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Controllability for Volterra Integro-Dynamic Sylvester Matrix Systems with Impulse on Time Scales

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Abstract:

This article presents the complete controllability for a Volterra integro-dynamic Sylvester matrix system with time scale impulses in a finite-dimensional space Rn. We utilized the Banach fixed point theorem and nonlinear functional analysis to determine the existence of a unique solution for the system, and conducted an analysis of complete controllability using the Gramian matrix and various parameter changes. We provided a numerical example using simulation to demonstrate the application of these conclusions for two different time scales, T=R and T=P_1,1.

Keywords: Controllability, impulses, Volterra integro-dynamic system, time scale.

AMS Subject Classification: 34A37, 93B05, 34H05, 34N05.

1. Introduction

There are numerous health issues that exhibit abrupt shifts in their states. We refer to these abrupt alterations as impulsive impacts within the system. Impulsive differential equations are those that incorporate the impact of impulses. These have substantial applications in several real-world situations, specifically in mechanical systems involving impact, biological systems like heartbeats and population dynamics, blood flow, ecology, medicine, control theory, and more. Within the current body of knowledge, there are two distinct categories of impulsive systems. There are two types of systems: impulsive and non-impulsive. In the impulsive system, the period of abrupt changes is significantly shorter compared to the overall duration of an evolutionary process, such as shocks, natural disasters, and non-impulsive events. The duration of these modifications persists throughout a limited time span. Insulin administration into the bloodstream is an important application of non-impulses is the administration of insulin into the bloodstream. This involves a sudden shift followed by a gradual absorption process, with the insulin remaining active for a specific period of time. Some references [1, 10, 14].

Kalman presented the idea of controllability and observability in the year 1960, and it quickly became a subject of examination for a significant number of researchers immediately after its introduction. In a general sense, controllability refers to the ability of a control dynamical system to guide itself from a state initial to the intended final by making use of a control that is accessible within the system. Recently, numerous authors have published their research articles [3, 8, 9, 11-13, 15-24].

We design with nonlinear time-varying complete controllability Volterra integro-dynamic Sylvester matrix system with an impulse control system in Rn

$$X^{\Delta}(t) = A(t)X(t) + X(t)B(t) + \int_{t_0}^{t}\big(K_1(t,s)X(s) + X(s)K_2(t,s)\big)\Delta s + C(t)U(t) + F\big(t, X(t)\big) \quad (1.1)$$

$$X(t) = \big(1 + D_j\big)X\big(t_j^-\big), \qquad j = 1,2, \dots, \quad (1.2)$$

Where A(t), B(t), C(t), $K_1(t)$ and $K_2(t)$ are rd-continuous matrices orders $n \times n$. $F: I \times \mathbb{R}^n \to \mathbb{R}^n$ is rd-continuous on $\mathbb{T}_0$. $D_j \in M_{n\times n}(\mathbb{R})$, $X(t) \in \mathbb{R}^n$ is state variable. $U(t) \in \mathbb{R}^m$ is the control input.

Time scale theory incorporates both discrete and continuous theories, as well as a hybrid of the two. Thus, in contrast to previous findings in the literature, our findings are more applicable to a wider range of situations. In this paper, the following structure is used: We lay the groundwork, provide some definitions, state some key lemmas and theorems in Section 2. Section 3 presents the results for complete controllability with Gramian matrix.

2. Preliminaries

Stefan Hilger's 1988 doctoral thesis was the first to present the time scales calculus. He is bringing together the system's discrete and continuous analysis. A time scale $\mathbb{T}$ is defined as a non-empty closed subset of $\mathbb{R}$. If max $\mathbb{T}$ exists, we define $\mathbb{T}^k = \mathbb{T}\{max\mathbb{T}\}$. But if that is not the case, $\mathbb{T}^k = \mathbb{T}$. According, we define $(a,b)_{\mathbb{T}}$, $[a,b)_{\mathbb{T}}$, $(a,b]_{\mathbb{T}}$ and so on as a time scale interval, where $[a,b]_{\mathbb{T}} = \{t \in \mathbb{T}: a \le t \le b\}$. With the substitution $sup\mathbb{T}$ for inf$\{\emptyset\}$, the forward jump operator $\sigma: \mathbb{T}^k \to \mathbb{T}$ is defined as $\sigma(t) = inf\{\, s \in \mathbb{T}: s > t\} \in \mathbb{T}$. The operator $\rho: \mathbb{T}^k \to \mathbb{T}$, which is defined as $\rho(t) = \sup\{\, s \in \mathbb{T}: s > t\} \in \mathbb{T}$, can be expanded with the substitution $\sup\{\emptyset\} = \inf \mathbb{T}$. At last, for $t \in \mathbb{T}$, the graininess function $\mu(t)$ follows the equation $\sigma(t) - t$.

when $t = sup\mathbb{T}$, choose τ such that mapping x from $\mathbb{T}$ to $\mathbb{R}$ is not left scattered. If $\varepsilon > 0$, then the generalized delta derivative of x(t), denoted as x^{Δ}(t), is of the form that. Given that U(t) is a neighbourhood, it follows that

$$\big|[x(\sigma(t) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]\big| \le \varepsilon|\sigma(t) - s|, \text{ for } s \in U.$$

The process of mapping x from $\mathbb{T}$ to $\mathbb{R}$ is known as the generalized delta derivative on time scales calculus, where x is delta derivative for every $t \in \mathbb{T}$.

The right dense points in $\mathbb{T}$ are considered to represent the origins of rd-continuous M mapping from $\mathbb{T}$ to $\mathbb{R}$, whereas the left dense points in $\mathbb{T}$ are the locations of its finite left sided limits. The set of rd-continuous functions M is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$. Assuming $M^{\Delta}(\tau) = M(\tau)$ for every $\tau \in \mathbb{T}^k$, the mapping from $\mathbb{T}^k$ to $\mathbb{R}$ is referred to as the anti- derivative of M from $\mathbb{T}^k$ to $\mathbb{R}$. We continue by creating the integral $\int_a^b m(t)\Delta t = M(b) - M(a)$.

Definition 2.1.[5]: The function M(t) that maps from $\mathbb{T}$ to $\mathbb{R}$ is regressive is defined as$1 + \mu(t)(t) \neq 0 \; \forall \, t \in \mathbb{T}$. The right dense continuous function $\mathscr{R} = \mathscr{R}(t) = \mathscr{R}(\mathbb{T}, \mathbb{R})$ is the sum of all regressive functions. Likewise, $\mathscr{R}^{+} = \mathscr{R}^{+}(\mathbb{T},\mathbb{R}) = \{M \in \mathscr{R}: 1 + \mu(t)M(t) > 0, \forall\, t \in \mathbb{T}\}$ denotes all positively regressive function.

