N}}\rangle}{\tilde{\alpha}^{i}}\right) \qquad (8)$$

Combining the result we have $\mathscr{R}\big(\langle(\tilde{g}^{\kappa})^{\mathcal{N}},(\tilde{g}^{i})^{\mathcal{N}}\rangle\big) = 0 = \mathscr{R}\big(\langle(\tilde{d}^{\kappa})^{\mathcal{N}},\tilde{\mathcal{Q}}(\tilde{d}^{\kappa})^{\mathcal{N}}\rangle\big)$, for all κ.

5. Algorithms

This section provides the step-by-step procedures for the Neutrosophic steepest descent and the Fletcher-Reeves methods. These algorithms are developed to iteratively improve solutions by considering uncertainties and inconsistencies during the process. They represent the principles of Neutrosophy and provide valuable insights into managing uncertainty during optimization.

5.1 Algorithm for neutrosophic Cauchy's Steepest Descent Method

Step 1. Let us take the SVNNLPP with unconstrained optimization problem incorporate the neutrosophic triangular coefficient $\tilde{g}^{\mathcal{N}}(\tilde{\upsilon}^{(\kappa)^{\mathcal{N}}})$.

Step 2. Convert the triangular neutrosophic number coeffecients into arithmetic neutrosophic number and then covert to parametric form.

Step 3. Compute first and second derivative for the given function, and choose any initial points.

Step 4. Input $\tilde{\upsilon}^{(0)^{\mathcal{N}}}$, let $\kappa \leftarrow 0$

Step 5. Calculate $\tilde{S}_i^{\mathcal{N}}$, $\tilde{S}_i^{\mathcal{N}} = -\nabla\tilde{g}_i^{\mathcal{N}}$, where $\tilde{g}_i^{\mathcal{N}} = \tilde{f}^{\mathcal{N}}(\tilde{\upsilon}), \nabla\tilde{g}_i^{\mathcal{N}} = \nabla\tilde{f}^{\mathcal{N}}(\tilde{\upsilon}^{\kappa})$.

Step 6. Find Hessian for $\tilde{g}_i^{\mathcal{N}}(x)$

Step 7. Find the $(\tilde{\upsilon}^{\kappa+1})^{\mathcal{N}} = (\tilde{\upsilon}^{\kappa})^{\mathcal{N}} - (\tilde{\gamma}_i\tilde{g}^{\kappa})^{\mathcal{N}}$ then $\kappa \leftarrow \kappa + 1$. where $\tilde{\gamma} = \frac{-(\nabla\tilde{g}_i^{\mathcal{N}}(x))^T\tilde{S}_i^{\mathcal{N}}}{(\tilde{S}_i^{\mathcal{N}})^T H_i^{\mathcal{N}}\tilde{S}_i^{\mathcal{N}}}$

Step 8. Repeat the process until $\| \nabla\tilde{g}_i^{\mathcal{N}}(\tilde{\upsilon}^{\kappa}) \| < \varepsilon$ (or) $\| (\tilde{\upsilon}^{\kappa})^{\mathcal{N}} - (\tilde{\upsilon}^{\kappa+1})^{\mathcal{N}} \| < \varepsilon$. Then halt the process if we get an optimal solution; otherwise, go to step 5.

Step 9. Verify the optimum.

5.2 Algorithm for neutrosophic fletcher reeves method

Step 1. Let us take the SVNNLPP with an unconstrained optimization problem and incorporate the neutrosophic triangular coefficient. $\tilde{g}^{\mathcal{N}}(\tilde{\upsilon}^{(\kappa)^{\mathcal{N}}})$.

Step 2. Convert the triangular neutrosophic number coeffecients into arithmetic neutrosophic number and then covert to parametric form.

Step 3. Compute first and second derivative for the given function, and choose any initial points.

