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On new algebraic structures approach towards $(\varsigma_1, \varsigma_2)$ intuitionistic Q fuzzy ideals of an ordered ternary semigroups

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Abstract

Introducing the concept of ς_1, ς_2-intuitionistic Q fuzzy ternary subsemigroup (IQFTSS), we explore some of the properties of these ordered ternary semigroups, including intuitionistic Q fuzzy left ideal (IQFLI), intuitionistic Q fuzzy right ideal (IQFRI), intuitionistic Q fuzzy lateral ideal (IQFLATI), intuitionistic Q fuzzy ideal (IQFI), and intuitionistic Q fuzzy bi-ideal (IQFBI). We provide a novel extension of IQFI over ternary semigroups $\mathcal{M}$: ς_1, ς_2-IQFI. A non-empty subset $\aleph_{\varsigma_1}$ is a $(\varsigma_1, \varsigma_2)$-IQFTSS (IQFLI, IQFRI, IQFLATI, IQFBI) of $\mathcal{M}$. Then the lower level set $\eth_{\varsigma_1}$ is an TSS $(TLI, TRI, TLATI, TBI)$ of $\mathcal{M}$, where $\eth_{\varsigma_1} = \{\ell \in \mathcal{M} | \eth(\ell, q) > \varsigma_1\}$ and $\mho_{\varsigma_1} = \{\ell \in \mathcal{M} | \eth(\ell, q) < \varsigma_1\}$. A subset $\aleph = [\eth, \mho]$ is a $(\varsigma_1, \varsigma_2) - IQFTSS[IQFLI, IQFRI, IQFLATI, IQFBI]$ of $\mathcal{M}$ if and only if each non-empty level subset $\aleph_t$ is a TSS $[TLI, TRI, TLATI, TBI]$ of $\mathcal{M}$ for all $t \in (\varsigma_1, \varsigma_2]$. We present a few instances to demonstrate our findings.

Keywords: Ordered ternary semigroups, intuitionistic Q fuzzy ideals, Q fuzzy bi-ideals, $(\varsigma_1, \varsigma_2)$ Q fuzzy bi-ideals.

1 Introduction

Triplex structures are ternary algebraic systems that were first conceptualized in 1932 by D. H. Lehmer.[1] In 1934, Vandiver invented the concept of semiring. Hestenes[2] developed the idea of ternary algebra in 1962, using matrices and linear transformation as examples. Lister discussed this algebraic system a ternary ring in 1971 after characterizing those additive subgroups of rings that are closed under the triple ring product. Algebraic structures are used extensively in a variety of fields, including discrete event dynamical systems, automata theory, mathematical modeling of quantum physics, information sciences, computer sciences, coding theory, topological spaces, combinatorics, functional analysis, graph theory, Euclidean geometry, probability theory, and commutative and non-commutative ring theory. Fuzzy set (FS) theory was developed by Zadeh[3] and is most effective in managing ambiguity and uncertainty. If an element in an FS has a single value inside the interval, it is regarded as a member. The degree of non-membership may not always equal one minus the degree of membership, though, as resistance might occur in real-world circumstances. An increasing number of hybrid fuzzy models are being created as FS theory develops swiftly. The uncertainties have led to the development of several uncertain theories, such as Pythagorean FS (PFS),[5] intuitionistic FS (IFS),[4] and FS.[3] MG sets, or sets with grades between 0 and 1, make up an FS. Despite the claim made by Atanassov[4] that non-membership grades (NMG) may only have a value of 1, IFS is categorized as MG. The total of MGs and NMGs may occasionally reach 1 throughout a decision-making process. Yager[5] used PFS logic to develop the generalized MG and NMG logic, which has a value not exceeding 1 and is determined by the square of the MGs and NMGs. As the neutral state is neither positive nor negative, these theories are unable to describe it.

An intuitionistic fuzzy normal subbisemiring of bisemiring was recently presented by Palaniku-mar et al.[6] The notion of bisemiring was created by Palanikumar et al.[7] utilizing bipolar-valued neutrosophic normal sets. Bi-ideals on ordered semigroups were the subject of dis-cussion by Hila et al.[8] The novel theories based on prime ideals and prime radicals of ternary semirings were presented by Dutta T.K. et al.[9] Different prime and semiprime bi-ideals of the rings were covered by Palanikumar et al.[11] The several ideals of semigroups, semirings, and ternary semirings were covered by Palanikumar et al.[12–16] We explore that ordered ternary semigroups' $(\varsigma_1, \varsigma_2)$-intuitionistic Q fuzzy ideals, and we illustrate some of their features.

2 Preliminaries

Definition 2.1. An ordered semigroup $\mathcal{M}$ together with an order relation $\leq$ such that $x \leq y$ implies $xz \leq yz$ and $zx \leq zy$ for all $x, y, z \in \mathcal{M}$.

Definition 2.2. Let $\Im$ and $\Im_1$ be two non empty subsets of $\mathcal{M}$. We denote

1. $(\Im] = \{t \in \mathcal{M} \,|\, t \leq h \, for\, some\, h \in \Im\}$,
2. $\Im\Im_1 = \{ab : a \in \Im, \, b \in \Im_1\}$,
3. $\Im_x = \{(y, z) \in \mathcal{M} \times \mathcal{M} \,|\, x \leq yz\}$.

Definition 2.3. Let $\aleph$ be a fuzzy subset of $\mathcal{M}$ and $t \in [0, 1]$. The set $\aleph_t = \{x \in \mathcal{M} | \aleph(x) \geq t\}$ is called the level subset of $\aleph$. Clearly $\aleph_t \subseteq \aleph_s$ whenever $t \geq s$.

Definition 2.4. A fuzzy $\aleph$ subset of $\mathcal{M}$ is called a fuzzy bi-ideal of $\mathcal{M}$ if

1. $a \leq b \implies \aleph(a) \geq \aleph(b)$ and $\aleph(xy) \geq \min\{\aleph(x), \aleph(y)\}$
2. $\aleph(xyz) \geq \min\{\aleph(x), \aleph(z)\}$ for all $x, y, z \in \mathcal{M}$.

Definition 2.5. A fuzzy subset $\hbar$ of an ordered semigroup $\mathcal{M}$ is called a fuzzy right (resp. left) ideal of $\mathcal{M}$ if

1. $x \leq y \Rightarrow \hbar(x) \geq \hbar(y)$ for all $x, y \in \mathcal{M}$,
2. $\hbar(xy) \geq \hbar(x)$ (resp. $\hbar(xy) \geq \hbar(y)$) for all $x, y \in \mathcal{M}$,

3. A fuzzy subset $\hbar$ of an ordered semigroup $\mathcal{M}$ is called a fuzzy ideal of $\mathcal{M}$, if it is both fuzzy left ideal and fuzzy right ideal.

Definition 2.6. Let A be a fuzzy set,if χ_A is the characteristic function of A, then $(\chi_A)^{\varrho}_{\sigma}$ is defined as

$$(\chi_A)^{\varrho}_{\sigma}(x) := \begin{cases} \varrho \, if \; x \in A, \\ \sigma \, if \; x \notin A. \end{cases}$$

Definition 2.7. [9] A non-empty set $\mathcal{M}$ together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition is said to be a ternary semiring if $(\mathcal{M}, +)$ is a commutative semigroup and ternary multiplication satisfies the following:

1. $(abc)de = a(bcd)e = ab(cde)$,
2. $(a + b)cd = acd + bcd$,
3. $a(b + c)d = abd + acd$,
4. $ab(c + d) = abc + abd$ for all $a, b, c, d, e \in \mathcal{M}$

Corollary 2.8. *If $\mathcal{M}$ is regular if and only if every RI $\Im$, every LATI $\Im_1$ and every LI $\Im_2$ of $\mathcal{M}$, then* $(\Im \cap \Im_1 \cap \Im_2] = (\Im \cdot \Im_1 \cdot \Im_2]$.