Lemma 2.1.[6]: When $M, N \in \mathscr{R}$ matrices on $\mathbb{T}$, thus

i. $e_M^{-1}(\tau,s) \equiv e^{*}_{\ominus M}(\tau,s)$;
ii. $e_0(\tau,s) \equiv I$ and $e_0(\tau,\tau) \equiv I$;
iii. $e_M(\sigma(\tau),s) \equiv \big(I + \mu(\tau)M(\tau)\big)e_M(\tau,s)$;
iv. $e_M(\tau,s) = e_M^{-1}(s,\tau) = e^{*}_{\ominus M^{*}}(s,\tau)$;
v. $e_M(\tau,s)e_N(\tau,s) = e_{M\oplus N}(\tau,s)$;
vi. $e_M(\tau,s)e_M(s,r) = e_M(\tau,r)$;

Lemma 2.2.[4]: Consider a matrix M of size $n \times n$ on a time scale. Assume that the mapping f from $\mathbb{T}$ to $\mathbb{R}^n$ is continuous and right dense. Given that t_0 belongs to the set $\mathbb{T}$ and p_0 belongs to $\mathbb{R}^n$, this implies the initial value problem (IVP).

$$p^{\Delta}(t) = M(t)p(t) + l(t), p(t_0) = p_0,$$

having one and only one solution p mapping from $\mathbb{T}$ to $\mathbb{R}$ is developed as

$$p(t) = f_M(t,t_0)p_0 + \int_{t_0}^{t} f_M\big(t,\sigma(\tau)\big)l(\tau)\Delta\tau.$$

Theorem 2.1. Let Z(t)=Vec X(t), $\widehat{U}(t)$=Vec U(t), and $f(t,z(t)) = VecF(t,X(t))$. Then the Volterra Integro-dynamic Sylvester matrix with an impulse control system (1.1), (1.2) is equivalent the system

$$z^{\Delta}(t) = P(t)z(t) + \int_0^t K(t,s)z(s)\Delta s + Q(t)\widehat{U}(t) + f\big(t,z(t)\big) \tag{2.1}$$

$$z(t) = [I_n \otimes R_j]z\big(t_j^{-}\big) \tag{2.2}$$

Where $P(t) = [B^{*} \otimes I_n + I_n \otimes A]$, $Q(t) = [I_n \otimes C]$, $K(t,s) = [K_2^{*} \otimes I_n) + (I_n \otimes K_1)$ and $R_j = \big(I + D_j\big)$ and I_n is the identity matrix.

Proof: We apply the Vec operator to the equation (1.1), (1.2) and using the above properties of Kronecker product [3], we have

$$z^{\Delta}(t) = P(t)z(t) + \int_0^t K(t,s)z(s)\Delta s + Q(t)\widehat{U}(t) + f\big(t,z(t)\big)$$

$$z(t) = [I_n \otimes R_j] z(t_j^-)$$

Lemma 2.3 [7]. For the system (2.2) with $P \in M_{n^2}(\mathbb{R})$ is a constant, there exists a scalar function $\gamma_0(t,s), \dots, \gamma_{n^2-1}(t,s) \in (\mathbb{T}^+, \mathbb{R})$ such that the only one solution has representation.

$$e_P(t,s) = \sum_{k=0}^{n^2-1} \gamma_k(t,s) P^k.$$

Theorem 2.2. Each $\forall\, t \in (s_j, t_{j+1}]_{\mathbb{T}}, j = 1,2,..$, implies the satisfying function is known as the solution of a system (2.1) represented by

$$z(t) = \Psi(t, s_j)[I_n \otimes R_j] z(t_j^-) + \int_{s_j}^{t} \Psi(t, \sigma(s)) \mathrm{K}(t,s) z(s) \Delta s + \int_{s_j}^{t} \Psi(t, \sigma(\tau)) [Q(\tau)\widehat{U}(\tau) + f(\tau, z(\tau))] \Delta\tau, \tag{2.3}$$

Proof: if $t \in [t_0, t_1]_{\mathbb{T}}$, then there exists an only one solution of (2.1), we have

$$z(t) = \Psi(t, t_0) z_0 + \int_{t_0}^{t} \Psi(t, \sigma(s)) \mathrm{K}(t,s) z(s) \Delta s + \int_{t_0}^{t} \Psi(t, \sigma(\tau)\, Q(\tau)\widehat{U}(\tau) \Delta\tau + \int_{t_0}^{t} \Psi(t, \sigma(\tau)\, f(\tau, z(\tau)) \Delta\tau$$

Next, j=1 then $t \in (s_1, t_2]_{\mathbb{T}}$ we have

$$z(t) = \Psi(t, t_0) z(s_1) + \int_{s_1}^{t} \Psi(t, \sigma(s)) \mathrm{K}(t,s) z(s) \Delta s + \int_{s_1}^{t} \Psi(t, \sigma(\tau)\, Q(\tau)\widehat{U}(\tau) \Delta\tau + \int_{s_1}^{t} \Psi(t, \sigma(\tau)\, f(\tau, z(\tau)) \Delta\tau.$$

Also, for $z(s_1) = [I_n \otimes R_1] z(t_1)$ substitute above equation, we get

$$z(t) = \Psi(t, t_0)[I_n \otimes R_1] z(t_1) + \int_{s_1}^{t} \Psi(t, \sigma(s)) \mathrm{K}(t,s) z(s) \Delta s + \int_{s_1}^{t} \Psi(t, \sigma(\tau)\, Q(\tau)\widehat{U}(\tau) \Delta\tau + \int_{s_1}^{t} \Psi(t, \sigma(\tau)\, f(\tau, z(\tau)) \Delta\tau.$$

Similarly, we are repeating the above same process for $t \in (s_j, t_{j+1}]_{\mathbb{T}}$, $j = 1,2, \dots, m$, we get

$$z(t) = \Psi(t, s_j)[I_n \otimes R_j] z(t_j^-) + \int_{s_j}^{t} \Psi(t, \sigma(s)) \mathrm{K}(t,s) z(s) \Delta s$$

$$+\int_{s_j}^{t} \Psi\big(t,\sigma(\tau)\big)\,[Q(\tau)\widehat{U}(t)+f(\tau,z(\tau))]\,\Delta\tau$$

Therefore, the equation (2.3) was derived.

3. CONTROLLABILITY

In this section, we provide necessary and sufficient conditions for complete controllability in the following system.

$$\begin{cases} z^{\Delta}(t)=P(t)z(t)+\int_0^t K(t,s)z(s)\Delta s+Q(t)\widehat{U}(t)+f\big(t,z(t)\big), & t\in\big(s_j,t_{j+1}\big]_{\mathbb{T}}, j=1,2,\dots \\ z(t)=\big[I_n\otimes R_j\big]z\big(t_j^-\big), & t\in\big(s_j,t_{j+1}\big]_{\mathbb{T}}, \quad j=0,1,\dots \\ z(t_0)=z_0,\ \ t_0\in\mathbb{T} \end{cases} \tag{3.1}$$

Definition 3.1: For any z_0 and $z_T\in\mathbb{R}^{n^2}$, there must be a piece-wise rd-continuous control function $\widehat{U}(t)\colon[t_0,T]_{\mathbb{T}}\to\mathbb{R}^{n^2}$, so that the solution of the system (3.1) satisfies $z(t_0)=z_0$ and $z(T)=z_T$. This system is called controllability on $[t_0,T]_{\mathbb{T}}$ with $t_0<T$.