Step 4. Input $\tilde{\upsilon}^{(0)^{\mathcal{N}}}$, let $\kappa \leftarrow 0$

Step 5. Calculate $\tilde{S}_i^{\mathcal{N}}$, $\tilde{S}_i^{\mathcal{N}} = -\nabla\tilde{g}_i^{\mathcal{N}}$, where $\tilde{g}_i^{\mathcal{N}} = \tilde{f}^{\mathcal{N}}(\tilde{\upsilon}), \nabla\tilde{g}_i^{\mathcal{N}} = \nabla\tilde{f}^{\mathcal{N}}(\tilde{\upsilon}^{\kappa})$. and $\tilde{S}_{i+1}^{\mathcal{N}} = -\nabla\tilde{g}_i^{\mathcal{N}} + \frac{|\nabla\tilde{g}_i^{\mathcal{N}}|^2}{|\nabla\tilde{g}_{i-1}^{\mathcal{N}}|^2}\tilde{S}_i^{\mathcal{N}}$

Step 6. Find Hessian for $\tilde{g}_i^{\mathcal{N}}(\tilde{\upsilon})$

Step 7. Find the $(\tilde{\upsilon}^{\kappa+1})^{\mathcal{N}} = (\tilde{\upsilon}^{\kappa})^{\mathcal{N}} - (\tilde{\gamma}_i\tilde{g}^{\kappa})^{\mathcal{N}}$ then $\kappa \leftarrow \kappa + 1$. where $\tilde{\gamma} = \frac{-(\nabla\tilde{g}_i^{\mathcal{N}}(\tilde{\upsilon}))^T\tilde{S}_i^{\mathcal{N}}}{(\tilde{S}_i^{\mathcal{N}})^T H_i^{\mathcal{N}}\tilde{S}_i^{\mathcal{N}}}$

Step 8. Repeat the process until $\| \nabla\tilde{g}_i^{\mathcal{N}}(\tilde{\upsilon}^{\kappa}) \| < \varepsilon$ (or) $\| (\tilde{\upsilon}^{\kappa})^{\mathcal{N}} - (\tilde{\upsilon}^{\kappa+1})^{\mathcal{N}} \| < \varepsilon$. Then halt the process if we get an optimal solution; otherwise, go to step 5.

Step 9. Verify the optimum.

6. Numerical Examples

This section presents numerical examples of two significant methods utilized in Neutrosophic optimization: the Fletcher-Reeves method (also known as the conjugate gradient method) and the Neutrosophic Steepest Descent. These examples provide an in-depth demonstration of how these algorithms are applied, discussing their convergence properties and their ability to traverse uncertain solution domains.

Example 6.1 Consider the problem of unconstrained optimization with single-valued neutrosophic triangular coefficients.

$$\begin{aligned} min\ \tilde{f}(\tilde{\upsilon}_1,\tilde{\upsilon}_2) &= (19,20,21);(0.69,0.51,0.52)\tilde{\upsilon}_1 + (25,26,27);(0.67,0.54,0.5)\tilde{\upsilon}_2 \\ &+(3,4,5);(0.74,0.75,0.68)\tilde{\upsilon}_1\tilde{\upsilon}_2 + (-5,-4,-3);(0.8,0.75,0.6)\tilde{\upsilon}_1^2 \\ &+(-4,-3,-2);(0.7,0.65,0.75)\tilde{\upsilon}_2^2 \end{aligned} \quad (9)$$

Starting with initial point

$$\tilde{Y}^0 = \begin{pmatrix}(0,1,2);(0.65,0.6,0.5)\\(0,1,2);(0.65,0.6,0.5)\end{pmatrix}$$

Solution: The parametric form of the given SVNNLPP (only triplet)(9) is given by

$$\begin{aligned} min\ \tilde{f}(\tilde{\upsilon}_1,\tilde{\upsilon}_2) &= (20,1-r,1-r);(0.69,0.51,0.52)\tilde{\upsilon}_1 + (26,1-r,1-r);\\ &(0.67,0.54,0.5)\tilde{\upsilon}_2 + (4,1-r,1-r);(0.74,0.75,0.68)\tilde{\upsilon}_1\tilde{\upsilon}_2 \\ &+(-4,1-r,1-r);(0.8,0.75,0.6)\tilde{\upsilon}_1^2 + (-3,1-r,1-r);\\ &(0.7,0.65,0.75)\tilde{\upsilon}_2^2 \end{aligned} \quad (10)$$

$$\tilde{Y}^0 = \begin{pmatrix}(1,1-r,1-r);(0.65,0.6,0.5)\\(1,1-r,1-r);(0.65,0.6,0.5)\end{pmatrix}$$