3 $(\varsigma_1, \varsigma_2)$ intuitionistic Q fuzzy ideals

In this section $\mathcal{M}$ denotes the ordered ternary semigroup. In what follows that $(\varsigma_1, \varsigma_2) \in [0, 1]$ be such that $0 \leq \varsigma_1 < \varsigma_2 \leq 1$ both $(\varsigma_1, \varsigma_2)$ are arbitrary fixed.

Definition 3.1. A intuitionistic fuzzy subset $A = [\eth_A, \mho_A]$ of $\mathcal{M}$ and Q is a any non-empty set, (the pair (A, Q) intuitionistic Q fuzzy set) is is called a $(\varsigma_1, \varsigma_2)$ IQTFSS of $\mathcal{M}$ if it satisfies the following conditions:

1. $\ell \leq \flat \Rightarrow \eth(\ell) \geq \eth(\flat)$.
2. $\max\{\eth(\ell\hbar\flat, q), \varsigma_1\} \geq \min\{\eth(\ell, q), \eth(\hbar, q), \eth(\flat, q), \varsigma_2\}$
3. $\min\{\mho(\ell\hbar\flat, q), \varsigma_1\} \leq \max\{\mho(\ell, q), \mho(\hbar, q), \mho(\flat, q), \varsigma_2\}$ for all $\ell, \hbar, \flat \in \mathcal{M}$ and $q \in Q$.

Example 3.2. Let $\mathcal{M} = \{\flat_1, \flat_2, \flat_3, \flat_4\}$ with the following Cayley table:

$\cdot$	$\flat_1$	$\flat_2$	$\flat_3$	$\flat_4$
$\flat_1$	a	a	a	a
$\flat_2$	a	b	c	d
$\flat_3$	a	c	c	c
$\flat_4$	a	c	c	c

$\cdot$	$\flat_1$	$\flat_2$	$\flat_3$	$\flat_4$
a	$\flat_1$	$\flat_1$	$\flat_1$	$\flat_1$
b	$\flat_1$	$\flat_2$	$\flat_3$	$\flat_4$
c	$\flat_1$	$\flat_3$	$\flat_3$	$\flat_3$
d	$\flat_1$	$\flat_3$	$\flat_3$	$\flat_3$

$\leq := \{(\flat_1,\flat_1),(\flat_1,\flat_2),(\flat_1,\flat_3),(\flat_1,\flat_4),(\flat_2,\flat_2),(\flat_2,\flat_3),(\flat_2,\flat_4),(\flat_3,\flat_3),(\flat_4,\flat_3),(\flat_4,\flat_4)\}$.
Define the mapping $A = [\eth_A, \mho_A] : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to [0,1]$.

$$\eth(\flat,q) = \begin{cases} 0.66 \ \ if \ \flat = \flat_1 \\ 0.46 \ \ if \ \flat = \flat_2 \\ 0.16 \ \ if \ \flat = \flat_3 \\ 0.26 \ \ if \ \flat = \flat_4 \end{cases} \quad \mho(\flat,q) = \begin{cases} 0.34 \ \ if \ \flat = \flat_1 \\ 0.41 \ \ if \ \flat = \flat_2 \\ 0.51 \ \ if \ \flat = \flat_3 \\ 0.46 \ \ if \ \flat = \flat_4 \end{cases}$$

Then A is a $(0.56, 0.71)$ $IQTFSS$ of $\mathcal{M}$.

Definition 3.3. A intuitionistic Q subset A of $\mathcal{M}$ is called a $(\varsigma_1,\varsigma_2)$-IQFBI of $\mathcal{M}$ if it satisfies the following conditions:

1. If $\ell \leq \flat$, then $\eth(\ell) \geq \eth(\flat)$ and $\mho(\ell) \leq \mho(\flat)$,
2. $\max\{\eth(\ell\hbar_1\flat,q),\varsigma_1\} \geq \min\{\eth(\ell,q),\eth(\flat,q),\varsigma_2\}$,
   $\min\{\mho(\ell\hbar_1\flat,q),\varsigma_1\} \leq \max\{\mho(\ell,q),\mho(\flat,q),\varsigma_2\}$,
3. $\max\{\eth(\ell\hbar_1\flat\hbar_2\gamma,q),\varsigma_1\} \geq \min\{\eth(\ell,q),\eth(\gamma,q),\varsigma_2\}$,
   $\min\{\mho(\ell\hbar_1\flat\hbar_2\gamma,q),\varsigma_1\} \leq \max\{\mho(\ell,q),\mho(\gamma,q),\varsigma_2\}$, for $\ell,\flat,\gamma,\hbar_1,\hbar_2 \in \mathcal{M}$ and $q \in Q$.

Example 3.4. Let $\mathcal{M} = \{\flat_1,\flat_2,\flat_3,\flat_4\}$ with the following Cayley tables:

$\cdot$	$\flat_1$	$\flat_2$	$\flat_3$	$\flat_4$
$\flat_1$	a	a	a	a
$\flat_2$	a	b	c	d
$\flat_3$	a	c	c	c
$\flat_4$	a	c	c	c

$\cdot$	$\flat_1$	$\flat_2$	$\flat_3$	$\flat_4$
a	$\flat_1$	$\flat_1$	$\flat_1$	$\flat_1$
b	$\flat_1$	$\flat_2$	$\flat_3$	$\flat_4$
c	$\flat_1$	$\flat_3$	$\flat_3$	$\flat_3$
d	$\flat_1$	$\flat_3$	$\flat_3$	$\flat_3$

$\cdot$	$\flat_1$	$\flat_2$	$\flat_3$	$\flat_4$
a	a	a	a	a
b	a	b	c	d
c	a	c	c	c
d	a	b	c	d

$\cdot$	$\flat_1$	$\flat_2$	$\flat_3$	$\flat_4$
a	$\flat_1$	$\flat_1$	$\flat_1$	$\flat_1$
b	$\flat_1$	$\flat_2$	$\flat_3$	$\flat_4$
c	$\flat_1$	$\flat_3$	$\flat_3$	$\flat_3$
d	$\flat_1$	$\flat_2$	$\flat_3$	$\flat_4$

$\leq := \{(\flat_1,\flat_1),(\flat_1,\flat_2),(\flat_1,\flat_3),(\flat_1,\flat_4),(\flat_2,\flat_2),(\flat_2,\flat_3),(\flat_2,\flat_4),(\flat_3,\flat_3),(\flat_4,\flat_3),(\flat_4,\flat_4)\}$. The mapping $\aleph = [\eth,\mho] : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to [0,1]$

$$\eth(\flat,q) = \begin{cases} 0.57 \ \ if \ \flat = \flat_1 \\ 0.38 \ \ if \ \flat = \flat_2 \\ 0.10 \ \ if \ \flat = \flat_3 \\ 0.19 \ \ if \ \flat = \flat_4 \end{cases} \quad \mho(\flat,q) = \begin{cases} 0.40 \ \ if \ \flat = \flat_1 \\ 0.45 \ \ if \ \flat = \flat_2 \\ 0.55 \ \ if \ \flat = \flat_3 \\ 0.50 \ \ if \ \flat = \flat_4 \end{cases}$$

Then $\aleph$ is a $(0.43, 0.58)$ IQFBI of $\mathcal{M}$.