Definition 3.2: For $j=1,2,\dots,m$, it is controllable on both $[t_0,t_1]_{\mathbb{T}}$ and $[s_j,t_{j+1}]_{\mathbb{T}}$, then system (3.1) is known as complete controllable in $[t_0,T]_{\mathbb{T}}$ with $t_0<T$.

The corresponding Gramian matrices are defined by.

$$\mathcal{N}_0(t_0,t_1)=\int_{t_0}^{t}\Psi\big(t_0,\sigma(\tau)\big)Q(\tau)Q^*(\tau)\Psi^*\big(t_0,\sigma(\tau)\big)\Delta\tau \tag{3.2}$$

$$\mathcal{N}_j\big(s_j,t_{j+1}\big)=\int_{s_j}^{t_{j+1}}\Psi\Big(s_j,\sigma(\tau)\Big)\,Q(\tau)Q^*(\tau)\Psi^*\Big(s_j,\sigma(\tau)\Big)\Delta\tau\,,\ j=1,2,\dots,m, \tag{3.3}$$

If $P(t)=P$ and $Q(t)=Q$ are constant matrices, then

$$\mathcal{N}_0(t_0,t_1)=\int_{t_0}^{t}e_P\big(t_0,\sigma(\tau)\big)QQ^*e_P^*\big(t_0,\sigma(\tau)\big)\Delta\tau \tag{3.4}$$

$$\mathcal{N}_j\big(s_j,t_{j+1}\big)=\int_{s_j}^{t_{j+1}}e_P\Big(s_j,\sigma(\tau)\Big)\,QQ^*e_P^*\Big(s_j,\sigma(\tau)\Big)\Delta\tau,\ j=1,2,\dots,m \tag{3.5}$$

Here $(.)^*$is represented as transpose of a matrix $(.)$.

We define $\widehat{U}(t)$ as

$$\widehat{U}(t)=\begin{cases} -Q^*(t)\Psi^*\big(t_0,\sigma(\tau)\big)\Phi_0\,, & t\in[t_0,t_1]_{\mathbb{T}} \\ -Q^*(t)\Psi^*\big(s_k,\sigma(\tau)\big)\Phi_k, & t\in\big(s_j,t_{j+1}\big]_{\mathbb{T}}, j=1,2,\dots,m. \end{cases} \tag{3.6}$$

Where

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$$\Phi_0 = \mathcal{N}_0^{-1}(t_0,t_1)\left[z_0 - \Psi(t_0,t_1)z_{t_1} + \int_{t_0}^{t_1} \Psi\big(t_0,\sigma(s)\big)\mathrm{K}(t,s)z(s)\Delta s + \int_{t_0}^{t_1}\Psi\big(t_0,\sigma(\tau)\big)f(\tau,z(\tau))\Delta\tau\right],$$

and

$$\Phi_j = \mathcal{N}_j^{-1}\big(s_j,t_{j+1}\big)\Bigg[\big[I_n\otimes R_j\big]z\big(t_j^-\big) - \Psi\big(s_j,t_{j+1}\big)z_{t_{j+1}} + \int_{s_j}^{t_{j+1}} \Psi\left(t_j,\sigma(s)\right)\mathrm{K}(t,s)z(s)\Delta s$$
$$+ \int_{s_j}^{t_{j+1}} \Psi\left(s_j,\sigma(\tau)\right) f(\tau,z(\tau))\Delta\tau\Bigg].$$

We need following the conditions:

(H1): The nonlinear function $f: J_1 \times \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}, J_1 = \cup_{j=0}^{m}[s_j,t_{j+1}]_{\mathbb{T}}$ is rd-continuous and there is exists $M_f > 0$ such that $\|f(t,z) - f(t,x)\| \le M_f\|z-x\|, \forall\, z,x \in \mathbb{R}^{n^2}, t \in J_1$. Also, there is exists $L_f > 0$ such that $\|f(t,z)\| \le L_f, \forall t \in J_1\ and\ z \in \mathbb{R}^{n^2}$.

(H2): The function $\big[I_n\otimes R_j\big] := [t_j,s_j]_{\mathbb{T}} \times \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$ are rd-continuous there is exists $M_{[I_n\otimes R_j]} > 0$ such that $\big\|\big[I_n\otimes R_j\big]z\big(t_j^-\big) - \big[I_n\otimes R_j\big]x\big(t_j^-\big)\big\| \le M_{[I_n\otimes R_j]}\|z-x\|, \forall\, z,x \in \mathbb{R}^{n^2}, t \in I_j$. Also, there is exists $L_{[I_n\otimes R_j]} > 0$ such that $\big\|\big[I_n\otimes R_j\big]z\big(t_j\big)\big\| \le L_{[I_n\otimes R_j]}, \forall t \in I_j\ and\ z \in \mathbb{R}^{n^2}$.

(H3): $M_\alpha = \max_{1\le k\le m}\{M_{\alpha_1}^0, M_{\alpha_1}^j, M_{[I_n\otimes R_j]}\} < 1$, were

$$M_{\alpha_1}^0 = LM_kM_ft_1\big(1 + L^2L_Q^2t_1\delta_0\big),$$

$$M_{\alpha_1}^j = LM_{[I_n\otimes R_j]} + L^2L_Q^2M_{[I_n\otimes R_j]}T\delta_j + LM_kM_fT\big(1 + L^2L_Q^2T\delta_j\big), j = 1,2,\dots,m$$

For notational accommodation, we get

$\delta_0 = \big\|\mathcal{N}_0^{-1}(t_0,t_1)\big\|, \delta_j = \big\|\mathcal{N}_j^{-1}\big(s_j,t_{j+1}\big)\big\|.$

$L = \max_{(t,s)\in I}\Psi(t,s), L_Q = \max_{t\in I}\|Q(t)\|, L_K = \max_{t\in I}\|K(t,s)\|$

$\mathscr{H}_0 = L\|z_0\| + LL_Kt_1 + LL_ft_1 + LL_QL_{\hat{U}}^0t_1.$

$L_{\hat{U}}^0 = LL_Q\delta_0\big(\|z_0\| + L\|z_{t_1}\| + LL_Kt_1 + LL_ft_1\big).$

$\mathscr{H}_1^j = LL_{[I_n\otimes R_j]} + LL_KT + LL_fT + LL_QL_{\hat{U}}^TT.$

$L_{\hat{U}}^j = LL_Q\delta_0\left(L_{[I_n\otimes R_j]} + L\left\|z_{t_{j+1}}\right\| + LL_Kt_{j+1} + LL_ft_{j+1}\right).$

$$\gamma \ge \max_{1\le k\le m}\left\{\mathscr{H}_0, \mathscr{H}_1^j, L_{[I_n\otimes R_j]}\right\}.$$

Theorem 3.1: Assuming that requirements **(H1) - (H3)** are satisfied, then there is only one solution to system (3.1).