The condition required for $\tilde{\upsilon}$ to be the optimum solution for (10) $\nabla\tilde{f}(\tilde{\upsilon}_1,\tilde{\upsilon}_2) \approx \tilde{0}$. Hence we have

$$\begin{aligned} \frac{\partial\tilde{f}}{\partial\tilde{\upsilon}_1} = &\ (20,1-r,1-r);(0.69,0.51,0.52) + (4,1-r,1-r);(0.74,0.75,0.68)\tilde{\upsilon}_2 \\ &+(-8,1-r,1-r);(0.8,0.75,0.6)\tilde{\upsilon}_1 \\ \frac{\partial\tilde{f}}{\partial\tilde{\upsilon}_2} = &\ (26,1-r,1-r);(0.67,0.54,0.5) + (4,1-r,1-r);(0.74,0.75,0.68)\tilde{\upsilon}_1 \\ &(-6,1-r,1-r);(0.7,0.65,0.75)\tilde{\upsilon}_2 \end{aligned}$$

The condition required for $\tilde{\upsilon}$ to be the optimum solution for (10) $\nabla^2\tilde{f}(\tilde{\upsilon}_1,\tilde{\upsilon}_2) \approx \tilde{0}$. Hence we have

$$\frac{\partial^2\tilde{f}}{\partial\tilde{\upsilon}_1^2} = (-8,1-r,1-r);(0.8,0.75,0.6),\ \frac{\partial^2\tilde{f}}{\partial\tilde{\upsilon}_2^2} = (-6,1-r,1-r);(0.7,0.65,0.75)$$

$$\frac{\partial^2\tilde{f}}{\partial\tilde{\upsilon}_1\,\partial\tilde{\upsilon}_2} = (4,1-r,1-r);(0.74,0.75,0.68),\ \frac{\partial^2\tilde{f}}{\partial\tilde{\upsilon}_2\,\partial\tilde{\upsilon}_1} = (4,1-r,1-r);(0.74,0.75,0.68)$$

Table 1: Cauchy's Steepest Descent Method

Iteration	$\tilde{Y}_i = (\tilde{\upsilon}_1, \tilde{\upsilon}_2)$	$\tilde{Y}_{i+1} = (\tilde{\upsilon}_1, \tilde{\upsilon}_2)$	$\nabla\tilde{f}_i$
1	$\begin{pmatrix}(1,1-r,1-r);\\(0.65,0.6,0.5)\\(1,1-r,1-r);\\(0.65,0.6,0.5)\end{pmatrix}$	$\begin{pmatrix}(6.4736,1-r,1-r);\\(0.8,0.75,0.75)\\(9.2104,1-r,1-r);\\(0.8,0.75,0.75)\end{pmatrix}$	$\begin{pmatrix}(5.0528,1-r,1-r);\\(0.8,0.75,0.75)\\(-3.368,1-r,1-r);\\(0.8,0.75,0.75)\end{pmatrix}$
2	$\begin{pmatrix}(6.4736,1-r,1-r);\\(0.8,0.75,0.75)\\(9.2104,1-r,1-r);\\(0.8,0.75,0.75)\end{pmatrix}$	$\begin{pmatrix}(6.9299,1-r,1-r);\\(0.8,0.75,0.75)\\(8.9063,1-r,1-r);\\(0.8,0.75,0.75)\end{pmatrix}$	$\begin{pmatrix}(0.186,1-r,1-r);\\(0.8,0.75,0.75)\\(0.2818,1-r,1-r);\\(0.8,0.75,0.75)\end{pmatrix}$
3	$\begin{pmatrix}(6.9299,1-r,1-r);\\(0.8,0.75,0.75)\\(8.9063,1-r,1-r);\\(0.8,0.75,0.75)\end{pmatrix}$	$\begin{pmatrix}(6.993,1-r,1-r);\\(0.8,0.75,0.75)\\(9.002,1-r,1-r);\\(0.8,0.75,0.75)\end{pmatrix}$	$\begin{pmatrix}(0.064,1-r,1-r);\\(0.8,0.75,0.75)\\(0.2818,1-r,1-r);\\(0.8,0.75,0.75)\end{pmatrix}$
4	$\begin{pmatrix}(6.993,1-r,1-r);\\(0.8,0.75,0.75)\\(9.002,1-r,1-r);\\(0.8,0.75,0.75)\end{pmatrix}$	$\begin{pmatrix}(6.9993,1-r,1-r);\\(0.8,0.75,0.75)\\(9.002,1-r,1-r);\\(0.8,0.75,0.75)\end{pmatrix}$	$\begin{pmatrix}(0.064,1-r,1-r);\\(0.8,0.75,0.75)\\(-0.04,1-r,1-r);\\(0.8,0.75,0.75)\end{pmatrix}$