Theorem 3.5. *A non-empty subset $\aleph_{\varsigma_1}$ is a $\eth_{\varsigma_1}$ is a $(\varsigma_1, \varsigma_2)$-IQTFSS (IQFLI,IQFLATI, IQFRI, IQFBI, (1,2)-ideal) of $\mathcal{M}$. Then the lower level set $\eth_{\varsigma_1}$ is an ternary subsemigroup (TLI,TLATI, TRI, TBI) of $\mathcal{M}$, where $\eth_{\varsigma_1} = \{\ell \in \mathcal{M} | \eth(\ell, q) > \varsigma_1\}$ and $\mho_{\varsigma_1} = \{\ell \in \mathcal{M} | \eth(\ell, q) < \varsigma_1\}$.*

Proof. Suppose that $\aleph_{\varsigma_1}$ is a $(\varsigma_1, \varsigma_2)$-IQTFSS of $\mathcal{M}$. Let $\ell, \hbar, \flat \in \mathcal{M}$ such that $\ell, \hbar, \flat \in \eth_{\varsigma_1}$. Then $\eth(\ell, q) > \varsigma_1, \eth(\hbar, q) > \varsigma_1, \eth(\flat, q) > \varsigma_1$.
Therefore $\max\{\eth(\ell\hbar\flat, q), \varsigma_1\} \geq \min\{\eth(\ell, q), \eth(\hbar, q), \eth(\flat, q), \varsigma_2\} > \min\{\varsigma_1, \varsigma_1, \varsigma_1, \varsigma_2\} = \varsigma_1$. Hence $\eth(\ell\hbar\flat, q) > \varsigma_1$. It shows that $\ell\hbar\flat \in \eth_{\varsigma_1}$. Therefore $\eth_{\varsigma_1}$ is a ternary subsemigroup of $\mathcal{M}$. Let $\ell, \hbar, \flat \in \mathcal{M}$ such that $\ell, \hbar, \flat \in \mho_{\varsigma_1}$. Then $\mho(\ell, q) < \varsigma_1, \mho(\hbar, q) < \varsigma_1 \mho(\flat, q) < \varsigma_1$. Therefore $\min\{\mho(\ell\hbar\flat, q), \varsigma_1\} \leq \max\{\mho(\ell, q), \mho(\hbar, q), \mho(\flat, q), \varsigma_2\} < \max\{\varsigma_1, \varsigma_1, \varsigma_1, \varsigma_2\} = \varsigma_2$. Hence $\mho(\ell\hbar\flat, q) < \varsigma_1$. It shows that $\ell\hbar\flat \in \mho_{\varsigma_1}$. Therefore $\mho_{\varsigma_1}$ is a ternary subsemigroup of $\mathcal{M}$. Therefore $\aleph_{\varsigma_1}$ is a ternary subsemigroup of $\mathcal{M}$.

Theorem 3.6. *A non-empty subset $\Im$ of $\mathcal{M}$ is a SS $[TLI, TLATI, TRI, TBI]$ of $\mathcal{M}$ if and only if the intuitionistic Q fuzzy subset $\aleph = [\eth, \mho]$ of $\mathcal{M}$ is defined as*

$$\eth(\ell, q) = \begin{cases} \geq \varsigma_2 \; for\, all \;\; \ell \in (\Im] \\ \varsigma_1 \; for\, all \;\; \ell \notin (\Im] \end{cases} \quad \mho(\ell, q) = \begin{cases} \leq \varsigma_2 \; for\, all \;\; \ell \in (\Im] \\ \varsigma_1 \; for\, all \;\; \ell \notin (\Im] \end{cases}$$

is a $(\varsigma_1, \varsigma_2) IQTFSS[IQFLI, IQFLATI, IQFRI, IQFBI]$ of $\mathcal{M}$.

Proof. Suppose that $\Im$ is an ternary subsemigroup of $\mathcal{M}$. Let $\ell, \hbar, \flat \in \mathcal{M}$ be such that $\ell, \hbar, \flat \in (\Im]$ then $\ell\hbar\flat \in (\Im]$. Hence $\eth(\ell\hbar\flat, q) \geq \varsigma_2$ and $\mho(\ell\hbar\flat, q) \leq \varsigma_2$. Thus, $\max\{\eth(\ell\hbar\flat, q), \varsigma_1\} \geq \varsigma_2 = \min\{\eth(\ell, q), \eth(\hbar, q), \eth(\flat, q), \varsigma_2\}$ and $\min\{\mho(\ell\hbar\flat, q), \varsigma_1\} \leq \varsigma_2 = \max\{\mho(\ell, q), \mho(\hbar, q), \mho(\flat, q), \varsigma_2\}$. If $\ell \notin (\Im]$ or $\hbar \notin (\Im]$ or $\flat \notin (\Im]$,
then $\min\{\eth(\ell, q), \eth(\hbar, q), \eth(\flat, q), \varsigma_2\} = \varsigma_1$ and $\max\{\mho(\ell, q), \mho(\hbar, q), \mho(\flat, q), \varsigma_2\} = \varsigma_2$.
That is $\max\{\eth(\ell\hbar\flat, q), \varsigma_1\} \geq \min\{\eth(\ell, q), \eth(\hbar, q), \eth(\flat, q), \varsigma_2\}$ and
$\min\{\mho(\ell\hbar\flat, q), \varsigma_1\} \leq \max\{\mho(\ell, q), \mho(\hbar, q), \mho(\flat, q), \varsigma_2\}$.
Therefore $\aleph$ is a $(\varsigma_1, \varsigma_2)$ IQTFSS of $\mathcal{M}$.
Conversely assume that $\aleph = [\eth, \mho]$ is a $(\varsigma_1, \varsigma_2)$-IQTFSS of $\mathcal{M}$. Let $\ell\hbar\flat \in (\Im]$. Then $\eth(\ell, q) \geq \varsigma_2, \eth(\hbar, q) \geq \varsigma_2, \eth(\flat, q) \geq \varsigma_2$ and $\mho(\ell, q) \leq \varsigma_2, \mho(\hbar, q) \leq \varsigma_2, \mho(\flat, q) \leq \varsigma_2$. Now $\aleph = [\eth, \mho]$ is a $(\varsigma_1, \varsigma_2)$-IQTFSS of $\mathcal{M}$.
Therefore $\max\{\eth(\ell\hbar\flat, q), \varsigma_1\} \geq \min\{\eth(\ell, q), \eth(\hbar, q), \eth(\flat, q), \varsigma_2\} \geq \min\{\varsigma_2, \varsigma_2, \varsigma_2, \varsigma_2\} = \varsigma_2$ and
$\min\{\mho(\ell\hbar\flat, q), \varsigma_1\} \leq \max\{\mho(\ell, q), \mho(\hbar, q), \mho(\flat, q), \varsigma_2\} \leq \max\{\varsigma_2, \varsigma_2, \varsigma_2, \varsigma_2\} = \varsigma_2$. It follows that $\ell\hbar\flat \in (\Im]$. Therefore $\Im$ is a ternary subsemigroup of $\mathcal{M}$.

Theorem 3.7. *A subset $\aleph = [\eth, \mho]$ is a $(\varsigma_1, \varsigma_2)-IQTFSS[IQFLI, IQFLATI, IQFRI, IQFBI]$ of $\mathcal{M}$ if and only if each non-empty level subset $\aleph_t$ is a ternary subsemigroup [TLI,TLATI,TRI,TBI] of $\mathcal{M}$ for all $t \in (\varsigma_1, \varsigma_2]$.*

Proof. Assume that $\aleph_t$ is a ternary subsemigroup of $\mathcal{M}$ for each $t \in [0, 1]$.
Let $t = \min\{\eth(\ell_1, q), \eth(\ell_2, q), \eth(\ell_3, q)\}$. Then $\ell_1, \ell_2, \ell_3 \in \eth_t$ for each $\ell_1, \ell_2, \ell_3 \in \mathcal{M}$.
Thus $\max\{\eth(\ell\hbar\flat, q), \varsigma_1\} \geq t = \min\{\eth(\ell_1, q), \eth(\ell_2, q), \eth(\ell_3, q), \varsigma_2\}$.
Let $t = \max\{\mho(\ell_1, q), \mho(\ell_2, q), \mho(\ell_3, q)\}$.
Then $\ell_1, \ell_2, \ell_3 \in \mho_t$ for each $\ell_1, \ell_2, \ell_3 \in \mathcal{M}$.
Thus $\min\{\mho(\ell\hbar\flat, q), \varsigma_1\} \leq t = \max\{\mho(\ell_1, q), \mho(\ell_2, q), \mho(\ell_3, q), \varsigma_2\}$.