Proof: The subset $\mathcal{D} \subseteq PC$ is defined as the set

$$\mathcal{D} = \{z \in PC: \|z\|_{PC} \leq \gamma\}.$$

Currently, we are defining the function $\mathcal{G}: \mathcal{D} \longrightarrow \mathcal{D}$, which means that

For, $t \in [0, t_1]_{\mathbb{T}}$

$$(\mathcal{G}z)(t) = \Psi(t, t_0)z_0 + \int_{t_0}^{t} \Psi\big(t, \sigma(s)\big)K(t, s)z(s)\Delta s + \int_{t_0}^{t} \Psi\big(t, \sigma(\tau)\big)(f(\tau, z(\tau)) + Q(\tau))\Delta\tau. \tag{3.8}$$

For $t \in \big(t_j, s_j\big]_{\mathbb{T}}, j = 1,2, \dots, m$

$$(\mathcal{G}z)(t) = \big[I_n \otimes R_j\big]z\big(t_j^-\big), \tag{3.8}$$

For $\forall t \in \big(s_j, t_{j+1}\big]_{\mathbb{T}},\ j = 1,2, \dots, m$

$$(\mathcal{G}z)(t) = \Psi\big(t, s_j\big)\big[I_n \otimes R_j\big]z\big(t_j^-\big) + \int_{s_j}^{t} \Psi\big(t, \sigma(s)\big)K(t, s)z(s)\Delta s + \int_{s_j}^{t} \Psi\big(t, \sigma(\tau)\big)\Big(f(\tau, z(\tau)) + Q(\tau)\widehat{U}(\tau)\Big)\Delta\tau. \tag{3.9}$$

It is evident that the solution is the Banach fixed point for $\mathcal{G}$. Let us now consider $t \in \big(s_j, t_{j+1}\big]_{\mathbb{T}}, j = 1,2, \dots, m,$ and $z \in \mathcal{D}$, we obtain

$$\begin{aligned}\|(\mathcal{G}z)(t)\| &\leq \big\|\Psi\big(t, s_j\big)\big\|\big\|\big[I_n \otimes R_j\big]z\big(t_j^-\big)\big\| + \int_{s_j}^{t} \big\|\Psi\big(t, \sigma(s)\big)\big\|\|K(t, s)\|\|z(s)\|\Delta s \\ &+ \int_{\square_\square}^{\square} \big\|\Psi\big(\square, \square(\square)\big)\big\|\big\|\square\big(\square, \square(\square)\big)\big\|\Delta\square \\ &+ \int_{s_j}^{t} \big\|\Psi\big(t, \sigma(\tau)\big)\big\|\big\|\widehat{\mathrm{U}}(\tau)\big\|\|Q(\tau)\|\,\Delta\tau \\ &\leq LL_{[I_n \otimes R_j]} + LL_K t_{j+1} + LL_f t_{j+1} + LL_Q L_{\widehat{U}}^{j} t_{j+1} \\ &\leq \mathscr{H}_1^j \leq \gamma.\end{aligned} \tag{3.10}$$

Similarly, for $t \in [0, t_1]_{\mathbb{T}}$ and $z \in \mathcal{D}$, then

$$\|(\mathcal{G}z)(t)\| \leq \|\Psi(t,t_0)\|\|z_0\| + \int_{t_0}^{t} \|\Psi(t,\sigma(s))\|\|K(t,s)\|\|z(s)\|\Delta s$$
$$+\int_{t_0}^{t} \|\Psi(t,\sigma(\tau))\|\|f(\tau,z(\tau))\|\Delta\tau \qquad (3.11)$$
$$+\int_{t_0}^{t} \|\Psi(t,\sigma(\tau))\|\|\widehat{\mathrm{U}}(\tau)\|\|Q(\tau)\|\,\Delta\tau$$
$$\leq L\|z_0\| + LL_K t + LL_f t + LL_Q L_{\widehat{U}}^{0} t$$
$$\leq \mathscr{H}_0 \leq \gamma.$$

Similarly, for $t \in (s_j, t_j]_{\mathbb{T}}$, and $z \in \mathcal{D}$, we get

$$\|(\mathcal{G}z)\|_{PC} \leq L_{[I_n \otimes R_j]} \leq \gamma. \qquad (3.12)$$

After succinct the above inequalities (3.10) - (3.12), we have

$$\|(\mathcal{G}z)\|_{PC} \leq \gamma.$$

Since, $\mathcal{G}: \mathcal{D} \longrightarrow \mathcal{D}$, For any $z, x \in \mathcal{D}$, $t \in (s_j, t_{j+1}]_{\mathbb{T}}$, $j = 1,2,\ldots,m$, we get

$$\|(\mathcal{G}z)(t) - (\mathcal{G}x)(t)\| \leq \|\Psi(t,s_j)\|\|[I_n \otimes R_j]z(t_j^-) - [I_n \otimes R_j]x(t_j^-)\|$$
$$+\int_{s_j}^{t} \|\Psi(t,\sigma(s))\|\|K(t,s)\|\|z(s) - x(s)\|\Delta s$$
$$+\int_{s_j}^{t} \|\Psi(t,\sigma(\tau))\|\|f(\tau,z(\tau)) - f(\tau,x(\tau))\|\Delta\tau$$
$$+\int_{s_j}^{t}\Bigg[\|\Psi(t,\sigma(\tau))\|\|Q(\tau)\|\|Q^*(\tau)\|\|\Psi^*(t,\sigma(\tau))\|$$
$$\times \|\mathcal{N}_j^{-1}(s_j,t_{j+1})\|\|[I_n \otimes R_j]z(t_j^-) - [I_n \otimes R_j]x(t_j^-)\|$$
$$+\int_{\square_\square}^{\square_\square+1} \|\Psi(\square,\square(\square))\|\|\square(\square,\square)\|\|\square(\square) - \square(\square)\|\Delta\square$$
$$+\int_{s_j}^{t_{j+1}} \left\|\Psi\left(s_j,\sigma(s)\right)\right\| \; \|f(s,z(s)) - f(s,x(s))\|\Delta s\Bigg]\Delta\tau$$