Table 2: Fletcher Reeves Method

Iteration	$\tilde{Y}_i = (\tilde{\upsilon}_1, \tilde{\upsilon}_2)$	$\tilde{Y}_{i+1} = (\tilde{\upsilon}_1, \tilde{\upsilon}_2)$	$\nabla\tilde{f}_i$
1	$\begin{pmatrix}(1,1-r,1-r);\\(0.65,0.6,0.5)\\(1,1-r,1-r);\\(0.65,0.6,0.5)\end{pmatrix}$	$\begin{pmatrix}(6.4736,1-r,1-r);\\(0.8,0.75,0.75)\\(9.2104,1-r,1-r);\\(0.8,0.75,0.75)\end{pmatrix}$	$\begin{pmatrix}(5.0528,1-r,1-r);\\(0.8,0.75,0.75)\\(-3.368,1-r,1-r);\\(0.8,0.75,0.75)\end{pmatrix}$
2	$\begin{pmatrix}(6.4736,1-r,1-r);\\(0.8,0.75,0.75)\\(9.2104,1-r,1-r);\\(0.8,0.75,0.75)\end{pmatrix}$	$\begin{pmatrix}(6.9975,1-r,1-r);\\(0.8,0.75,0.75)\\(9.000,1-r,1-r);\\(0.8,0.75,0.75)\end{pmatrix}$	$\begin{pmatrix}(0.02,1-r,1-r);\\(0.8,0.75,0.75)\\(-0.01,1-r,1-r);\\(0.8,0.75,0.75)\end{pmatrix}$
3	$\begin{pmatrix}(6.9975,1-r,1-r);\\(0.8,0.75,0.75)\\(9.000,1-r,1-r);\\(0.8,0.75,0.75)\end{pmatrix}$	$\begin{pmatrix}(6.999,1-r,1-r);\\(0.8,0.75,0.75)\\(8.999,1-r,1-r);\\(0.8,0.75,0.75)\end{pmatrix}$	$\begin{pmatrix}(0.001,1-r,1-r);\\(0.8,0.75,0.75)\\(0.003,1-r,1-r);\\(0.8,0.75,0.75)\end{pmatrix}$

From the above tables(1),(2), we observed that the SDM converges at iteration 4 while the FRM at iteration 3. Hence the optimal solution of the given SVNNLPP (9) is

$\tilde{\upsilon}_1 = (6.999,1-\mathrm{r},1-\mathrm{r});(0.8,0.75,0.75)$, $\tilde{\upsilon}_2 = (8.999,1-\mathrm{r},1-\mathrm{r});(0.8,0.75,0.75)$ with $\tilde{\mathrm{f}}(\tilde{\upsilon}_1,\tilde{\upsilon}_2) = (186.999,1-\mathrm{r},1-\mathrm{r});(0.8,0.75,0.75)$

That is the optimal solution of the SVNNLPP (9) is

$\tilde{\upsilon}_1 = (5.999+\mathrm{r},6.999,7.999-\mathrm{r});(0.8,0.75,0.75)$, $\tilde{\upsilon}_2 = (7.999+\mathrm{r},8.999,9.999-\mathrm{r});(0.8,0.75,0.75)$ with $min\ \tilde{\mathrm{f}}(\tilde{\upsilon}_1,\tilde{\upsilon}_2) = (185.999+\mathrm{r},186.999,187.999-\mathrm{r});(0.8,0.75,0.75)$.