This shows that $\aleph^t$ is IQTFSS of $\mathcal{M}$.

Conversely, assume that $\aleph^t$ is a IQTFSS of $\mathcal{M}$. For each $t \in [0, 1]$ and $\ell_1, \ell_2, \ell_3 \in \eth_t$. We have $\eth(\ell_1, q) \geq t$, $\eth(\ell_2, q) \geq t$, $\eth(\ell_3, q) \geq t$. Since $\eth$ is a ternary subsemigroup of $\mathcal{M}$, $\max\{\eth(\ell_1\ell_2\ell_3, q), \varsigma_1\} \geq \min\{\eth(\ell_1, q), \eth(\ell_2, q), \eth(\ell_3, q), \varsigma_2\} \geq t$. This implies that $\ell_1\ell_2\ell_3 \in \eth_t$. We have $\mho(\ell_1, q) \leq t$, $\mho(\ell_2, q) \leq t$, $\mho(\ell_3, q) \leq t$. Since $\mho$ is a ternary subsemigroup of $\mathcal{M}$,$\min\{\mho(\ell_1\ell_2\ell_3, q), \varsigma_1\} \leq \max\{\mho(\ell_1, q), \mho(\ell_2, q), \mho(\ell_3, q), \varsigma_2\} \leq t$. This implies that$\ell_1\ell_2\ell_3 \in \mho_t$. Therefore $\aleph_t$ is a ternary subsemigroup of $\mathcal{M}$ for each $t \in (\varsigma_1, \varsigma_2]$. Similar proofs holds.

Example 3.8. Every IQTFSS $\aleph$ of $\mathcal{M}$ is a $(\varsigma_1, \varsigma_2)$-IQTFSS of $\mathcal{M}$, but converse is not true.

For the Example 3.2, we define subset $\aleph$ by

$$\eth(\flat, q) = \begin{cases} 0.46 & if\ \flat = \flat_1 \\ 0.39 & if\ \flat = \flat_2 \\ 0.29 & if\ \flat = \flat_3 \\ 0.34 & if\ \flat = \flat_4 \end{cases} \quad \mho(\flat, q) = \begin{cases} 0.31 & if\ \flat = \flat_1 \\ 0.36 & if\ \flat = \flat_2 \\ 0.46 & if\ \flat = \flat_3 \\ 0.41 & if\ \flat = \flat_4 \end{cases}$$

Then $\aleph$ is a $(0.37, 0.51)$-IQTFSS of $\mathcal{M}$, but not a IQTFSS.
Since $\eth(\flat_4\hbar\flat_4, q) = 0.29 \ngeq \min\{\eth(\flat_4, q), \eth(\flat_4, q)\} = 0.34$ and
$\mho(\flat_4\hbar\flat_4, q) = 0.46 \nleq \max\{\mho(\flat_4, q), \mho(\flat_4, q)\} = 0.41$.

Example 3.9. Every IQFBI $\aleph = [\eth, \mho]$ of $\mathcal{M}$ is a $(\varsigma_1, \varsigma_2)$-IQFBI of $\mathcal{M}$, but converse need not be true by the following Example.

For the Example 3.4, we define subset $\aleph$ by,

$$\eth(\flat, q) = \begin{cases} 0.57 & if\ \flat = \flat_1 \\ 0.38 & if\ \flat = \flat_2 \\ 0.10 & if\ \flat = \flat_3 \\ 0.19 & if\ \flat = \flat_4 \end{cases} \quad \mho(\flat, q) = \begin{cases} 0.23 & if\ \flat = \flat_1 \\ 0.38 & if\ \flat = \flat_2 \\ 0.48 & if\ \flat = \flat_3 \\ 0.43 & if\ \flat = \flat_4 \end{cases}$$

Then $\aleph$ is a $(0.33, 0.58)$IQFBI, but not a IQFBI. Since $\eth(\flat_4\hbar_1\flat_4\hbar_2\flat_4, q) = \eth(\flat_3, q) = 0.10 \ngeq \min\{\eth(\flat_4, q), \eth(\flat_4, q)\} = 0.19$ and
$\mho(\flat_4\hbar_1\flat_4\hbar_2\flat_4, q) = \mho(\flat_3, q) = 0.48 \nleq \max\{\eth(\flat_4, q), \eth(\flat_4, q)\} = 0.43$.

Definition 3.10. If $\chi_{\Im}$ is the characteristic function of $\Im$, then $(\chi_{\Im})_{\varsigma_1}^{\varsigma_2}$ is defined as

$$(\chi_{\Im}^{\mathscr{T}})_{\varsigma_1}^{\varsigma_2}(\ell, q) = \begin{cases} \varsigma_2\, if\ \ell \in (\Im] \\ \varsigma_1\, if\ \ell \notin (\Im] \end{cases} \quad (\chi_{\Im}^{\mathscr{F}})_{\varsigma_1}^{\varsigma_2}(\ell, q) = \begin{cases} \varsigma_1\, if\ \ell \in (\Im] \\ \varsigma_2\, if\ \ell \notin (\Im] \end{cases}$$

Theorem 3.11. *A non empty subset $\Im$ of $\mathcal{M}$ is a ternary subsemigroup $[TLI, TLATI, TRI, TBI]$ of $\mathcal{M}$ if and only if subset $\chi_{(\Im]}$ is a $(\varsigma_1, \varsigma_2)$-$IQTFSS[IQFLI, IQFLATI, IQFRI, IQFBI]$ of $\mathcal{M}$.*

Proof. Assume that $\Im$ is a ternary subsemigroup of $\mathcal{M}$. Then $\chi_{(\Im]}$ is a IQTFSS of $\mathcal{M}$ and hence $\chi_{(\Im]}$ is an $(\varsigma_1, \varsigma_2)$-IQTFSS of $\mathcal{M}$.

Conversely, Let $\chi_{(\Im]}$ is an $(\varsigma_1, \varsigma_2)$-IQTFSS of $\mathcal{M}$. Let $\ell, \hbar, \flat \in \mathcal{M}$ be such that $\ell, \hbar, \flat \in (\Im]$. Then $\chi^{\mathscr{T}}_{(\Im]}(\ell, q) = \varsigma_2, \chi^{\mathscr{T}}_{(\Im]}(\hbar, q) = \varsigma_2, \chi^{\mathscr{T}}_{(\Im]}(\flat, q) = \varsigma_2$. Since $\chi^{\mathscr{T}}_{(\Im]}$ is a $(\varsigma_1, \varsigma_2)$IQTFSS. Consider

$$\begin{aligned}\max\{\chi^{\mathscr{T}}_{(\Im]}(\ell\hbar\flat, q), \varsigma_1\} &\geq \min\{\chi^{\mathscr{T}}_{(\Im]}(\ell, q), \chi^{\mathscr{T}}_{(\Im]}(\hbar, q), \chi^{\mathscr{T}}_{(\Im]}(\flat, q), \varsigma_2\}\\ &= \min\{\varsigma_2, \varsigma_2, \varsigma_2, \varsigma_2\}\\ &= \varsigma_2\end{aligned}$$

as $\varsigma_1 < \varsigma_2$, this implies that $\chi^{\mathscr{T}}_{(\Im]}(\ell\hbar\flat, q) \geq \varsigma_2$. Thus $\ell\hbar\flat \in (\Im]$. Thus $\ell\hbar\flat \in (\Im]$.

