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Pythagorean Fuzzy Ideals in Semiring

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Abstract:

In this paper, we introduce the notion of Pythagorean fuzzy ideals in semiring some interesting properties, results are discussed in this paper.

Keywords: Pythagorean, Fuzzy set, Semiring

1 INTRODUCTION

Nobusawa[5] studied the concept of gamma semiring as a generalization of ring after that Sen introduced the gamma semigroups as a generalization of gamma groups. Murali Krishna Rao[6] in 1995 introduced the notion of gamma semiring as a generalization of gamma ring, ring, ternary semiring and semiring.The important reason for development of gamma semiring is a generalization of results of rings, gamma rings, semirings, semigroup and ternary semirings.

Zadeh[12] studied the notion of fuzzy set theory. Atanassov [2] introduced intuitionistic fuzzy sets as a generalization of fuzzy sets. In intuitionistic, the sum of membership degree and non-membership degree should not exceed one. Yager [10] initially introduced the concept of Pythagorean fuzzy sets. In a Pythagorean fuzzy sets, the sum of the squared membership and non-membership degrees satisfies the condition. More recently, Yager [10, 11] proposed Pythagorean fuzzy sets as a powerful tool for effectively managing uncertainty or imprecise information in real-world scenarios. These sets enforce a constraint where the sum of squares of membership and non-membership degrees is less than or equal to 1.

Pythagorean fuzzy sets have showcased remarkable efficacy in navigating uncertainties, prompting a surge of scholarly exploration across diverse research avenues, resulting in significant progress. The conceptualization of Pythagorean fuzzy sets facilitates a more comprehensive and accurate portrayal of uncertain information when juxtaposed with intuitionistic fuzzy sets. Across various disciplines, academics have meticulously examined the algebraic attributes of Pythagorean fuzzy sets, shedding light on their practical applications and foundational theoretical constructs. Many authors studied the algebraic structures of Pythagorean fuzzy sets

This paper is structured into three sections. The first and second sections serve as the introduction and cover basic results pertinent to the paper's topic. In the third section, we introduce Pythagorean fuzzy ideals in semirings and some interesting properties this ideals are discussed.

2 Preliminaries

In this section we present the basic concepts related to this paper.

Definition 2.1 *A nonempty set S is said to be a semi-ring with respect to two binary compositions, addition and multiplication defined on it, if the following conditions are satisfied:*

1. $(S,+)$ is a commutative semigroup with zero.
2. $(S,.)$ is a semigroup.
3. for any three elements $a,b,c \in S$, the left distributive law

   $a.(b+c)=a.b+a.c$ and the right distributive law $(b+c).a=b.a+c.a$.
4. $s.0=0.s$, for all $s \in S$.

Definition 2.2 *A nonempty subset $\mathcal{J}$ of a semi-ring S is called an ideal if*

1. $a,b \in \mathcal{J}$ implies $a+b \in \mathcal{J}$
2. $a \in \mathcal{J}, s \in S$ implies $s.a \in \mathcal{J}$ and $a.s \in \mathcal{J}$

Definition 2.3 *Let μ be a nonempty fuzzy subset of a semi-ring S. Then μ is called a fuzzy left(right) ideal of S if for all $i,j \in S$.*

1. $\mu(i+j) \geq \min\{\mu(i),\mu(j)\}$
2. $\mu(ij) \geq \mu(j)(resp.right)\ \mu(ij) \geq \mu(i))$

A fuzzy ideal of a semi-ring S is a nonempty fuzzy subset of S which is both a fuzzy left ideal and a fuzzy right ideal of S.

3 Pythagorean fuzzy ideals in semiring

In this section S denotes Semiring(S).

Definition 3.1 *Let $P=(\mu_P,\vartheta_P)$ be a Pythagorean fuzzy subset of a semiring S and $\forall x,y \in S$.*

(i) $\mu_P(x+y) \geq \min\{\mu_P(x),\mu_P(y)\}$; $\vartheta_P(x+y) \leq \max\{\vartheta_P(x),\vartheta_P(y)\}$

(ii) $\mu_P(xy) \geq \min\{\mu_P(x),\mu_P(y)\}$; $\vartheta_P(xy) \leq \max\{\vartheta_P(x),\vartheta_P(y)\}$

Then $P=(\mu_P,\vartheta_P)$ is called a Pythagorean fuzzy subsemiring of R.