Example 6.2 Consider the problem of unconstrained optimization with single-valued neutrosophic triangular fuzzy coefficients,

$$\begin{aligned} min\ \tilde{\mathrm{f}}(\tilde{\upsilon}_1,\tilde{\upsilon}_2) = \ & (0,1,2);(0.68,0.51,0.55)\tilde{\upsilon}_1 - (0,1,2)(0.67,0.53,0.54)\tilde{\upsilon}_2 + (1,2,3); \\ & (0.7,0.65,0.72)\tilde{\upsilon}_1^2 + (1,2,3);(0.69,0.51,0.52)\tilde{\upsilon}_1\tilde{\upsilon}_2 \\ & +(0,1,2);(0.67,0.5,0.51)\tilde{\upsilon}_2^2 \end{aligned} \tag{11}$$

Starting with initial point

$$\tilde{\mathrm{Y}}^0 = \begin{pmatrix}(0,0,0);(0.66,0.52,0.53)\\(0,0,0);(0.66,0.52,0.53)\end{pmatrix}$$

Solution: The parametric form of the given SVNNLPP (only triplet)(11) is given by

$$\begin{aligned} min\ \tilde{\mathrm{f}}(\tilde{\upsilon}_1,\tilde{\upsilon}_2) = \ & (1,1-\mathrm{r},1-\mathrm{r});(0.68,0.51,0.55)\tilde{\upsilon}_1 - (1,1-\mathrm{r},1-\mathrm{r}); \\ & (0.67,0.53,0.54)\tilde{\upsilon}_2 + (2,1-\mathrm{r},1-\mathrm{r});(0.7,0.65,0.72)\tilde{\upsilon}_1^2 \\ & +(2,1-\mathrm{r},1-\mathrm{r});(0.69,0.51,0.52)\tilde{\upsilon}_1\tilde{\upsilon}_2 + (1,1-\mathrm{r},1-\mathrm{r}); \\ & (0.67,0.5,0.51)\tilde{\upsilon}_2^2 \end{aligned} \tag{12}$$

$$\tilde{\mathrm{Y}}^0 = \begin{pmatrix}(0,1-\mathrm{r},1-\mathrm{r});(0.66,0.52,0.53)\\(0,1-\mathrm{r},1-\mathrm{r});(0.66,0.52,0.53)\end{pmatrix}$$

The condition required for $\tilde{\upsilon}$ to be the optimum solution for (12) $\nabla\tilde{\mathrm{f}}(\tilde{\upsilon}_1,\tilde{\upsilon}_2) \approx \tilde{0}$. Hence we have

$$\begin{aligned} \frac{\partial\tilde{\mathrm{f}}}{\partial\tilde{\upsilon}_1} = \ & (1,1-\mathrm{r},1-\mathrm{r});(0.68,0.51,0.55) + (4,1-\mathrm{r},1-\mathrm{r});(0.7,0.65,0.72)\tilde{\upsilon}_1 \\ & +(2,1-\mathrm{r},1-\mathrm{r});(0.69,0.51,0.52)\tilde{\upsilon}_2 \\ \frac{\partial\tilde{\mathrm{f}}}{\partial\tilde{\upsilon}_2} = \ & -(1,1-\mathrm{r},1-\mathrm{r});(0.67,0.53,0.54) + (2,1-\mathrm{r},1-\mathrm{r});(0.69,0.51,0.52)\tilde{\upsilon}_1 \\ & (2,1-\mathrm{r},1-\mathrm{r});(0.67,0.5,0.51)\tilde{\upsilon}_2 \end{aligned}$$