Let $\ell, \hbar, \flat \in \mathcal{M}$ be such that $\ell, \hbar, \flat \in (\Im]$. Then $\chi^{\mathscr{F}}_{(\Im]}(\ell, q) = \varsigma_1, \chi^{\mathscr{F}}_{(\Im]}(\hbar, q) = \varsigma_1, \chi^{\mathscr{F}}_{(\Im]}(\flat, q) = \varsigma_1$. Since $\chi^{\mathscr{F}}_{(\Im]}$ is a $(\varsigma_1, \varsigma_2)$IQTFSS. Consider

$$\begin{aligned}\min\{\chi^{\mathscr{F}}_{(\Im]}(\ell\hbar\flat, q), \varsigma_1\} &\leq \max\{\chi^{\mathscr{F}}_{(\Im]}(\ell, q), \chi^{\mathscr{F}}_{(\Im]}(\hbar, q), \chi^{\mathscr{F}}_{(\Im]}(\flat, q), \varsigma_2\}\\ &= \max\{\varsigma_1, \varsigma_1, \varsigma_1, \varsigma_2\}\\ &= \varsigma_2\end{aligned}$$

as $\varsigma_1 < \varsigma_2$, this implies that $\chi^{\mathscr{F}}_{(\Im]}(\ell\hbar\flat, q) \leq \varsigma_1$. Thus $\ell\hbar\flat \in (\Im]$. Thus $\ell\hbar\flat \in (\Im]$.
Therefore $\Im$ is a ternary subsemigroup of $\mathcal{M}$.
Let $\ell, \hbar, \flat \in \mathcal{M}$ be such that $\ell, \hbar, \flat \notin (\Im]$. Then $\chi^{\mathscr{T}}_{(\Im]}(\ell, q) = \varsigma_1, \chi^{\mathscr{T}}_{(\Im]}(\hbar, q) = \varsigma_1, \chi^{\mathscr{T}}_{(\Im]}(\flat, q) = \varsigma_1$. Since $\chi^{\mathscr{T}}_{(\Im]}$ is a $(\varsigma_1, \varsigma_2)$IQTFSS.

$$\begin{aligned}\max\{\chi^{\mathscr{T}}_{(\Im]}(\ell\hbar\flat, q), \varsigma_1\} &\geq \min\{\chi^{\mathscr{T}}_{(\Im]}(\ell, q), \chi^{\mathscr{T}}_{(\Im]}(\hbar, q), \chi^{\mathscr{T}}_{(\Im]}(\flat, q), \varsigma_2\}\\ &= \min\{\varsigma_1, \varsigma_1, \varsigma_1, \varsigma_2\}\\ &= \varsigma_1\end{aligned}$$

as $\varsigma_1 < \varsigma_2$, this implies that $\chi^{\mathscr{T}}_{(\Im]}(\ell\hbar\flat, q) \geq \varsigma_1$. Thus $\ell\hbar\flat \notin (\Im]$.

Let $\ell, \hbar, \flat \in \mathcal{M}$ be such that $\ell, \hbar, \flat \notin (\Im]$. Then $\chi^{\mathscr{F}}_{(\Im]}(\ell, q) = \varsigma_2, \chi^{\mathscr{F}}_{(\Im]}(\hbar, q) = \varsigma_2, \chi^{\mathscr{F}}_{(\Im]}(\flat, q) = \varsigma_2$. Since $\chi^{\mathscr{F}}_{(\Im]}$ is a $(\varsigma_1, \varsigma_2)$IQTFSS.

$$\begin{aligned}\min\{\chi^{\mathscr{F}}_{(\Im]}(\ell\hbar\flat, q), \varsigma_1\} &\leq \max\{\chi^{\mathscr{F}}_{(\Im]}(\ell, q), \chi^{\mathscr{F}}_{(\Im]}(\hbar, q), \chi^{\mathscr{F}}_{(\Im]}(\flat, q), \varsigma_2\}\\ &= \max\{\varsigma_2, \varsigma_2, \varsigma_2, \varsigma_2\}\\ &= \varsigma_2\end{aligned}$$

as $\varsigma_1 < \varsigma_2$, this implies that $\chi^{\mathscr{F}}_{(\Im]}(\ell\hbar\flat, q) \leq \varsigma_2$. Thus $\ell\hbar\flat \notin (\Im]$. Therefore $\Im$ is a ternary subsemigroup of $\mathcal{M}$. Similar to proof holds.

Definition 3.12. For three intuitionistic Q fuzzy subsets $\aleph$, $\hbar$ and λ of $\mathcal{M}$, their product $\aleph \cdot \hbar \cdot \lambda$ is defined as

$$(\aleph^{\mathscr{T}} \cdot \hbar^{\mathscr{T}} \cdot \lambda^{\mathscr{T}})(\ell, q) = \begin{cases} \sup\limits_{(r,s,t) \in \Im_\ell} \{\aleph^{\mathscr{T}}(r, q) \wedge \hbar^{\mathscr{T}}(s, q) \wedge \lambda^{\mathscr{T}}(t, q)\} \; if \; \Im_\ell \neq 0 \\ 0 \; otherwise \end{cases}$$

$$(\aleph^{\mathscr{F}} \cdot \hbar^{\mathscr{F}} \cdot \lambda^{\mathscr{F}})(\ell, q) = \begin{cases} \inf\limits_{(r,s,t) \in \Im_\ell} \{\aleph^{\mathscr{F}}(r, q) \vee \hbar^{\mathscr{F}}(s, q) \vee \lambda^{\mathscr{F}}(t, q)\} \; if \; \Im_\ell \neq 0 \\ 1 \; otherwise \end{cases}$$

Definition 3.13. Let $\aleph$ be subset of $\mathcal{M}$, we define the subset $(\eth)^{\varsigma_2}_{\varsigma_1}(\ell, q) = \{\eth(\ell, q) \wedge \varsigma_2\} \vee \varsigma_1$ and $(\mho)^{\varsigma_2}_{\varsigma_1}(\ell, q) = \{\mho(\ell, q) \vee \varsigma_2\} \wedge \varsigma_1$, for all $\ell \in \mathcal{M}$.

Lemma 3.14. *Let $\Im$, $\Im_1$ and $\Im_2$ be non-empty subsets of $\mathcal{M}$. Then the following hold:*

1. $(\chi_{(\Im]} \wedge \chi_{(\Im_1]} \wedge \chi_{(\Im_2]})^{\varsigma_2}_{\varsigma_1} = (\chi_{(\Im\cap\Im_1\cap\Im_2]})^{\varsigma_2}_{\varsigma_1}$,
2. $(\chi_{(\Im]} \vee \chi_{(\Im_1]} \vee \chi_{(\Im_2]})^{\varsigma_2}_{\varsigma_1} = (\chi_{(\Im\cup\Im_1\cup\Im_2]})^{\varsigma_2}_{\varsigma_1}$,
3. $(\chi_{(\Im]} \cdot \chi_{(\Im_1]} \cdot \chi_{(\Im_2]})^{\varsigma_2}_{\varsigma_1} = (\chi_{(\Im\Im_1\Im_2]})^{\varsigma_2}_{\varsigma_1}$.