Definition 3.2 *Let $P=(\mu_P,\vartheta_P)$ of S is called a Pythagorean fuzzy left ideal of S, if P satisfies the following conditions*

(i) $\mu_P(x+y) \geq \min\{\mu_P(x),\mu_P(y)\}$; $\vartheta_P(x+y) \leq \max\{\vartheta_P(x),\vartheta_P(y)\}$

(ii) $\mu_P(xy) \geq \mu_P(y)$; $\vartheta_P(xy) \leq \vartheta_P(y)$

Definition 3.3 *Let $P=(\mu_P,\vartheta_P)$ of S is called a Pythagorean fuzzy right ideal of S, if P satisfies the following conditions*

(i) $\mu_P(x+y) \geq \min\{\mu_P(x),\mu_P(y)\}$; $\vartheta_P(x+y) \leq \max\{\vartheta_P(x),\vartheta_P(y)\}$

(ii) $\mu_P(xy) \geq \mu_P(x)$; $\vartheta_P(xy) \leq \vartheta_P(x)$.

Theorem 3.4 *Intersection of a non empty collection of Pythagorean fuzzy right (resp. left) ideals is also a Pythagorean fuzzy right (resp. left) ideal of* S.

Proof. Let $\{P_i = (\mu_i, \vartheta_i) | i \in I\}$ be a non empty family of Pythagorean fuzzy right ideals of S and $x, y \in S$.

Then

$$\bigcap_{i\in I} \mu_i(x+y) = \inf_{i\in I}\{\mu_i(x+y)\}$$

$$\geq \inf_{i\in I}\{\min\{\mu_i(x), \mu_i(y)\}\}$$

$$= \min\{\inf_{i\in I}\mu_i(x), \inf_{i\in I}\mu_i(y)\}$$

$$= \min\{\bigcap_{i\in I} \mu_i(x), \bigcap_{i\in I} \mu_i(y)\}.$$

Also

$$\bigcap_{i\in I} \vartheta_i(x+y) = \sup_{i\in I}\{\vartheta_i(x+y)\}$$

$$\leq \sup_{i\in I}\{\max\{\vartheta_i(x), \vartheta_i(y)\}\}$$

$$= \max\{\sup_{i\in I}\vartheta_i(x), \sup_{i\in I}\vartheta_i(y)\}$$

$$= \max\{\bigcap_{i\in I} \vartheta_i(x), \bigcap_{i\in I} \vartheta_i(y)\}.$$

Moreover

$$\bigcap_{i\in I} \mu_i(xy) = \inf_{i\in I}\{\mu_i(xy)\}$$

$$\geq \inf_{i\in I}\{\mu_i(x)\}$$

$$= \bigcap_{i\in I} \mu_i(x).$$

Finally

$$\bigcap_{i\in I} \vartheta_i(xy) = \sup_{i\in I}\{\vartheta_i(xy)\}$$

$$\leq \sup_{i\in I}\{\vartheta_i(x)\}$$

$$= \bigcap_{i\in I} \vartheta_i(x).$$

Hence $\bigcap_{i\in I} P_i$ is a Pythagorean fuzzy right ideals of S.

Similarly, we can prove the result for Pythagorean fuzzy left ideal also.

Theorem 3.5 *Union of a non empty collection of Pythagorean fuzzy right (resp. left) ideals is also a*

Pythagorean fuzzy right (resp. left) ideal of S.

Proof. Let $\{P_i = (\mu_i, \vartheta_i) | i \in I\}$ be a non empty family of Pythagorean fuzzy right ideals of S and $x, y \in S$.
Then

$$\bigcup_{i\in I} \mu_i(x+y) = \sup_{i\in I}\{\mu_i(x+y)\}$$

$$\leq \sup_{i\in I}\{\max\{\mu_i(x), \mu_i(y)\}\}$$

$$= \max\{\sup_{i\in I}\mu_i(x), \sup_{i\in I}\mu_i(y)\}$$

$$= \max\{\cup_{i\in I} \mu_i(x), \cup_{i\in I} \mu_i(y)\}.$$

Also

$$\bigcup_{i\in I} \vartheta_i(x+y) = \inf_{i\in I}\{\vartheta_i(x+y)\}$$

$$\geq \inf_{i\in I}\{\min\{\vartheta_i(x), \vartheta_i(y)\}\}$$

$$= \min\{\inf_{i\in I}\vartheta_i(x), \inf_{i\in I}\vartheta_i(y)\}$$

$$= \min\{\cup_{i\in I} \vartheta_i(x), \cup_{i\in I} \vartheta_i(y)\}.$$

Moreover

$$\bigcup_{i\in I} \mu_i(xy) = \sup_{i\in I}\{\mu_i(xy)\}$$

$$\leq \sup_{i\in I}\{\mu_i(x)\}$$

$$= \cup_{i\in I} \mu_i(x).$$

Finally

$$\bigcup_{i\in I} \vartheta_i(xy) = \inf_{i\in I}\{\vartheta_i(xy)\}$$

$$\geq \inf_{i\in I}\{\vartheta_i(x)\}$$

$$= \cup_{i\in I} \vartheta_i(x).$$

Hence $\cup_{i\in I} P_i$ is a Pythagorean fuzzy right ideals of S.