The condition required for $\tilde{\upsilon}$ to be the optimum solution for (12) $\nabla\tilde{\mathrm{f}}(\tilde{\upsilon}_1,\tilde{\upsilon}_2) \approx \tilde{0}$. Hence we have

$$\begin{aligned} \frac{\partial^2\mathrm{f}}{\partial\tilde{\upsilon}_1^2} = \ & (4,1-\mathrm{r},1-\mathrm{r});(0.7,0.65,0.72),\ \frac{\partial^2\mathrm{f}}{\partial\tilde{\upsilon}_2^2} = (2,1-\mathrm{r},1-\mathrm{r});(0.67,0.5,0.51) \\ \frac{\partial^2\mathrm{f}}{\partial\tilde{\upsilon}_1\,\partial\tilde{\upsilon}_2} = \ & (2,1-\mathrm{r},1-\mathrm{r});(0.69,0.51,0.52),\ \frac{\partial^2\mathrm{f}}{\partial\tilde{\upsilon}_2\,\partial\tilde{\upsilon}_1} = (2,1-\mathrm{r},1-\mathrm{r});(0.69,0.51,0.52) \end{aligned}$$

Table 3: Cauchy's Steepest Descent Method

Iteration	$\tilde{Y}_i = (\tilde{\upsilon}_1, \tilde{\upsilon}_2)$	$\tilde{Y}_{i+1} = (\tilde{\upsilon}_1, \tilde{\upsilon}_2)$	$\nabla\tilde{f}_i$
1	$\begin{pmatrix}(0,1-r,1-r); \\ (0.66,0.52,0.53) \\ (0,1-r,1-r); \\ (0.66,0.52,0.53)\end{pmatrix}$	$\begin{pmatrix}(1,1-r,1-r); \\ (0.7,0.65,0.72) \\ (1,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$	$\begin{pmatrix}(-1,1-r,1-r); \\ (0.7,0.65,0.72) \\ (-1,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$
2	$\begin{pmatrix}(1,1-r,1-r); \\ (0.7,0.65,0.72) \\ (1,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$	$\begin{pmatrix}(-0.8,1-r,1-r); \\ (0.7,0.65,0.72) \\ (1.2,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$	$\begin{pmatrix}(0.2,1-r,1-r); \\ (0.7,0.65,0.72) \\ (-0.2,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$
3	$\begin{pmatrix}(-0.8,1-r,1-r); \\ (0.7,0.65,0.72) \\ (1.2,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$	$\begin{pmatrix}(-1,1-r,1-r); \\ (0.7,0.65,0.72) \\ (1.4,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$	$\begin{pmatrix}(-0.2,1-r,1-r); \\ (0.7,0.65,0.72) \\ (-0.2,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$
4	$\begin{pmatrix}(-1,1-r,1-r); \\ (0.7,0.65,0.72) \\ (1.4,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$	$\begin{pmatrix}(-0.96,1-r,1-r); \\ (0.7,0.65,0.72) \\ (1.44,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$	$\begin{pmatrix}(0.04,1-r,1-r); \\ (0.7,0.65,0.72) \\ (-0.04,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$
5	$\begin{pmatrix}(-0.96,1-r,1-r); \\ (0.7,0.65,0.72) \\ (1.44,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$	$\begin{pmatrix}(-1,1-r,1-r); \\ (0.7,0.65,0.72) \\ (1.48,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$	$\begin{pmatrix}(-0.04,1-r,1-r); \\ (0.7,0.65,0.72) \\ (-0.04,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$
6	$\begin{pmatrix}(-1,1-r,1-r); \\ (0.7,0.65,0.72) \\ (1.48,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$	$\begin{pmatrix}(-0.992,1-r,1-r); \\ (0.7,0.65,0.72) \\ (1.488,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$	$\begin{pmatrix}(0.008,1-r,1-r); \\ (0.7,0.65,0.72) \\ (-0.008,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$

