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A Connection Between Knot Theory and Linear Algebra

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Abstract:
In knot theory literature, a lot of references treated the topic of computing the Alexander polynomial and proved that the Alexander polynomial is a knot invariant. In this paper, we concentrate on an unconventional connection between knot theory and linear algebra in computing the Alexander polynomial and introduce an approach depending linear algebra tools to show that the Alexander polynomial is a knot invariant. Also, we connect this approach to the way of Fox coloring to suggest another way to prove that the 3-coloring is a knot invariant and confirm what is already known about the 3-coloring of the trefoil and the figure eight knots..

Keywords: Knot Theory, Linear Algebra.

1. Introduction

Alexander polynomial is the first polynomial knot invariant discovered. It is a well understood classical knot invariant discovered by J. W. Alexander in 1927. There are many ways to compute the Alexander polynomial, involving algebraic techniques and others have more geometric approaches such as [1], [2], [4], [6],... In this paper, we will see connections between knot theory and linear algebra regarding two knot invariants, the Alexander polynomial and the *3*-coloring. To prove that the Alexander polynomial is a knot invariant is not a new topic, it is already proved in several references such as [1], [2], [3], [4], [7]. We claim that we touch a different approach to prove such a well-known property of the Alexander polynomial and suggest a similar proof for the same property of the *3*-coloring. Also, we treated the *3*-coloring invariant from the linear algebra point of view depending on the Fox coloring.

We organized this paper as follows: In section *2*, we give a linear algebra approach of computing the Alexander polynomial of a knot and apply it on the trefoil knot. In section *3*, we prepare the reader to accept that playing in the given oriented diagram of a knot in a specific way may change the form of the corresponding homogeneous system of equations that can be obtained from a knot diagram but does not change its space of solutions and then the Alexander polynomial of the given knot is unchanged. In section *4*, we put an approach to prove that the Alexander polynomial is a knot invariant. In section *5*, we connect the previous approach to the Fox coloring way to suggest a proof of the assumption that the *3*-coloring is a knot invariant and give a way to check if a given knot diagram is *3*-coloring or not. Then we apply this approach on the trefoil and the figure eight knots depending on the system obtained in section *2*.

2. The Alexander polynomial

Given an oriented diagram of a knot K of n crossings. We assign the variables: $x_1, x_2, \dots, x_n$ to its arcs. At any crossing, we see three arcs in the position as in Figure *2.1*.

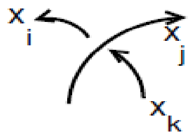


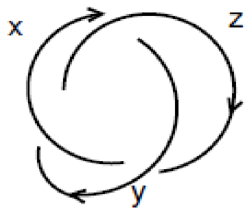
Fig. 2.1: A crossing

For such crossing, we write the equation : $tx_i + (1-t)x_i = x_k, t \in \mathbb{C}$, then we construct the following homogeneous system of n linear equations in the variables: $x_1, x_2, \dots, x_n$.

$$tx_i + (1-t)x_i = x_k, i, k = \overline{1, n}, t \in \mathbb{C} \qquad (2.1)$$

This system can be also written in the matrix form, $AX = \vec{0}$. Because of this way of construction, the entries of any row of A are $t, 1-t, -1,$ and some possible zeros then A is singular for every t. In the 1920s, Alexander showed that all $(n-1)$-minors are equal up to multiplication by the monomial $\pm t^k, k \in \mathbb{Z}$. So, these minors which are Laurent polynomials in t can be normalized for some consideration by dividing a minor by $\pm t^k$. Any one of these polynomials is called an Alexander polynomial of the Knot K, denoted by $\nabla_K(t)$ with $\nabla_O(t) = 1$. One often fixes a particular form as the first Alexander's choice of the normalization so that the term of the polynomial of the least degree of t is a positive constant.

A simple example is now introduced to visualize the process of computing the Alexander polynomial using this approach. let us use this approach to compute the Alexander polynomial of the trefoil.



The corresponding linear system is

$$tz + (1-t)x = z$$
$$tx + (1-t)y = z$$
$$ty + (1-t)z = x$$

Which can be rewritten as:

$$\begin{bmatrix} 1-t & -1 & t \\ t & 1-t & -1 \\ -1 & t & 1-t \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Take any 2 $-$minor, say $\begin{vmatrix} t & -1 \\ -1 & 1-t \end{vmatrix} = t - t^2 + 1$, or alternatively, if we take $\begin{vmatrix} t & -1 \\ -1 & 1-t \end{vmatrix} = t - t^2 - 1$, which can be normalized to the previous one.

3. A useful definition

We claim that getting a homogeneous linear system of a knot and preserving its solutions from the change guarantees that the Alexander polynomial of the same knot will not change up to multiplication of $\pm t^k$, for some integer k. A very close claim is mentioned and proved in [2]. As in the basics of linear algebra, where we can check if two given matrices of the same size are row equivalent or not, the following definition from [7] helps us to extend the concept of equivalency regarding the solutions of the corresponding homogeneous linear systems of equations.