Similarly, we can prove the result for Pythagorean fuzzy left ideal also.

Definition 3.6 *Let* $P_1 = (\mu_1, \vartheta_1)$ *and* $P_2 = (\mu_2, \vartheta_2)$ *Pythagorean fuzzy subsets of* S. *The cartesian product of* P_1 *and* P_2 *is defined by*

(i) $\mu_1 \times \mu_2(x, y) = \min\{\mu_1(x), \mu_2(x)\}$

(ii) $\vartheta_1 \times \vartheta_2(x, y) = \max\{\vartheta_1(x), \vartheta_2(x)\}$, for all $x, y \in S$.

Theorem 3.7 *Let P_1 and P_2 be a Pythagorean fuzzy left ideals of semiring S. Then $P_1 \times P_2$ is a Pythagorean fuzzy left ideal of $S \times S$.*

Proof. Let $(x_1, x_2), (y_1, y_2) \in S \times S$.

Then

$$(\mu_1 \times \mu_2)((x_1, x_2) + (y_1, y_2)) = (\mu_1 \times \mu_2)(x_1 + y_1, x_2 + y_2)$$
$$= \min\{\mu_1(x_1 + y_1), \mu_2(x_2 + y_2)\}$$
$$\geq \min\{\min\{\mu_1(x_1), \mu_1(y_1)\}, \min\{\mu_2(x_2), \mu_2(y_2)\}\}$$
$$= \min\{\min\{\mu_1(x_1), \mu_2(x_2)\}, \min\{\mu_1(y_1), \mu_2(y_2)\}\}$$
$$= \min\{(\mu_1 \times \mu_2)(x_1, x_2), (\mu_1 \times \mu_2)(y_1, y_2)\}.$$

$$(\mu_1 \times \mu_2)((x_1, x_2)(y_1, y_2)) = (\mu_1 \times \mu_2)(x_1 y_1, x_2 y_2)$$
$$= \min\{\mu_1(x_1 y_1), \mu_2(x_2 y_2)\}$$
$$\geq \min\{\mu_1(y_1), \mu_2(y_2)\}$$
$$= (\mu_1 \times \mu_2)(y_1, y_2).$$

$$(\vartheta_1 \times \vartheta_2)((x_1, x_2) + (y_1, y_2)) = (\vartheta_1 \times \vartheta_2)(x_1 + y_1, x_2 + y_2)$$
$$= \max\{\vartheta_1(x_1 + y_1), \vartheta_2(x_2 + y_2)\}$$
$$\leq \max\{\max\{\vartheta_1(x_1), \vartheta_1(y_1)\}, \max\{\vartheta_2(x_2), \vartheta_2(y_2)\}\}$$
$$= \max\{\max\{\vartheta_1(x_1), \vartheta_2(x_2)\}, \max\{\vartheta_1(y_1), \vartheta_2(y_2)\}\}$$
$$= \max\{(\vartheta_1 \times \vartheta_2)(x_1, x_2), (\vartheta_1 \times \vartheta_2)(y_1, y_2)\}.$$

$$(\vartheta_1 \times \vartheta_2)((x_1, x_2)(y_1, y_2)) = (\vartheta_1 \times \vartheta_2)(x_1 y_1, x_2 y_2)$$
$$= \max\{\vartheta_1(x_1 y_1), \vartheta_2(x_2 y_2)\}$$
$$\leq \max\{\vartheta_1(y_1), \vartheta_2(y_2)\}$$
$$= (\vartheta_1 \times \vartheta_2)(y_1, y_2).$$

Therefore $P_1 \times P_2$ is a Pythagorean fuzzy left ideal of $S \times S$.

Theorem 3.8 *Let P be a Pythagorean fuzzy subset of semiring. Then P is a Pythagorean fuzzy left ideal of S if and only if $P \times P$ is a Pythagorean fuzzy left ideal of $S \times S$.*

Proof. Consider P is a Pythagorean fuzzy left ideal of S. Then by Previous theorem $P \times S$.

Conversely $P \times P$ is a Pythagorean fuzzy left ideal of $S \times S$, for all $x_1, x_2, y_1, y_2 \in S$.