Proof. (i) and (ii) Straightforward.
(iii) Let $\ell \in \mathcal{M}$. If $\ell \in (\Im\Im_1\Im_2]$, then $(\chi_{(\Im\Im_1\Im_2]})(\ell, q) = \varsigma_2$.
Since $\ell \le abc$ for some $a \in (\Im]$,$b \in (\Im_1]$ and $c \in (\Im_2]$. We have $(a, b, c) \in \Im_\ell$ and $\Im_\ell \neq 0$.

$$\begin{aligned}(\chi^{\mathscr{T}}_{(\Im]} \cdot \chi^{\mathscr{T}}_{(\Im_1]} \cdot \chi^{\mathscr{T}}_{(\Im_2]})(\ell, q) &= \sup_{\ell=xyz} \min\{\chi^{\mathscr{T}}_{(\Im]}(x, q), \chi^{\mathscr{T}}_{(\Im_1]}(y, q), \chi^{\mathscr{T}}_{(\Im_2]}(z, q)\}\\ &\ge \min\{\chi^{\mathscr{T}}_{(\Im]}(a, q), \chi^{\mathscr{T}}_{(\Im_1]}(b, q), \chi^{\mathscr{T}}_{(\Im_2]}(c, q)\}\\ &= \varsigma_2\end{aligned}$$

$$\begin{aligned}(\chi^{\mathscr{F}}_{(\Im]} \cdot \chi^{\mathscr{F}}_{(\Im_1]} \cdot \chi^{\mathscr{F}}_{(\Im_2]})(\ell, q) &= \inf_{\ell=xyz} \max\{\chi^{\mathscr{F}}_{(\Im]}(x, q), \chi^{\mathscr{F}}_{(\Im_1]}(y, q), \chi^{\mathscr{F}}_{(\Im_2]}(z, q)\}\\ &\le \max\{\chi^{\mathscr{F}}_{(\Im]}(a, q), \chi^{\mathscr{F}}_{(\Im_1]}(b, q), \chi^{\mathscr{F}}_{(\Im_2]}(c, q)\}\\ &= \varsigma_1\end{aligned}$$

Therefore $(\chi_{(\Im]} \cdot \chi_{(\Im_1]} \cdot \chi_{(\Im_2]})(\ell, q) = (\chi_{(\Im\Im_1\Im_2]})(\ell, q)$.

If $\ell \notin (\Im\Im_1\Im_2]$ then $(\chi^{\mathscr{T}}_{(\Im\Im_1\Im_2]})(\ell, q) = \varsigma_1$ and $(\chi^{\mathscr{F}}_{(\Im\Im_1\Im_2]})(\ell, q) = \varsigma_2$. Since $\ell \le abc$ for some $a \notin (\Im]$, $b \notin (\Im_1]$ and $c \notin (\Im_2]$. We have

$$\begin{aligned}(\chi^{\mathscr{T}}_{(\Im]} \cdot \chi^{\mathscr{T}}_{(\Im_1]} \cdot \chi^{\mathscr{T}}_{(\Im_2]})(\ell, q) &= \sup_{\ell=xyz} \min\{\chi^{\mathscr{T}}_{(\Im]}(x, q), \chi^{\mathscr{T}}_{(\Im_1]}(y, q), \chi^{\mathscr{T}}_{(\Im_2]}(z, q)\}\\ &\ge \min\{\chi^{\mathscr{T}}_{(\Im]}(a, q), \chi^{\mathscr{T}}_{(\Im_1]}(b, q), \chi^{\mathscr{T}}_{(\Im_2]}(c, q)\}\\ &= \varsigma_1\end{aligned}$$

$$\begin{aligned}(\chi^{\mathscr{F}}_{(\Im]} \cdot \chi^{\mathscr{F}}_{(\Im_1]} \cdot \chi^{\mathscr{F}}_{(\Im_2]})(\ell, q) &= \inf_{\ell=xyz} \max\{\chi^{\mathscr{F}}_{(\Im]}(x, q), \chi^{\mathscr{F}}_{(\Im_1]}(y, q), \chi^{\mathscr{F}}_{(\Im_2]}(z, q)\}\\ &\le \max\{\chi^{\mathscr{F}}_{(\Im]}(a, q), \chi^{\mathscr{F}}_{(\Im_1]}(b, q), \chi^{\mathscr{F}}_{(\Im_2]}(c, q)\}\\ &= \varsigma_2\end{aligned}$$

Hence $(\chi_{(\Im]} \cdot \chi_{(\Im_1]} \cdot \chi_{(\Im_2]})(\ell, q) = (\chi_{(\Im\Im_1\Im_2]})(\ell, q)$.

Theorem 3.15. *For* $\Im, \Im_2 \subseteq \mathcal{M}$ *and* $\{\Im_j | j \in J\}$ *be a family of subsets of* $\mathcal{M}$ *then*
(i) $(\Im] \subseteq (\Im_1]$ *if and only if* $(\chi_{(\Im]})^{\varsigma_2}_{\varsigma_1} \leq (\chi_{(\Im_1]})^{\varsigma_2}_{\varsigma_1}$.
(ii) $(\cap_{j\in J}\chi_{(\Im_j]})^{\varsigma_2}_{\varsigma_1} = (\chi_{\cap_{j\in J}(\Im_j]})^{\varsigma_2}_{\varsigma_1}$.
(iii) $(\cup_{j\in J}\chi_{(\Im_j]})^{\varsigma_2}_{\varsigma_1} = (\chi_{\cup_{j\in J}(\Im_j]})^{\varsigma_2}_{\varsigma_1}$.

Theorem 3.16. *If* $\Im$ *is a* $(\varsigma_1, \varsigma_2)$*-IQFLI[IQTFSS,IQFLATI,IQFRI] of* $\mathcal{M}$*, then* $(\Im)^{\varsigma_2}_{\varsigma_1}$ *is a IQFLI[IQTFSS, IQFLATI,IQFRI] of* $\mathcal{M}$.

Proof. Assume that $\Im$ is a $(\varsigma_1, \varsigma_2)$IQFLATI of $\mathcal{M}$. If there exist $\ell, \hbar, \flat \in \mathcal{M}$. Now

$$\begin{aligned}
\max\{(\eth)^{\varsigma_2}_{\varsigma_1}(\ell\hbar\flat, q), \varsigma_1\} &= \max\{(\{\eth(\ell\hbar\flat, q) \wedge \varsigma_2\} \vee \varsigma_1), \varsigma_1\}\\
&= \{\eth(\ell\hbar\flat, q) \wedge \varsigma_2\} \vee \varsigma_1\\
&= \{\eth(\ell\hbar\flat, q) \vee \varsigma_1\} \wedge \{\varsigma_2 \vee \varsigma_1\}\\
&= \{(\eth(\ell\hbar\flat, q) \vee \varsigma_1) \vee \varsigma_1\} \wedge \varsigma_2\\
&\geq \{(\eth(\hbar, q) \wedge \varsigma_2) \vee \varsigma_1\} \wedge \varsigma_2\\
&= \{(\eth(\hbar, q) \wedge \varsigma_2) \wedge \varsigma_2\} \vee (\varsigma_1 \wedge \varsigma_2)\\
&= \{(\eth(\hbar, q) \wedge \varsigma_2) \wedge \varsigma_2\} \vee \varsigma_1\\
&\geq (\eth)^{\varsigma_2}_{\varsigma_1}(\hbar, q) \wedge \varsigma_2.
\end{aligned}$$

$$\begin{aligned}
\min\{(\mho)^{\varsigma_2}_{\varsigma_1}(\ell\hbar\flat, q), \varsigma_1\} &= \min\{(\{\mho(\ell\hbar\flat, q) \vee \varsigma_2\} \wedge \varsigma_1), \varsigma_1\}\\
&= \{\mho(\ell\hbar\flat, q) \vee \varsigma_2\} \wedge \varsigma_1\\
&= \{\mho(\ell\hbar\flat, q) \wedge \varsigma_1\} \vee \{\varsigma_2 \wedge \varsigma_1\}\\
&= \{(\mho(\ell\hbar\flat, q) \wedge \varsigma_1) \wedge \varsigma_1\} \vee \varsigma_1\\
&\leq \{(\mho(\hbar, q) \vee \varsigma_2) \wedge \varsigma_1\} \vee \varsigma_1\\
&= \{(\mho(\hbar, q) \vee \varsigma_2) \vee \varsigma_2\} \wedge \varsigma_1\\
&\leq (\mho)^{\varsigma_2}_{\varsigma_1}(\hbar, q) \vee \varsigma_2
\end{aligned}$$

Hence $\Im = [(\eth)^{\varsigma_2}_{\varsigma_1}, (\mho)^{\varsigma_2}_{\varsigma_1}]$ is a IQFLATI of $\mathcal{M}$.