$$\leq LM_{[I_n\otimes R_j]}\|z(t_j^-) - x(t_j^-)\| + LM_K\int_{s_j}^{t}\|z(s)-x(s)\|\Delta s + LM_f\int_{s_j}^{t}\|z(\tau)-x(\tau)\|\Delta\tau + L^2L_Q^2\delta_j$$
$$\times\int_{s_j}^{t}[M_{[I_n\otimes R_j]}\|z(t_j^-) - x(t_j^-)\| + M_K\int_{s_j}^{t_{j+1}}\|z(s)-x(s)\|\Delta s + M_f\int_{s_j}^{t_{j+1}}\|z(s)-x(s)\|\Delta s]\Delta\tau$$
$$\leq LM_{[I_n\otimes R_j]}\|z-x\|_{PC} + LM_K\|z-x\|_{PC}(t-s_j) + LM_f\|z-x\|_{PC}(t-s_j)$$
$$+L^2L_Q^2(t-s_j)\delta_j[M_{[I_n\otimes R_j]} + LM_K(t_{j+1}-s_j)]\,\|z-x\|_{PC} + LM_f(t_{j+1}-s_j)]\,\|z-x\|_{PC}$$

$$\leq M_{\alpha_1}^{j}\|z-x\|_{PC} \leq M_{\alpha}\|z-x\|_{PC} \tag{3.13}$$

For any $z, x \in \mathcal{D}, t \in [0, t_1]_{\mathbb{T}}$, we get

$$\begin{aligned}
&\|(\square\square)(\square) - (\square\square)(\square)\| \\
&\leq \int_0^{\square} \|\Psi(\square, \square(\square))\| \|\square(\square, \square)\| \|\square(\square) - \square(\square)\| \Delta\square \\
&+ \int_0^{t} \|\Psi(t, \sigma(\tau))\| \|f(\tau, z(\tau)) - f(\tau, x(\tau))\| \Delta\tau \\
&+ \int_0^{t} [\|\Psi(t, \sigma(\tau))\| \|Q(\tau)\| \|Q^*(\tau)\| \|\Psi^*(t, \sigma(\tau))\| \\
&\times \|\mathcal{N}_0^{-1}(t_0, t_1)\| [\int_0^{t_1} \|\Psi(t, \sigma(s))\| \|K(t, s)\| \|z(s) - x(s)\| \Delta s \\
&+ \int_{t_0}^{t_1} \|\Psi(s_k, \sigma(s))\| \|f(s, z(s)) - f(s, x(s))\| \Delta s] \Delta\tau \\
&\leq M_{\alpha_1}^{0}\|z-x\|_{PC} \leq M_{\alpha}\|z-x\|_{PC}.
\end{aligned} \tag{3.14}$$

Similarly, for $t \in (s_j, t_j]_{\mathbb{T}}$, we have

$$\|(\mathcal{G}z)(t) - (\mathcal{G}x)(t)\| \leq M_{[I_n \otimes R_j]}\|z-x\|_{PC} \leq M_{\alpha}\|z-x\|_{PC} \tag{3.15}$$

After succinct the inequalities (3.13) - (3.15), for $t \in I$, we have

$$\|(\mathcal{G}z) - (\mathcal{G}x)\|_{PC} \leq M_{\alpha}\|z-x\|_{PC}.$$

Thus, according to Banach's fixed point theorem, there is only one solution to system (3.1). Because of this, $\mathcal{G}$ is a mapping that strictly contracts.

Theorem 3.2: Assuming that requirements **(H1) - (H3)** are satisfied; the system (3.1) is complete controllable in $[t_0, T]_{\mathbb{T}}$ if and only if the matrices $\mathcal{N}_0(t_0, t_1)$ and $\mathcal{N}_j(s_j, t_{j+1})$ are invertible.

Proof: Let $\mathcal{N}_0(t_0, t_1)$ and $\mathcal{N}_j(s_j, t_{j+1})$ are invertible. Then, for the given $z_{t_1}$and $z_{t_{j+1}}$, and the input control $\widehat{\mathrm{U}}$ (t) given by (3.6). Now, put $t = t_1$, in the system (3.1), we have

$$\begin{aligned}
z(t_1) = \Psi(t_1, t_0)z_0 + \int_{t_0}^{t_1} \Psi(t_1, \sigma(s))K(t, s)z(s)\Delta s + \int_{t_0}^{t_1} \Psi(t_1, \sigma(\tau))f(\tau, z(\tau))\Delta\tau \\
- \int_{t_0}^{t_1} \|\Psi(t_1, \sigma(\tau))\| \|Q(\tau)\| \|Q^*(\tau)\| \|\Psi^*(t_0, \sigma(\tau))\| z_0 \Delta\tau
\end{aligned}$$

$$= \Psi(t_1,t_0)z_0 + \int_{t_0}^{t_1} \Psi\big(t_1,\sigma(s)\big)K(t,s)z(s)\Delta s + \int_{t_0}^{t_1} \Psi\big(t_1,\sigma(\tau)\big)f(\tau,z(\tau))\Delta\tau$$

$$- \Psi(t_1,t_0)\mathcal{N}_0(t_0,t_1)\mathcal{N}_0^{-1}(t_0,t_1)\Big[z_0 - \Psi(t_0,t_1)z_{t_1} + \int_{t_0}^{t_1} \Psi\big(t_1,\sigma(s)\big)K(t,s)z(s)\Delta s$$

$$+ \int_{t_0}^{t_1} \Psi\big(t_0,\sigma(\tau)\big)f(\tau,z(\tau))\Delta\tau\Big]$$

$$= z_{t_1}$$

Similarly, for $t \in (s_j, t_{j+1}]_{\mathbb{T}}, , j = 1,2,\ldots,m,$. we replace $t = t_{j+1}$, in the solution of (3.1), we have

$$z(t_{j+1}) = \Psi(t_{j+1},s_j)[I_n \otimes R_j]z(t_j^-) + \int_{s_j}^{t_{j+1}} \Psi\Big(t_{j+1},\sigma(s)\Big)K(t,s)z(s)\Delta s$$

$$+ \int_{s_j}^{t_{j+1}} \Psi\Big(t_{j+1},\sigma(\tau)\Big)\Big[f\big(\tau,z(\tau)\big) - Q(\tau)Q^*(\tau)\Psi^*\Big(s_j,\sigma(\tau)\Big)z_j\Big]\Delta\tau$$

$$= \Psi(t_{j+1},s_j)[I_n \otimes R_j]z(t_j^-) + \int_{s_j}^{t_{j+1}} \Psi\Big(t_{j+1},\sigma(s)\Big)K(t,s)z(s)\Delta s$$

$$+ \int_{\square_\square}^{\square_{\square+1}} \Psi\big(\square_{\square+1},\square(\square)\big)\square\big(\square,\square(\square)\big)\Delta\square$$

$$- \Psi(t_{j+1},s_j)\mathcal{N}_j(s_j,t_{j+1})\mathcal{N}_j^{-1}(s_j,t_{j+1})$$

$$\times\Bigg[[I_n \otimes R_j]z(t_j^-) - \Psi(s_j,t_{j+1})z_{t_{j+1}} + \int_{s_j}^{t_{j+1}} \Psi\Big(t_{j+1},\sigma(s)\Big)K(t,s)z(s)\Delta s$$

$$+ \int_{s_j}^{t_{j+1}} \Psi\Big(t_{j+1},\sigma(\tau)\Big)f(\tau,z(\tau))\Delta\tau\Bigg]$$

$$= z_{t_{j+1}}.$$

Hence, for in $[t_0,T]_{\mathbb{T}}$ the system (3.1) is complete controllable

Conversely, on the interval $[t_0,T]_{\mathbb{T}}$, we presume that system (3.1) is complete controllable. Therefore, the matrices $\mathcal{N}_0(t_0,t_1)$ and $\mathcal{N}_j(s_j,t_{j+1})$ are not invertible.