Then

$$\min\{\mu(x_1 + y_1), \mu(x_2 + y_2)\} = \mu \times \mu(x_1 + y_1, x_2 + y_2)$$
$$= (\mu \times \mu)\{(x_1, x_2) + (y_1, y_2)\}$$
$$\geq \min\{(\mu \times \mu)(x_1, x_2), (\mu \times \mu)(y_1, y_2)\}$$

$$= \min\{\min\{\mu(x_1),\mu(x_2)\},\min\{\mu(y_1),\mu(y_2)\}\}$$

Next,we have

$$\min\{\mu(x_1y_1),\mu(x_2y_2)\} = (\mu\times\mu)(x_1y_1,x_2y_2)$$
$$= (\mu\times\mu)\{(x_1,x_2)(y_1,y_2)\}$$
$$= (\mu\times\mu)\{y_1,y_2\}$$
$$= \min\{\mu(y_1),\mu(y_2)\}$$

Also

$$\max\{\vartheta(x_1+y_1),\vartheta(x_2+y_2)\} = \vartheta\times\vartheta(x_1+y_1,x_2+y_2)$$
$$= (\vartheta\times\vartheta)\{(x_1,x_2)+(y_1,y_2)\}$$
$$\leq \max\{(\vartheta\times\vartheta)(x_1,x_2),(\vartheta\times\vartheta)(y_1,y_2)\}$$
$$= \max\{\max\{\vartheta(x_1),\vartheta(x_2)\},\max\{\vartheta(y_1),\vartheta(y_2)\}\}$$

and

$$\max\{\vartheta(x_1y_1),\vartheta(x_2y_2)\} = (\vartheta\times\vartheta)(x_1y_1,x_2y_2)$$
$$= (\vartheta\times\vartheta)\{(x_1,x_2)(y_1,y_2)\}$$
$$= (\vartheta\times\vartheta)\{y_1,y_2\}$$
$$= \max\{\vartheta(y_1),\vartheta(y_2)\}$$

Hence P is a Pythagorean fuzzy left ideal of S.

Theorem 3.9 *If P_1,P_2 be any two Pythagorean fuzzy ideals of semiring S, then P_1+P_2 is also so.*

Proof. Consider P_1,P_2 are any two Pythagorean fuzzy ideals of semiring S and $x,y\in S$.

Then

$$(\mu_1+\mu_2)(x+y) = \sup_{x+y\leq c+d}\{\min\{\mu_1(c),\mu_2(d)\}\}$$
$$\geq \sup_{x+y\leq(a_1+b_1)+(a_2+b_2)=(a_1+a_2)+(b_1+b_2)}\{\min\{\mu_1(a_1+a_2),\mu_2(b_1+b_2)\}\}$$
$$\geq \sup\{\min\{\mu_1(a_1),\mu_2(a_2)\},\min\{\mu_2(b_1),\mu_2(b_2)\}\}$$
$$= \min\{\sup_{x\leq a_1+b_1}\{\min\{\mu_1(a_1),\mu_2(b_1)\}\},\sup_{y\leq a_2+b_2}\{\min\{\mu_1(a_2),\mu_2(b_2)\}\}\}$$
$$= \min\{(\mu_1+\mu_2)(x),(\mu_1+\mu_2)(y)\}$$

Also

$$(\vartheta_1+\vartheta_2)(x+y) = \inf_{x+y\leq c+d}\{\max\{\vartheta_1(c),\vartheta_2(d)\}\}$$
$$\leq \inf_{x+y\leq(a_1+b_1)+(a_2+b_2)=(a_1+a_2)+(b_1+b_2)}\{\max\{\vartheta_1(a_1+a_2),\vartheta_2(b_1+b_2)\}\}$$
$$\leq \inf\{\max\{\vartheta_1(a_1),\vartheta_2(a_2)\},\max\{\vartheta_2(b_1),\vartheta_2(b_2)\}\}$$

$$= \max\{\inf_{x\leq a_1+b_1}\{\max\{\vartheta_1(a_1),\vartheta_2(b_1)\}\}, \inf_{y\leq a_2+b_2}\{\max\{\vartheta_1(a_2),\vartheta_2(b_2)\}\}\}$$

$$= \max\{(\vartheta_1+\vartheta_2)(x),(\vartheta_1+\vartheta_2)(y)\}$$

Now let as consider P_1, P_2 are Pythagorean fuzzy right ideals and we have

$$(\mu_1+\mu_2)(xy) = \sup_{x+y\leq c+d}\{\min\{\mu_1(c),\mu_2(d)\}\}$$

$$\geq \sup_{xy\leq(x_1+x_2)y}\{\min\{\mu_1(x_1y),\mu_2(x_2y)\}\}$$

$$\geq \sup_{x\leq(x_1+x_2)}\{\min\{\mu_1(x_1),\mu_2(x_2)\}\}$$

$$= (\mu_1+\mu_2)(x).$$

and

$$(\vartheta_1+\vartheta_2)(xy) = \inf_{x+y\leq c+d}\{\max\{\vartheta_1(c),\vartheta_2(d)\}\}$$