Theorem 3.17. *Let* $\Im$ *be an* $(\varsigma_1, \varsigma_2)$*IQFRI,* $\Im_1$ *be an* $(\varsigma_1, \varsigma_2)$*IQFLATI and* $\Im_2$ *be an* $(\varsigma_1, \varsigma_2)$*IQFLI of* $\mathcal{M}$ *then* $((\Im \cdot \Im_1 \cdot \Im_2])^{\varsigma_2}_{\varsigma_1} \subseteq (\Im \cap \Im_1 \cap \Im_2]^{\varsigma_2}_{\varsigma_1}$.

Proof. Let $\Im = [\eth_\Im, \mho_\Im]$ be an $(\varsigma_1, \varsigma_2)$IQFRI, $\Im_1 = [\eth_{\Im_1}, \mho_{\Im_1}]$ be an $(\varsigma_1, \varsigma_2)$IQFLATI and $\Im_2 = [\eth_{\Im_2}, \mho_{\Im_2}]$ be an $(\varsigma_1, \varsigma_2)$IQFLI of $\mathcal{M}$. Let $(\ell, \hbar, \flat) \in X_\gamma$. If $X_\gamma \neq \emptyset$, then $\gamma \leq \ell\hbar\flat$.
Thus $\eth_\Im(\gamma, q) \geq \eth_\Im(\ell\hbar\flat, q) \geq \eth_\Im(\ell, q)$ and $\mho_\Im(\gamma, q) \leq \mho_\Im(\ell\hbar\flat, q) \leq \mho_\Im(\ell, q)$.
Similarly $\eth_{\Im_1}(\gamma, q) \geq \eth_{\Im_1}(\ell\hbar\flat, q) \geq \eth_{\Im_1}(\hbar, q)$ and $\mho_{\Im_1}(\gamma, q) \leq \mho_{\Im_1}(\ell\hbar\flat, q) \leq \mho_{\Im_1}(\hbar, q)$.
Similarly, $\eth_{\Im_2}(\gamma, q) \geq \eth_{\Im_2}(\ell\hbar\flat, q) \geq \eth_{\Im_2}(\flat, q)$ and $\mho_{\Im_2}(\gamma, q) \leq \mho_{\Im_2}(\ell\hbar\flat, q) \leq \mho_{\Im_2}(\flat, q)$.

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We have

$$\begin{aligned}
&(\eth_{(\Im\cdot\Im_1\cdot\Im_2]})_{\varsigma_1}^{\varsigma_2}(\gamma,q)\\
&=(\eth_{(\Im\cdot\Im_1\cdot\Im_2]}(\gamma,q)\wedge\varsigma_2)\vee\varsigma_1\\
&=\Big[\big[\sup_{\gamma\leq\ell\hbar\flat}\{\eth_{\Im}(\ell,q)\wedge\eth_{\Im_1}(\hbar,q)\wedge\eth_{\Im_2}(\flat,q)\}\wedge\varsigma_2\big]\Big]\vee\varsigma_1\\
&=\Big[\sup_{\gamma\leq\ell\hbar\flat}\{\eth_{\Im}(\ell,q)\wedge\eth_{\Im_1}(\hbar,q)\wedge\eth_{\Im_2}(\flat,q)\}\wedge\varsigma_2\wedge\varsigma_2\wedge\varsigma_2\wedge\varsigma_2\Big]\vee\varsigma_1\\
&=\Big[\sup_{\gamma\leq\ell\hbar\flat}\{(\eth_{\Im}(\ell,q)\wedge\varsigma_2)\wedge(\eth_{\Im_1}(\hbar,q)\wedge\varsigma_2)\wedge(\eth_{\Im_2}(\flat,q)\wedge\varsigma_2)\}\wedge\varsigma_2\Big]\vee\varsigma_1\\
&\leq(\{(\eth_{\Im}(\gamma,q)\vee\varsigma_1)\wedge(\eth_{\Im_1}(\gamma,q)\vee\varsigma_1)\wedge(\eth_{\Im_2}(\gamma,q)\vee\varsigma_1)\}\wedge\varsigma_2)\vee\varsigma_1\\
&=\{((\eth_{\Im}(\gamma,q)\wedge\eth_{\Im_1}(\gamma,q)\wedge\eth_{\Im_2}(\gamma,q))\vee\varsigma_1)\wedge\varsigma_2\}\vee\varsigma_1\\
&=\{((\eth_{\Im}\wedge\eth_{\Im_1}\wedge\eth_{\Im_2})(\gamma,q)\wedge\varsigma_2\}\vee\varsigma_1\\
&=(\eth_{\Im\cap\Im_1\cap\Im_2})_{\varsigma_1}^{\varsigma_2}(\gamma,q)
\end{aligned}$$

$$\begin{aligned}
&(\mho_{(\Im\cdot\Im_1\cdot\Im_2]})_{\varsigma_1}^{\varsigma_2}(\gamma,q)\\
&=(\mho_{(\Im\cdot\Im_1\cdot\Im_2]}(\gamma,q)\vee\varsigma_2)\wedge\varsigma_1\\
&=\Big[\big[\inf_{\gamma\leq\ell\hbar\flat}\{\mho_{\Im}(\ell,q)\vee\mho_{\Im_1}(\hbar,q)\vee\mho_{\Im_2}(\flat,q)\}\vee\varsigma_2\big]\Big]\wedge\varsigma_1\\
&=\Big[\inf_{\gamma\leq\ell\hbar\flat}\{\mho_{\Im}(\ell,q)\vee\mho_{\Im_1}(\hbar,q)\vee\mho_{\Im_2}(\flat,q)\}\vee\varsigma_2\vee\varsigma_2\vee\varsigma_2\vee\varsigma_2\Big]\wedge\varsigma_1\\
&=\Big[\inf_{\gamma\leq\ell\hbar\flat}\{(\mho_{\Im}(\ell,q)\vee\varsigma_2)\vee(\mho_{\Im_1}(\hbar,q)\vee\varsigma_2)\vee(\mho_{\Im_2}(\flat,q)\vee\varsigma_2)\}\vee\varsigma_2\Big]\wedge\varsigma_1\\
&\geq(\{(\mho_{\Im}(\gamma,q)\wedge\varsigma_1)\vee(\mho_{\Im_1}(\gamma,q)\wedge\varsigma_1)\vee(\mho_{\Im_2}(\gamma,q)\wedge\varsigma_1)\}\vee\varsigma_2)\wedge\varsigma_1\\
&=\{((\mho_{\Im}(\gamma,q)\vee\mho_{\Im_1}(\gamma,q)\vee\mho_{\Im_2}(\gamma,q))\wedge\varsigma_1)\vee\varsigma_2\}\wedge\varsigma_1\\
&=\{((\mho_{\Im}\vee\mho_{\Im_1}\vee\mho_{\Im_2})(\gamma,q)\vee\varsigma_2\}\wedge\varsigma_1\\
&=(\mho_{\Im\cup\Im_1\cup\Im_2})_{\varsigma_1}^{\varsigma_2}(\gamma,q)
\end{aligned}$$