Table 4: Fletcher-Reeves Method

Iteration	$\tilde{Y}_i = (\tilde{\upsilon}_1, \tilde{\upsilon}_2)$	$\tilde{Y}_{i+1} = (\tilde{\upsilon}_1, \tilde{\upsilon}_2)$	$\nabla\tilde{f}_i$
1	$\begin{pmatrix}(0,1-r,1-r); \\ (0.66,0.52,0.53) \\ (0,1-r,1-r); \\ (0.66,0.52,0.53)\end{pmatrix}$	$\begin{pmatrix}(1,1-r,1-r); \\ (0.7,0.65,0.72) \\ (1,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$	$\begin{pmatrix}(-1,1-r,1-r); \\ (0.7,0.65,0.72) \\ (-1,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$
2	$\begin{pmatrix}(1,1-r,1-r); \\ (0.7,0.65,0.72) \\ (1,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$	$\begin{pmatrix}(-1,1-r,1-r); \\ (0.7,0.65,0.72) \\ (1.5,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$	$\begin{pmatrix}(0,1-r,1-r); \\ (0.7,0.65,0.72) \\ (0,1-r,1-r); \\ (0.7,0.65,0.72)\end{pmatrix}$

From the above tables(3),(4), we observed that the CSDM converges at iteration 6 and the FRM converges at iteration 2. Hence the optimal solution of the given SVNNLPP (11) is

$\tilde{\upsilon}_1 = (-1,1-r,1-r);(0.7,0.65,0.72)$, $\tilde{\upsilon}_2 = (1.5,1-r,1-r);(0.7,0.65,0.72)$ with

$\tilde{f}(\tilde{\upsilon}_1,\tilde{\upsilon}_2) = (-1.25,1-r,1-r);(0.7,0.65,0.72)$

That is the optimal solution of the SVNNLPP (11) is $\tilde{\upsilon}_1 = (-2+r,-1,0-r);(0.7,0.65,0.72)$, $\tilde{\upsilon}_2 = (0.5+r,1.5,2.5-r);(0.7,0.65,0.72)$ with $\tilde{f}(\tilde{\upsilon}_1,\tilde{\upsilon}_2) = (-2.25+r,-1.25,-0.25-r);(0.7,0.65,0.72)$

7. Result and Discussion

we focused on using the Fletcher-Reeves method Table (2) Table (4) and the Cauchy's Steepest Descent Method Table (1), Table (3) to solve two different problems, each with a unique set of notations. Our study determined that both methods produced the same solution, but the Fletcher-Reeves method converged to the solution in fewer iterations than the Steepest Descent Method, making it the more efficient method for obtaining the single-valued neutrosophic optimal solution to the problems.

Table (5),(6) represent the single-valued neutrosophic optimal solution of the SVNNLPP (9) for various values of r.

Table 5: Neutrosophic the optimum technique to solve the provided neutrosophic a triangular coefficients for different values of r ∈ [0,1]

r	$\tilde{\upsilon}_1$	$\tilde{\upsilon}_2$
0	(5.999,6.999,7.999); (0.8,0.75,0.75)	(7.999,8.999,9.999); (0.8,0.75,0.75)
0.25	(6.249,6.999,7.749); (0.8,0.75,0.75)	(8.249,8.999,9.749); (0.8,0.75,0.75)
0.5	(6.499,6.999,7.499); (0.8,0.75,0.75)	(8.499,8.999,9.499); (0.8,0.75,0.75)
0.75	(6.749,6.999,7.249); (0.8,0.75,0.75)	(8.749,8.999,9.249); (0.8,0.75,0.75)
1	(6.999,6.999,6.999); (0.8,0.75,0.75)	(8.999,8.999,8.999); (0.8,0.75,0.75)

Table 6: Continue on the table (5)

r	$\tilde{f}(\tilde{\upsilon}_1,\tilde{\upsilon}_2)$
0	(185.999,186.999,187.999); (0.8,0.75,0.75)
0.25	(186.249,186.999,187.749); (0.8,0.75,0.75)
0.5	(186.499,186.999,187.499); (0.8,0.75,0.75)
0.75	(186.749,186.999,187.249); (0.8,0.75,0.75)
1	(186.999,186.999,186.999); (0.8,0.75,0.75)