$$\leq \inf_{xy\leq(x_1+x_2)y}\{\max\{\vartheta_1(x_1y),\vartheta_2(x_2y)\}\}$$

$$\leq \inf_{x\leq(x_1+x_2)}\{\max\{\vartheta_1(x_1),\vartheta_2(x_2)\}\}$$

$$= (\vartheta_1+\vartheta_2)(x).$$

Similarly assuming P_1, P_2 are Pythagorean fuzzy left ideal,

we can show that $(P_1+P_2)(xy) \geq (P_1+P_2)(y)$

Also

$$(\mu_1+\mu_2)(x) = \sup_{x\leq x_1+x_2}\{\min\{\mu_1(x_1),\mu_2(x_2)\}\}$$

$$\geq \sup_{x\leq y\leq y_1+y_2}\{\min\{\mu_1(y_1),\mu_2(y_2)\}\}$$

$$= \sup_{y\leq y_1+y_2}\{\min\{\mu_1(y_1),\mu_2(y_2)\}\}$$

$$= (\mu_1+\mu_2)(y)$$

and

$$(\vartheta_1+\vartheta_2)(x) = \inf_{x\leq x_1+x_2}\{\max\{\vartheta_1(x_1),\vartheta_2(x_2)\}\}$$

$$\leq \inf_{x\leq y\leq y_1+y_2}\{\max\{\vartheta_1(y_1),\vartheta_2(y_2)\}\}$$

$$= \inf_{y\leq y_1+y_2}\{\max\{\vartheta_1(y_1),\vartheta_2(y_2)\}\}$$

$$= (\vartheta_1+\vartheta_2)(y)$$

Hence P_1+P_2 is a Pythagorean fuzzy ideal of S.

Theorem 3.10 *If P_1, P_2 be any two Pythagorean fuzzy ideals of semiring S, then $P_1 \circ P_2$ is also so.*

Proof. Let P_1, P_2 are any two Pythagorean fuzzy ideals of semiring S and $x, y \in S$.

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Then

$$(\mu_1 \circ \mu_2)(x+y) = \sup_{x+y \le c+d} \{\min\{\mu_1(c), \mu_2(d)\}\}$$

$$\ge \sup_{x+y \le (c_1 d_1)+(c_2 d_2) \le (c_1+c_2)(d_1+d_2)} \{\min\{\mu_1(c_1+c_2), \mu_2(d_1+d_2)\}\}$$

$$\ge \sup\{\min\{\mu_1(c_1), \mu_1(c_2)\}, \min\{\mu_2(d_1), \mu_2(d_2)\}\}$$

$$= \min\{\sup_{x \le c_1 d_1} \{\min\{\mu_1(c_1), \mu_2(d_1)\}\}, \sup_{y \le c_2 d_2} \{\min\{\mu_1(c_2), \mu_2(d_2)\}\}\}$$

$$= \min\{(\mu_1 \circ \mu_2)(x), (\mu_1 \circ \mu_2)(y)\}$$

Also

$$(\vartheta_1 \circ \vartheta_2)(x+y) = \inf_{x+y \le c+d} \{\max\{\vartheta_1(c), \vartheta_2(d)\}\}$$

$$\le \inf_{x+y \le (c_1 d_1)+(c_2 d_2) \le (c_1+c_2)(d_1+d_2)} \{\max\{\vartheta_1(c_1+c_2), \vartheta_2(d_1+d_2)\}\}$$

$$\le \inf\{\max\{\vartheta_1(c_1), \vartheta_1(c_2)\}, \max\{\vartheta_2(d_1), \vartheta_2(d_2)\}\}$$

$$= \max\{\inf_{x \le c_1 d_1} \{\max\{\vartheta_1(c_1), \vartheta_2(d_1)\}\}, \inf_{y \le c_2 d_2} \{\max\{\vartheta_1(c_2), \vartheta_2(d_2)\}\}\}$$

$$= \max\{(\vartheta_1 \circ \vartheta_2)(x), (\vartheta_1]circ\vartheta_2)(y)\}$$