Then, there exists a non-zero vector $z_\alpha, z_{\alpha_j} \in \mathbb{R}^{n^2}$ such that

$$z_\alpha^*\mathcal{N}_0(t_0,t_1)z_\alpha = 0 \text{ and } \quad z_{\alpha_j}^*\mathcal{N}_j(s_j,t_{j+1})z_{\alpha_j} = 0. \tag{3.16}$$

From the equations (3.2), (3.3) and (3.16), we get

$$\int_{t_0}^{t} z_\alpha^*\Psi\big(t_0,\sigma(\tau)\big)Q(\tau)Q^*(\tau)\Psi^*\big(t_0,\sigma(\tau)\big)z_\alpha\Delta\tau = 0. \tag{3.17}$$

$$\int_{s_j}^{t_{j+1}} z_{\alpha_j}^{*}\Psi\big(t_0,\sigma(\tau)\big)Q(\tau)Q^{*}(\tau)\Psi^{*}\big(t_0,\sigma(\tau)\big)z_{\alpha_j}\Delta\tau = 0. \tag{3.18}$$

On solving the above equations (3.17) and (3.18), we have

$z_{\alpha}^{*}\Psi\big(t_0,\sigma(\tau)\big)Q(\tau) = 0, \tau \in [t_0,t_1]_{\mathbb{T}}$

$z_{\alpha_j}^{*}\Psi\big(s_j,\sigma(\tau)\big)\,Q(\tau) = 0, \tau \in \big(s_j,t_{j+1}\big]_{\mathbb{T}}, j = 1,2,\ldots,m.$

Therefore, the system (3.1) is complete controllable on$[t_0,t_1]_{\mathbb{T}}$,

So, if we choose

$$z_0 = z_{\alpha} + \Psi(t_0,t_1)z_{\alpha_j} - \int_{t_0}^{t_1}\Psi\big(t_0,\sigma(s)\big)K(t,s)z(s)\Delta s - \int_{t_0}^{t_1}\Psi\big(t_0,\sigma(\tau)\big)f\big(\tau,z(\tau)\big)\Delta\tau,$$

in $[t_0,t_1]_{\mathbb{T}}$. In that case, there exist a piece-wise rd-continuous control $\widehat{\mathrm{U}}(t)$ that

$$z_{\alpha_1} = \Psi(t_1,t_0)\left(z_{\alpha} + \Psi(t_0,t_1)z_{\alpha_1} - \int_{t_0}^{t_1}\Psi\big(t_0,\sigma(s)\big)K(t,s)z(s)\Delta s - \int_{t_0}^{t_1}\Psi\big(t_0,\sigma(\tau)\big)f\big(\tau,z(\tau)\big)\Delta\tau\right) + \int_{t_0}^{t_1}\Psi\big(t_0,\sigma(s)\big)K(t,s)z(s)\Delta s + \int_{t_0}^{t_1}\Psi\big(t_0,\sigma(\tau)\big)\Big(f\big(\tau,z(\tau)\big) + Q(\tau)\widehat{\mathrm{U}}(\tau)\Big)\Delta\tau,$$

Which gives $z_{\alpha}^{*}z_{\alpha} = 0$. Similarly, we have

$$[I_n\otimes R_j]z\big(t_j^{-}\big) = z_{\alpha_j} + \Psi\big(s_j,t_{j+1}\big)z_{t_{j+1}} - \int_{s_j}^{t_{j+1}}\Psi\big(t_0,\sigma(s)\big)K(t,s)z(s)\Delta s - \int_{s_j}^{t_{j+1}}\Psi\big(t_0,\sigma(\tau)\big)f\big(\tau,z(\tau)\big)\Delta\tau.$$

It can be shown that $z_{\alpha_j}^{*}z_{\alpha_j} = 0,$ which contradicts the fact that $z_{\alpha}^{*}z_{\alpha} \neq 0$,Therefore, the matrices $\mathcal{N}_0(t_0,t_1)$ and $\mathcal{N}_j\big(s_j,t_{j+1}\big)$ are invertible.

Theorem 3.3: Assuming that requirements **(H1) - (H3)** are satisfied; the time-invariant case of system (3.1) is said to be complete controllable in interval $[t_0,T]_{\mathbb{T}}$ if and only if the rank of the matrix

$$[Q \;\; PQ \;\; P^2Q \ldots P^{n-1}Q] \;= n^2 \tag{3.19}$$

Proof: Assume that system (3.1) is to be complete controllable in $[t_0,T]_{\mathbb{T}}$. But the rank of $C \neq n^2$($\because$ $\big[Q \;\; PQ \;\; P^2Q \ldots P^{n^2-1}Q\big] = C$), then there exists non-zero vector $z_{\alpha} \in \mathbb{R}^{n^2}$such that

$$z_{\alpha}^{*}P^{i}B = 0, \qquad i = 0,1,\ldots,n^2-1. \tag{3.20}$$

Furthermore, based on equations (3.4) and (3.5), we can deduce

$$z_{\alpha}^{*}\mathcal{N}_0(t_0,t_1)z_{\alpha} = \int_{t_0}^{t} z_{\alpha}^{*}e_P\big(t_0,\sigma(\tau)\big)QQ^{*}e_P^{*}\big(t_0,\sigma(\tau)\big)z_{\alpha}\Delta\tau \tag{3.21}$$

$$z_\alpha^* \mathcal{N}_k\big(s_j, t_{j+1}\big) z_\alpha = \int_{s_j}^{t_{j+1}} z_\alpha^* e_P\left(s_j, \sigma(\tau)\right) Q Q^* e_P^*\left(s_j, \sigma(\tau)\right) z_\alpha \Delta\tau\,. \tag{3.22}$$

Now, we are using Theorem 2.4. and from equation (3.20) in the above equation (3.21) and (3.22), we have

$$z_\alpha^* \mathcal{N}_0(t_0, t_1) z_\alpha = \int_{t_0}^{t} \left[\sum_{j=0}^{n^2-1} \gamma_j\left(t_0, \sigma(\tau)\right) z_\alpha^* P^i Q\right] Q^* e_P^*\big(t_0, \sigma(\tau)\big) z_\alpha \Delta\tau = 0$$

$$z_\alpha^* \mathcal{N}_j\big(s_j, t_{j+1}\big) z_\alpha = \int_{s_j}^{t_{j+1}} \left[\sum_{j=0}^{n^2-1} \gamma_j\left(s_j, \sigma(\tau)\right) z_\alpha^* P^i Q\right] Q^* e_P^*\left(s_j, \sigma(\tau)\right) z_\alpha \Delta\tau = 0.$$

Thus, $\mathcal{N}_0(t_0, t_1)$ and $\mathcal{N}_j\big(s_j, t_{j+1}\big)$, are not invertible. Theorem 3.1states that system (3.1) is not completely controllable. Therefore, it contradicts. The rank of $C = n^2$.