When r = 1, we see that $\tilde{\upsilon}_1$ = (6.999,6.999,6.999); (0.8,0.75,0.75),

$\tilde{\upsilon}_2$ = (8.999,8.999,8.999); (0.8,0.75,0.75) and

$min\,\tilde{f}(\tilde{\upsilon}_1,\tilde{\upsilon}_2)$ = (186.999,186.999,186.999); (0.8,0.75,0.75)

Table (7), (8) represent the single-valued neutrosophic optimal solution of the SVNNLPP (11) for various values of r.

Table 7: Neutrosophic the optimum technique to solve the provided neutrosophic a triangular coefficients for different values of r ∈ [0,1]

r	$\tilde{\mathbf{\upsilon}}_1$	$\tilde{\mathbf{\upsilon}}_2$
0	$(-2,-1,0);(0.7,0.65,0.72)$	$(0.5,1.5,2.5);(0.7,0.65,0.72)$
0.25	$(-1.75,-1,-0.25);(0.7,0.65,0.72)$	$(0.75,1.5,2.25);(0.7,0.65,0.72)$
0.5	$(-1.5,-1,-0.5);(0.7,0.65,0.72)$	$(1,1.5,2);(0.7,0.65,0.72)$
0.75	$(-1.25,-1,-0.75);(0.7,0.65,0.72)$	$(1.25,1.5,1.75);(0.7,0.65,0.72)$
1	$(-1,-1,-1);(0.7,0.65,0.72)$	$(1.5,1.5,1.5);(0.7,0.65,0.72)$

Table 8: Continue on the table (7)

r	$\tilde{\mathbf{f}}(\tilde{\mathbf{\upsilon}}_1,\tilde{\mathbf{\upsilon}}_2)$
0	$(-2.25,-1.25,-0.25);(0.7,0.65,0.72)$
0.25	$(-2,-1.25,-0.5);(0.7,0.65,0.72)$
0.5	$(-1.75,-1.25,-0.75);(0.7,0.65,0.72)$
0.75	$(-1.25,-1.25,-1);(0.7,0.65,0.72)$
1	$(-1.25,-1.25,-1.25);(0.7,0.65,0.72)$

When r = 1, we see that $\tilde{\mathbf{\upsilon}}_1 = (-1,-1,-1);(0.7,0.65,0.72)$,

$\tilde{\mathbf{\upsilon}}_2 = (1.5,1.5,1.5);(0.7,0.65,0.72)$ and

$min\,\tilde{\mathbf{f}}(\tilde{\mathbf{\upsilon}}_1,\tilde{\mathbf{\upsilon}}_2) = (-1.25,-1.25,-1.25);(0.7,0.65,0.72)$

The above tables (5), (6), (7), and (8) illustrate that the proposed method provides decision makers with the flexibility to select their preferred solutions by making appropriate choices of r.

8. Conclusion

In this paper, we discussed a solution concept for SVNNLPP involving neutrosophic triangular numbers. First, the given SVNNLPP is expressed in terms of its location index number , left and right fuzziness index functions. In the parametric forms of neutrosophic numbers, a new type of neutrosophic arithmetic and neutrosophic ranking are introduced and utilized. The neutrosophic quadratic and neutrosophic conjugate gradient theorems for SVNNLPP are established. The neutrosophic versions of the CSDM and the FRM are used, and the neutrosophic optimal solution of the SVNNLPP is obtained without having to convert the given problem. The neutrosophic fuzzy optimal solution of the given SVNNLPP is tabulated for different values of r ∈ [0,1]. It is important to note that by utilizing the suggested procedure and selecting an appropriate value for r ∈ [0,1], the decision maker has the flexibility to select his or her preferred optimal solution based on the situation. The numerical solutions have been presented and discussed. Both methods provide same solution but FRM requires less number itreration comparing with CSDM.

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