Now let as consider P_1, P_2 are Pythagorean fuzzy right ideals and we have

$$(\mu_1 \circ \mu_2)(xy) = \sup_{xy \le cd} \{\min\{\mu_1(c), \mu_2(d)\}\}$$

$$\ge \sup_{xy \le (x_1 x_2)y} \{\min\{\mu_1(x_1 y), \mu_2(x_2 y)\}\}$$

$$\ge \sup_{x \le (x_1 x_2)} \{\min\{\mu_1(x_1), \mu_2(x_2)\}\}$$

$$= (\mu_1 \circ \mu_2)(x).$$

and

$$(\vartheta_1 \circ \vartheta_2)(xy) = \inf_{xy \le cd} \{\max\{\vartheta_1(c), \vartheta_2(d)\}\}$$

$$\le \inf_{xy \le (x_1 x_2)y} \{\max\{\vartheta_1(x_1 y), \vartheta_2(x_2 y)\}\}$$

$$\le \inf_{x \le (x_1 x_2)} \{\max\{\vartheta_1(x_1), \vartheta_2(x_2)\}\}$$

$$= (\vartheta_1 \circ \vartheta_2)(x).$$

Similarly assuming P_1, P_2 are Pythagorean fuzzy left ideal,

we can show that $(P_1 \circ P_2)(xy) \ge (P_1 \circ P_2)(y)$

Also

$$(\mu_1 \circ \mu_2)(x) = \sup_{x \le x_1 x_2} \{\min\{\mu_1(x_1), \mu_2(x_2)\}\}$$

$$\ge \sup_{x \le y \le y_1 y_2} \{\min\{\mu_1(y_1), \mu_2(y_2)\}\}$$

$$= \sup_{y \leq y_1 y_2} \{\min\{\mu_1(y_1), \mu_2(y_2)\}\}$$

$$= (\mu_1 \circ \mu_2)(y)$$

and

$$(\vartheta_1 \circ \vartheta_2)(x) = \inf_{x \leq x_1 x_2} \{\max\{\vartheta_1(x_1), \vartheta_2(x_2)\}\}$$

$$\leq \inf_{x \leq y \leq y_1 y_2} \{\max\{\vartheta_1(y_1), \vartheta_2(y_2)\}\}$$

$$= \inf_{y \leq y_1 y_2} \{\max\{\vartheta_1(y_1), \vartheta_2(y_2)\}\}$$

$$= (\vartheta_1 \circ \vartheta_2)(y)$$

Hence $P_1 \circ P_2$ is a Pythagorean fuzzy ideal of S.

Definition 3.11 *A Pythagorean fuzzy subset* $P = (\mu, \vartheta)$ *is called a Pythagorean fuzzy bi-ideal of* S, *for all* $x, y, z \in S$.

(i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}; \vartheta(x + y) \leq \max\{\vartheta(x), \vartheta(y)\}$

(ii) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}; \vartheta(xy) \leq \max\{\vartheta(x), \vartheta(y)\}$

(iii) $\mu(xyz) \geq \min\{\mu(x), \mu(z)\}; \vartheta(xyz) \leq \max\{\vartheta(x), \vartheta(z)\}$

Theorem 3.12 *Intersection of a non empty collection of Pythagorean fuzzy bi-ideals is also Pythagorean fuzzy bi-ideal of* S.

Proof. Let $\{P_i = (\mu_i, \vartheta_i) | i \in I\}$ be a family of Pythagorean fuzzy bi-ideals of S and $x, y \in S$.

Then

$$\bigcap_{i \in I} \mu_i(xyz) = \inf_{i \in I}\{\mu_i(xyz)\}$$

$$\geq \inf_{i \in I}\{\min\{\mu_i(x), \mu_i(z)\}\}$$

$$= \min\{\inf_{i \in I}\mu_i(x), \inf_{i \in I}\mu_i(z)\}$$

$$= \min\{\cap_{i \in I} \mu_i(x), \cap_{i \in I} \mu_i(z)\}.$$

Finally

$$\bigcap_{i \in I} \vartheta_i(xyz) = \sup_{i \in I}\{\vartheta_i(xyz)\}$$

$$\leq \sup_{i \in I}\{\max\{\vartheta_i(x), \vartheta_i(z)\}\}$$

$$= \max\{\sup_{i \in I}\vartheta_i(x), \sup_{i \in I}\vartheta_i(z)\}$$

$$= \max\{\cap_{i \in I} \vartheta_i(x), \cap_{i \in I} \vartheta_i(z)\}.$$

Hence P_i is a Pythagorean fuzzy bi-ideal of S.