Conversely, the matrices $\mathcal{N}_0(t_0, t_1)$ and $\mathcal{N}_j\big(s_j, t_{j+1}\big)$, are not invertible and we assume that the rank of $C = n^2$. the system (3.1) is not to be complete controllable. This means that there exists non-zero vectors $z_\alpha, z_{\alpha_j} \in \mathbb{R}^{n^2}$, such that

$$z_\alpha^* \mathcal{N}_0(t_0, t_1) z_\alpha = 0. \tag{3.23}$$

and

$$z_{\alpha_j}^* \mathcal{N}_j\big(s_j, t_{j+1}\big) z_{\alpha_j} = 0, j = 1,2 \dots, m, \tag{3.24}$$

Now, from the equations (3.4), (3.5), (3.23) and (3.24), we have

$$z_\alpha^* e_P(t_0, t_1) Q = 0, \forall\, t \in [t_0, t_1]_{\mathbb{T}} \tag{3.25}$$

and

$$z_\alpha^* e_P\big(s_j, t_{j+1}\big) Q = 0, \forall\, t \in \big(s_j, t_{j+1}\big]_{\mathbb{T}},, j = 1,2, \dots, m, \tag{3.26}$$

Now, for $j = 1,2, \dots, m,$ the $e_P(t_0, .), e_P\big(s_j, .\big)$ are rd-continuous and $\sigma([t_0, t_1]_{\mathbb{T}})$, $\sigma\big((s_j, t_{j+1}]_{\mathbb{T}}\big)$ are density argument $[\sigma(t_0), \sigma(t_1)]_{\mathbb{T}} = [t_0, t_1]_{\mathbb{T}}, (\sigma(s_j), \sigma(t_{j+1})]_{\mathbb{T}} = (s_j, t_{j+1}]_{\mathbb{T}}$, Hence, from the above equations (3.25) and (3.26), we have

$$z_\alpha^* e_P(t_0, t) Q = 0, \forall\, t \in [t_0, t_1]_{\mathbb{T}}. \tag{3.27}$$

$$z_\alpha^* e_P\big(s_j, t\big) Q = 0, \forall\, t \in \big(s_j, t_{j+1}\big]_{\mathbb{T}}, j = 1,2, \dots, m, \tag{3.28}$$

At $t = t_0$, an equation (3.27) becomes $z_\alpha^* Q = 0$. Also, $e_P(t_0, .)$ is delta differentiable, we get

$$e_P^{\Delta t}(t_0, t) = -e_P\big(t_0, \sigma(t)\big) P.$$

Then subsequent derivatives and the density equations of (3.27) give

$$(-1)^i z_\alpha^* e_P(t_0, t) P^{i-1} Q = 0, i = 0,1,2, \dots, n^2 - 1, t \in [t_0, t_1]_{\mathbb{T}}. \tag{3.29}$$

Put $t = t_0$ in the above equation (3.29), we have

$$z_{\alpha}^{*}P^{i-1}Q=0, i=0,1,2,\dots,n^{2}-1.$$

Therefore, $z_{\alpha}^{*}\left[Q\ \ PQ\ \ P^{2}Q\ \dots\ P^{n^{2}-1}Q\right]=0,$ hence, our assumption is wrong: Therefore, it is a contradictory that the rank of $C=n^{2}$.Similarly, we iterate the procedure on equation (3.28), yielding

$$z_{\alpha_k}^{*}\left[Q\ \ PQ\ \ P^{2}Q\ \dots\ P^{n^{2}-1}Q\right]=0$$

Once again, the contradiction demonstrate that system (3.1) is completely controllable throughout the time interval $[t_0,T]_{\mathbb{T}}$.

Example 3.1: The following non-linear Kreneker product of Volterra integro-dynamic with an impulse control system

$$\begin{cases} z^{\Delta}(t)=P(t)z(t)+\int_{0}^{t}K(t,s)z(s)\Delta s+Q(t)\widehat{U}(t)+f\big(t,z(t)\big), & t\in\left(s_j,t_{j+1}\right]_{\mathbb{T}}, j=0,1,2, \\ z(t)=[I_n\otimes R_k]z\left(t_j^{-}\right), & t\in\left(s_j,t_{j+1}\right]_{\mathbb{T}},\ j=1,2,\dots \\ z(t_0)=z_0,\ \ z_0\in\mathbb{R}^2 \end{cases} \tag{3.30}$$

Where $z(t)=\begin{bmatrix} z_{11}(t)\\ z_{12}(t)\\ z_{21}(t)\\ z_{22}(t)\end{bmatrix}, t_0=s_0=0, t_1=0.8, s_1=0.9, t_2=2.1, s_2=2.2, t_3=T=3,\ P(t)=[B^{*}\otimes \mathrm{I_n}+\mathrm{I_n}\otimes A]=\begin{bmatrix} -2 & 0 & 0 & 0\\ 0 & -2 & 0 & 0\\ 0 & 0 & -3 & 0\\ 0 & 0 & 0 & -3\end{bmatrix},$

$$K(t,s)=[K_2^{*}\otimes I_n)+(I_n\otimes K_1)\ =\begin{bmatrix} sint & 0 & 0 & 0\\ 0 & cost & 0 & 0\\ 0 & 0 & sint & 0\\ 0 & 0 & 0 & cost\end{bmatrix},$$

$$Q(t)=[\mathrm{I_n}\otimes C]=\begin{bmatrix} 1 & 0\\ \frac{2}{25}e^{1}(\sigma(t),0) & 0\\ 0 & 1\\ 0 & \frac{2}{25}e^{1}(\sigma(t),0)\end{bmatrix}, f(t,z(t)=\frac{1}{35}\begin{bmatrix} \frac{\sin\left(z_{22}(t)\right)}{e^{t^2+2}}\\ 0\\ 0\\ \frac{\cos\left(z_{11}(t)\right)}{e^{t^2+2}}\end{bmatrix},$$

$$[I_n\otimes R_j]z\left(t_j^{-}\right)=\frac{1}{20}\begin{bmatrix} \frac{z_2(t_k^{-})}{e^{t^2+2}(1+it)} & 0\\ \frac{z_1(t_k^{-})}{e^{t^2+3}(1+it^2)} & 0\\ 0 & \frac{z_2(t_k^{-})}{e^{t^2+2}(1+it)}\\ 0 & \frac{z_1(t_k^{-})}{e^{t^2+3}(1+it^2)}\end{bmatrix}, j=1,2.$$