Definition. Two matrices are equivalent if one can be obtained from the other by a finite sequence of the following operations and their inverse operations:

1. Permute two rows(columns).

2. Adjoin a new row of zeros.

3. Add a multiple of a row(column) to any row(column).

4. Multiply a row(column) by a unit.

5. Replace the $m \times n$ matrix A by the extended $(m + 1) \times (n + 1)$-matrix:

$$\begin{bmatrix} A & & & 0 \\ & & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

4. A linear algebra approach to the proof of an old theorem

Theorem. $\nabla_K(t)$ is a knot invariant.

Proof. To show that $\nabla_K(t)$ is a knot invariant it is sufficient to show that it does not change under Reidemeister moves. For the first Reidemeister move, a twist is introduced to any arc of the given oriented knot diagram, say, x_i as in Figure *4.1*.

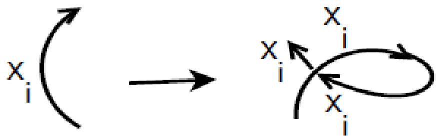


Fig. 4.1: The first Reidemeister move

Keeping the whole diagram fixed except this move at the arc x_i. To see the contribution of this move to the system *2.1* of the given diagram, the new twist adds the equation: $(1 - t)x_i + tx_i = x_i$, which does not affect the solutions of the system. It is typically, as one adjoined a row of zeros to the coefficients matrix of the system $AX = \vec{0}$. Then $\nabla_K(t)$ is unchanged under the first Reidemeister move. Note. The other type of the twist will add the same equation to the system *2.1*. For the second Reidemeister move, two new crossings are introduced to any two arcs say, x_i and x_k as in Figure *4.2*.

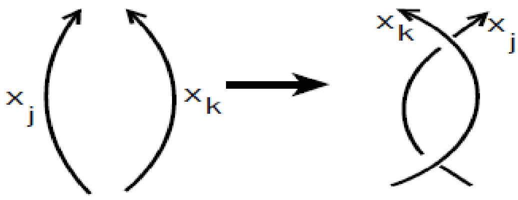


Fig. 4.2: The second Reidemeister move

Again, keeping the diagram fixed except this move at the arcs x_j and x_k. The contribution of this move to the system *2.1* is given by adding two equations of the form: $(1 - t)x_i + tx_i = x_i$. Each one of these two equations can be rewritten in the form: $(1 - t)x_i + tx_i = 0$, which do not affect the solution of the system *2.1* as we mentioned in Section *3*. So, $\nabla_K(t)$ is unchanged under the second Reidemeister move. Note. The other position of the second Reidemeister move will add the same equations to the system *2.1*. For the third Reidemeister move, the contribution of the three crossings before and after the move are the same, so $\nabla_K(t)$ is unchanged under the third Reidemeister move. Finally, we point out that we used the first and the second Reidemeister moves as introducing a twist and two new crossings, respectively, not removing them. Now we will treat the case of removing. Since we have proved that after introducing a twist or two crossings, the Alexander polynomial $\nabla_K(t)$ does not change then if we take the last diagram after introducing a twist or two new crossings as the original stage and we remove a twist or two crossings by the first or the second Reidemeister move, respectively, we return to the main diagram which has the same Alexander polynomial, $\nabla_K(t)$.

5. A connection with Fox coloring

In [6], Fox defined tricolorability or 3-coloring of a knot diagram as a coloring of one of its diagrams with three colors such that: (i) At least two different colors are used. (ii) At each crossing, the three arcs are of the same color or all are of distinct colors.

Note. This definition does not require the given diagram to be oriented. Furthermore, it is proved that 3-coloring is a knot invariant. It seems that 3-coloring is not numerical knot invariant, but this view may change when we connect 3-coloring with the system of homogeneous linear equations 2.1, where 3-coloring becomes a linear algebra problem ignoring the orientation which is necessary for the system 2.1.

Recall one equation from the system 2.1 say, $tx_i + (1-t)x_i = x_k$. If we put $t = -1$ we get $2x_i - x_i - x_k = 0$. This equation is called Fox coloring condition of the crossing with an upper arc x_j and two lower arcs x_i and x_k. So, we can assign any given knot diagram with a new system of linear homogeneous equations over $\mathbb{Z}$ called the coloring system corresponds to the system 2.1 with $t = -1$, which is always consistent. If we consider this new system of homogeneous linear equations over $\mathbb{Z}_3$, the *3*-coloring problem becomes the problem of the existence or non-existence of nontrivial solutions of the system $AX = \vec{0}$, where A is any 3×3-matrix obtained by omitting some rows and the same number of columns from the coefficients matrix of the coloring system.

Now why not? We can collect what we did in the proof of the main theorem in section *4* with what we mentioned about the *3*-coloring property that it depends on the solutions of a system obtained from the system 2.1 of the given knot diagram to state that Reidemeister moves do not change the solutions space of the system 2.1 and then *3*-coloring of the given diagram. This can be considered as a proof of a similar theorem to the theorem in section *4* that is *3*-coloring is a knot invariant.

Applications.

5.1 The trefoil: We can return to the obtained system for the trefoil in section *2* to solve the *3*-coloring system that is

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the system has non-trivial solutions then the trefoil is *3*-coloring.

5.2 The figure eight: Given any oriented diagram of the figure eight, we can easily

check that its coloring system is

$$\begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since any 3×3-matrix A obtained from the coefficient matrix of the last system by omitting one row and one column is non-singular then the system $AX = \vec{0}$ has only the trivial solution for each A. This confirms that the figure eight is not 3-coloring.

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