Theorem 3.13 *Let P_1 and P_2 be two Pythagorean fuzzy bi-ideal of S. Then P is a Pythagorean fuzzy bi-ideal of S.*

Proof. Let $x, y, z \in S$

$$
\begin{aligned}
(\mu_1 \cdot \mu_2)(x+y) &= \min\{\mu_1(x+y), \mu_2(x+y)\} \\
&\geq \min\{\min\{\mu_1(x), \mu_1(y)\}, \min\{\mu_2(x), \mu_2(y)\}\} \\
&= \min\{\min\{\mu_1(x), \mu_2(x)\}, \min\{\mu_1(y), \mu_2(y)\}\} \\
&= \min\{(\mu_1 \cdot \mu_2)(x), (\mu_1 \cdot \mu_2)(y)\}
\end{aligned}
$$

and

$$
\begin{aligned}
(\vartheta_1 \cdot \vartheta_2)(x+y) &= \max\{\vartheta_1(x+y), \vartheta_2(x+y)\} \\
&\leq \max\{\max\{\vartheta_1(x), \vartheta_1(y)\}, \max\{\vartheta_2(x), \vartheta_2(y)\}\} \\
&= \max\{\max\{\vartheta_1(x), \vartheta_2(x)\}, \max\{\vartheta_1(y), \vartheta_2(y)\}\} \\
&= \max\{(\vartheta_1 \cdot \vartheta_2)(x), (\vartheta_1 \cdot \vartheta_2)(y)\}
\end{aligned}
$$

Next

$$
\begin{aligned}
(\mu_1 \cdot \mu_2)(xy) &= \min\{\mu_1(xy), \mu_2(xy)\} \\
&\geq \min\{\min\{\mu_1(x), \mu_1(y)\}, \min\{\mu_2(x), \mu_2(y)\}\} \\
&= \min\{\min\{\mu_1(x), \mu_2(x)\}, \min\{\mu_1(y), \mu_2(y)\}\} \\
&= \min\{(\mu_1 \cdot \mu_2(x)), (\mu_1 \cdot \mu_2(y))\}
\end{aligned}
$$

and

$$
\begin{aligned}
(\vartheta_1 \cdot \vartheta_2)(xy) &= \max\{\vartheta_1(xy), \vartheta_2(xy)\} \\
&\leq \max\{\max\{\vartheta_1(x), \vartheta_1(y)\}, \max\{\vartheta_2(x), \vartheta_2(y)\}\} \\
&= \max\{\max\{\vartheta_1(x), \vartheta_2(x)\}, \max\{\vartheta_1(y), \vartheta_2(y)\}\} \\
&= \max\{(\vartheta_1 \cdot \vartheta_2(x)), (\vartheta_1 \cdot \vartheta_2(y))\}
\end{aligned}
$$

Also

$$
\begin{aligned}
(\mu_1 \cdot \mu_2)(xyz) &= \min\{\mu_1(xyz), \mu_2(xyz)\} \\
&\geq \min\{\min\{\mu_1(x), \mu_1(z)\}, \min\{\mu_2(x), \mu_2(z)\}\} \\
&= \min\{\min\{\mu_1(x), \mu_2(x)\}, \min\{\mu_1(z), \mu_2(z)\}\} \\
&= \min\{(\mu_1 \cdot \mu_2(x)), (\mu_1 \cdot \mu_2(x))\}
\end{aligned}
$$

and

$$
\begin{aligned}
(\vartheta_1 \cdot \vartheta_2)(xyz) &= \max\{\vartheta_1(xyz), \vartheta_2(xyz)\} \\
&\leq \max\{\max\{\vartheta_1(x), \vartheta_1(z)\}, \max\{\vartheta_2(x), \vartheta_2(z)\}\} \\
&= \max\{\max\{\vartheta_1(x), \vartheta_2(x)\}, \max\{\vartheta_1(z), \vartheta_2(z)\}\} \\
&= \max\{(\vartheta_1 \cdot \vartheta_2(x)), (\vartheta_1 \cdot \vartheta_2(z))\}
\end{aligned}
$$

Hence P_1 and P_2 is a Pythagorean fuzzy bi-ideal of S.

Definition 3.14 *The product of P_1 and P_2 is a Pythagorean fuzzy subset $P_1 \circ P_2 : S \rightarrow [0,1]$ by*

$$(\mu_1 \circ \mu_2)(a) = \sup_{a=bc} \{\min\{\mu_1(b), \mu_2(c)\}\}$$

$$(\vartheta_1 \circ \vartheta_2)(a) = \inf_{a=bc} \{\max\{\vartheta_1(b), \vartheta_2(c)\}\}$$

Theorem 3.15 *If P_1, P_2 be any two Pythagorean fuzzy bi-ideals of semiring S, then $P_1 \circ P_2$ is a Pythagorean fuzzy bi-ideal of S.*

Proof. Let P_1, P_2 are any two Pythagorean fuzzy ideals of semiring S and $x, y \in S$.