The matrix that provides the fundamental solution to the system (3.30) is

$$e^{P}(t,0)=\begin{bmatrix} e^{-2}(t,0) & 0 & 0 & 0 \\ 0 & e^{-2}(t,0) & 0 & 0 \\ 0 & 0 & e^{-3}(t,0) & 0 \\ 0 & 0 & 0 & e^{-3}(t,0) \end{bmatrix}.$$

Therefore $e^{P}(0,t)=\begin{bmatrix} e^{2}(0,t) & 0 & 0 & 0 \\ 0 & e^{2}(0,t) & 0 & 0 \\ 0 & 0 & e^{3}(0,t) & 0 \\ 0 & 0 & 0 & e^{3}(0,t) \end{bmatrix}$. Also we can easily compute

$$\mathcal{N}(0,\sigma(\tau))=\Psi(0,\sigma(\tau))Q(\tau)Q^{*}(\tau)\Psi^{*}(0,\sigma(\tau))$$

$$=\begin{bmatrix} \left(e^{-2}(0,\sigma(\tau))+\frac{8}{625}\right)^{2} & 0 & 0 & 0 \\ 0 & \left(e^{-2}(0,\sigma(\tau))+\frac{8}{625}\right)^{2} & 0 & 0 \\ 0 & 0 & \left(e^{-3}(0,\sigma(\tau))+\frac{8}{625}\right)^{2} & 0 \\ 0 & 0 & 0 & \left(e^{-3}(0,\sigma(\tau))+\frac{8}{625}\right)^{2} \end{bmatrix}.$$

$$\mathcal{N}(s_1,\sigma(\tau))=\Psi(s_1,\sigma(\tau))Q(\tau)Q^{*}(\tau)\Psi^{*}(s_1,\sigma(\tau))$$

$$=\begin{bmatrix} \left(e^{-2}(s_1,\sigma(\tau))+\frac{8}{625}\right)^{2} & 0 & 0 & 0 \\ 0 & \left(e^{-2}(s_1,\sigma(\tau))+\frac{8}{625}\right)^{2} & 0 & 0 \\ 0 & 0 & \left(e^{-3}(s_1,\sigma(\tau))+\frac{8}{625}\right)^{2} & 0 \\ 0 & 0 & 0 & \left(e^{-3}(s_1,\sigma(\tau))+\frac{8}{625}\right)^{2} \end{bmatrix}.$$

$$\mathcal{N}(s_2,\sigma(\tau))=\Psi(s_2,\sigma(\tau))Q(\tau)Q^{*}(\tau)\Psi^{*}(s_2,\sigma(\tau))$$

$$=\begin{bmatrix} \left(e^{-2}(s_2,\sigma(\tau))+\frac{8}{625}\right)^{2} & 0 & 0 & 0 \\ 0 & \left(e^{-2}(s_2,\sigma(\tau))+\frac{8}{625}\right)^{2} & 0 & 0 \\ 0 & 0 & \left(e^{-3}(s_2,\sigma(\tau))+\frac{8}{625}\right)^{2} & 0 \\ 0 & 0 & 0 & \left(e^{-3}(s_2,\sigma(\tau))+\frac{8}{625}\right)^{2} \end{bmatrix}.$$

Now consider the following two cases:

Case (1): If $\mathbb{T}=\mathbb{R}$, then $e^{a}(t,0)=e^{at}$. Therefore,

$$\mathcal{N}_0(0,t_1)=\int_0^{t}\mathcal{N}(0,\sigma(\tau))d\tau=\begin{bmatrix} 7.2665 & 0 & 0 & 0 \\ 0 & 7.2665 & 0 & 0 \\ 0 & 0 & 24.156 & 0 \\ 0 & 0 & 0 & 24.156 \end{bmatrix}.$$

$$\mathcal{N}_1(s_1,t_2)=\int_{s_1}^{t_2}\mathcal{N}\big(s_1,\sigma(\tau)\big)\mathrm{d}\tau=\begin{bmatrix}16.513 & 0 & 0 & 0\\ 0 & 16.513 & 0 & 0\\ 0 & 0 & 74.52 & 0\\ 0 & 0 & 0 & 74.52\end{bmatrix}.$$

$$\mathcal{N}_2(s_2,T)=\int_{s_2}^{T}\mathcal{N}\big(s_2,\sigma(\tau)\big)\mathrm{d}\tau=\begin{bmatrix}53.762 & 0 & 0 & 0\\ 0 & 53.762 & 0 & 0\\ 0 & 0 & 137.24 & 0\\ 0 & 0 & 0 & 137.24\end{bmatrix}.$$

It follows that the matrices $\mathcal{N}_0(0,t_1)$, $\mathcal{N}_1(s_1,t_2)$, and $\mathcal{N}_2(s_2,T)$ are all invertible. In addition, all three assumptions **(H1) − (H3)** hold. with M_α = max {0.9104, 0.2328, 0.7531, 0.01718} <1. The system (3.30) is complete controllable, since all the criteria of Theorem 3.2, are satisfied.

Case (2): If $\mathbb{T}=\mathbb{P}_{1,1}=\cup_{j=0}^{\infty}[2j.2j+1]$, then $e^{a}(t,0)=(1+a)^{j}e^{a(t-j)}$. Therefore,

$$\mathcal{N}_0(0,t_1)=\int_{0}^{\mathrm{t}}\mathcal{N}\big(0,\sigma(\tau)\big)\mathrm{d}\tau=\begin{bmatrix}7.2665 & 0 & 0 & 0\\ 0 & 7.2665 & 0 & 0\\ 0 & 0 & 24.156 & 0\\ 0 & 0 & 0 & 24.156\end{bmatrix}.$$

$$\mathcal{N}_1(s_1,t_2)=\int_{s_1}^{t_2}\mathcal{N}\big(s_1,\sigma(\tau)\big)\mathrm{d}\tau=\begin{bmatrix}6.781 & 0 & 0 & 0\\ 0 & 6.781 & 0 & 0\\ 0 & 0 & 15.167 & 0\\ 0 & 0 & 0 & 15.167\end{bmatrix}.$$

$$\mathcal{N}_2(s_2,T)=\int_{s_2}^{T}\mathcal{N}\big(s_2,\sigma(\tau)\big)\mathrm{d}\tau=\begin{bmatrix}73.2665 & 0 & 0 & 0\\ 0 & 73.2665 & 0 & 0\\ 0 & 0 & 827.24 & 0\\ 0 & 0 & 0 & 827.24\end{bmatrix}.$$

It follows that the matrices $\mathcal{N}_0(0,t_1)$, $\mathcal{N}_1(s_1,t_2)$, and $\mathcal{N}_2(s_2,T)$ are all invertible. In addition, all three assumptions **(H1) − (H3)** hold. with M_α = max {0.9480, 0.2107, 0.9513, 0.00367} <1. The system (3.30) is complete controllable, since all the criteria of Theorem 3.1, are satisfied.

Declarations

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