Then

$$(\mu_1 \circ \mu_2)(x+y) = \sup_{x+y \le c+d} \{\min\{\mu_1(c), \mu_2(d)\}\}$$

$$\geq \sup_{x+y \le (c_1 d_1)+(c_2 d_2) \le (c_1+c_2)(d_1+d_2)} \{\min\{\mu_1(c_1+c_2), \mu_2(d_1+d_2)\}\}$$

$$\geq \sup\{\min\{\mu_1(c_1), \mu_1(c_1)\}, \min\{\mu_2(d_1), \mu_2(d_2)\}\}$$

$$= \min\{\sup_{x \le c_1 d_1} \{\min\{\mu_1(c_1), \mu_2(d_1)\}\}, \sup_{y \le c_2 d_2} \{\min\{\mu_1(c_2), \mu_2(d_2)\}\}\}$$

$$= \min\{(\mu_1 \circ \mu_2)(x), (\mu_1 \circ \mu_2)(y)\}$$

Also

$$(\vartheta_1 \circ \vartheta_2)(x+y) = \inf_{x+y \le c+d} \{\max\{\vartheta_1(c), \vartheta_2(d)\}\}$$

$$\leq \inf_{x+y \le (c_1 d_1)+(c_2 d_2) \le (c_1+c_2)(d_1+d_2)} \{\max\{\vartheta_1(c_1+c_2), \vartheta_2(d_1+d_2)\}\}$$

$$\leq \inf\{\max\{\vartheta_1(c_1), \vartheta_1(c_2)\}, \max\{\vartheta_2(d_1), \vartheta_2(d_2)\}\}$$

$$= \max\{\inf_{x \le c_1 d_1} \{\max\{\vartheta_1(c_1), \vartheta_2(d_1)\}\}, \inf_{y \le c_2 d_2} \{\max\{\vartheta_1(c_2), \vartheta_2(d_2)\}\}\}$$

$$= \max\{(\vartheta_1 \circ \vartheta_2)(x), (\vartheta_1]circ\vartheta_2)(y)\}$$

Now let as consider P_1, P_2 are Pythagorean fuzzy right ideals and we have

$$(\mu_1 \circ \mu_2)(xy) = \sup_{xy \le cd} \{\min\{\mu_1(c), \mu_2(d)\}\}$$

$$\geq \sup_{xy \le (x_1 x_2) y} \{\min\{\mu_1(x_1 y), \mu_2(x_2 y)\}\}$$

$$\geq \sup_{x \le (x_1 x_2)} \{\min\{\mu_1(x_1), \mu_2(x_2)\}\}$$

$$= (\mu_1 \circ \mu_2)(x).$$

and

$$(\vartheta_1 \circ \vartheta_2)(xy) = \inf_{xy \le cd} \{\max\{\vartheta_1(c), \vartheta_2(d)\}\}$$

$$\leq \inf_{xy \le (x_1 x_2) y} \{\max\{\vartheta_1(x_1 y), \vartheta_2(x_2 y)\}\}$$

$$\leq \inf_{x \le (x_1 x_2)} \{\max\{\vartheta_1(x_1), \vartheta_2(x_2)\}\}$$

$$= (\vartheta_1 \circ \vartheta_2)(x).$$

Similarly assuming P_1, P_2 are Pythagorean fuzzy left ideal,

we can show that $(P_1 \circ P_2)(xy) \geq (P_1 \circ P_2)(y)$

Also

$$(\mu_1 \circ \mu_2)(x) = \sup_{x \leq x_1 x_2} \{\min\{\mu_1(x_1), \mu_2(x_2)\}\}$$

$$\geq \sup_{x \leq y \leq y_1 y_2} \{\min\{\mu_1(y_1), \mu_2(y_2)\}\}$$

$$= \sup_{y \leq y_1 y_2} \{\min\{\mu_1(y_1), \mu_2(y_2)\}\}$$

$$= (\mu_1 \circ \mu_2)(y)$$

and

$$(\vartheta_1 \circ \vartheta_2)(x) = \inf_{x \leq x_1 x_2} \{\max\{\vartheta_1(x_1), \vartheta_2(x_2)\}\}$$

$$\leq \inf_{x \leq y \leq y_1 y_2} \{\max\{\vartheta_1(y_1), \vartheta_2(y_2)\}\}$$

$$= \inf_{y \leq y_1 y_2} \{\max\{\vartheta_1(y_1), \vartheta_2(y_2)\}\}$$

$$= (\vartheta_1 \circ \vartheta_2)(y)$$

Hence $P_1 \circ P_2$ is a Pythagorean fuzzy ideal of S.

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