Let $\ell,\hbar,\flat\notin X_\gamma$. If $X_\gamma=\emptyset$, then $(\eth_{\Im}\cdot\Im_1\cdot\eth_{\Im_2})(\gamma,q)=0$ and $(\mho_{\Im}\cdot\Im_1\cdot\mho_{\Im_2})(\gamma,q)=1$ such that $\gamma\leq\ell\hbar\flat$.

$$\begin{aligned}
(\eth_{(\Im\cdot\Im_1\cdot\Im_2]})_{\varsigma_1}^{\varsigma_2}(\gamma,q)&=(\eth_{(\Im\cdot\Im_1\cdot\Im_2]}(\gamma,q)\wedge\varsigma_2)\vee\varsigma_1\\
&=0\vee\varsigma_1\\
&\leq(\eth_{\Im\cap\Im_1\cap\Im_2}(\gamma,q)\wedge\varsigma_2)\vee\varsigma_1\\
&=(\eth_{\Im\cap\Im_1\cap\Im_2}(\gamma,q)\wedge\varsigma_2)
\end{aligned}$$

$$\begin{aligned}
(\mho_{(\Im\cdot\Im_1\cdot\Im_2]})_{\varsigma_1}^{\varsigma_2}(\gamma,q)&=(\mho_{(\Im\cdot\Im_1\cdot\Im_2]}(\gamma,q)\vee\varsigma_2)\wedge\varsigma_1\\
&=1\wedge\varsigma_1\\
&=\varsigma_1\\
&\geq(\mho_{\Im\cup\Im_1\cup\Im_2}(\gamma,q)\vee\varsigma_2)\wedge\varsigma_1\\
&=(\mho_{\Im\cup\Im_1\cup\Im_2}(\gamma,q)\vee\varsigma_2)
\end{aligned}$$

Therefore $((\Im\cdot\Im_1\cdot\Im_2])_{\varsigma_1}^{\varsigma_2}\subseteq((\Im\cap\Im_1\cap\Im_2])_{\varsigma_1}^{\varsigma_2}$.

Theorem 3.18. *An ordered -semigroup $\mathcal{M}$ is regular, $\Im$ be an $(\varsigma_1, \varsigma_2)$IQFRI, $\Im_1$ be an $(\varsigma_1, \varsigma_2)$IQFLATI and $\Im_2$ be an $(\varsigma_1, \varsigma_2)$IQFLI of $\mathcal{M}$ if and only if* $((\Im \cdot \Im_1 \cdot \Im_2])_{\varsigma_1}^{\varsigma_2} = ((\Im \cap \Im_1 \cap \Im_2])_{\varsigma_1}^{\varsigma_2}$.

Proof. Let $\mathcal{M}$ be an ordered -regular ternary semigroup and $\Im$ be an $(\varsigma_1, \varsigma_2)$IQFRI, $\Im_1$ be an $(\varsigma_1, \varsigma_2)$IQFLATI and $\Im_2$ be an $(\varsigma_1, \varsigma_2)$IQFLI of $\mathcal{M}$. Let $(\ell, \flat) \in X_\gamma$. If $X_\gamma \neq \emptyset$, then $\gamma \leq \ell\hbar\flat$. Thus $\eth_\Im(\gamma, q) \geq \eth_\Im(\ell\hbar\flat, q) \geq \eth_\Im(\ell, q)$ and $\mho_\Im(\gamma, q) \leq \mho_\Im(\ell\hbar\flat, q) \leq \mho_\Im(\ell, q)$.
Similarly $\eth_{\Im_1}(\gamma, q) \geq \eth_{\Im_1}(\ell\hbar\flat, q) \geq \eth_{\Im_1}(\hbar, q)$ and $\mho_{\Im_1}(\gamma, q) \leq \mho_{\Im_1}(\ell\hbar\flat, q) \leq \mho_{\Im_1}(\hbar, q)$.
Similarly, $\eth_{\Im_2}(\gamma, q) \geq \eth_{\Im_2}(\ell\hbar\flat, q) \geq \eth_{\Im_2}(\flat, q)$ and $\mho_{\Im_2}(\gamma, q) \leq \mho_{\Im_2}(\ell\hbar\flat, q) \leq \mho_{\Im_2}(\flat, q)$.
For $\gamma \in \mathcal{M}$, there exists $x \in \mathcal{M}$ such that $\gamma \leq \gamma x_1\gamma x_2\gamma x_3\gamma$. Then $\gamma, (x_1\gamma x_2\gamma x_3), \gamma \in X_\gamma$.
We have

$$\begin{aligned}
&(\eth_{(\Im\cdot\Im_1\cdot\Im_2]})_{\varsigma_1}^{\varsigma_2}(\gamma, q)\\
&= (\eth_{(\Im\cdot\Im_1\cdot\Im_2]}(\gamma, q) \wedge \varsigma_2) \vee \varsigma_1\\
&= \Big[\big[\sup_{\gamma\leq\gamma x_1\gamma x_2\gamma x_3\gamma}\{\eth_\Im(\ell, q) \wedge \eth_{\Im_1}(\hbar, q) \wedge \eth_{\Im_2}(\flat, q)\} \wedge \varsigma_2\big]\Big] \vee \varsigma_1\\
&= \Big[\sup_{\gamma\leq\gamma x_1\gamma x_2\gamma x_3\gamma}\{\eth_\Im(\ell, q) \wedge \eth_{\Im_1}(\hbar, q) \wedge \eth_{\Im_2}(\flat, q)\} \wedge \varsigma_2 \wedge \varsigma_2 \wedge \varsigma_2 \wedge \varsigma_2\Big] \vee \varsigma_1\\
&= \Big[\sup_{\gamma\leq\gamma x_1\gamma x_2\gamma x_3\gamma}\{(\eth_\Im(\ell, q) \wedge \varsigma_2) \wedge (\eth_{\Im_1}(\hbar, q) \wedge \varsigma_2) \wedge (\eth_{\Im_2}(\flat, q) \wedge \varsigma_2)\} \wedge \varsigma_2\Big] \vee \varsigma_1\\
&\geq (\{(\eth_\Im(\gamma, q) \vee \varsigma_1) \wedge (\eth_{\Im_1}(x_1\gamma x_2\gamma x_3) \vee \varsigma_1) \wedge (\eth_{\Im_2}(\gamma, q) \vee \varsigma_1)\} \wedge \varsigma_2) \vee \varsigma_1\\
&\geq (\{(\eth_\Im(\gamma, q) \vee \varsigma_1) \wedge (\eth_{\Im_1}(\gamma, q) \vee \varsigma_1) \wedge (\eth_{\Im_2}(\gamma, q) \vee \varsigma_1)\} \wedge \varsigma_2) \vee \varsigma_1\\
&= \{((\eth_\Im(\gamma, q) \wedge \eth_{\Im_1}(\gamma, q) \wedge \eth_{\Im_2}(\gamma, q)) \vee \varsigma_1) \wedge \varsigma_2\} \vee \varsigma_1\\
&= \{((\eth_\Im \wedge \eth_{\Im_1} \wedge \eth_{\Im_2})(\gamma, q) \wedge \varsigma_2\} \vee \varsigma_1\\
&= (\eth_{\Im\cap\Im_1\cap\Im_2})_{\varsigma_1}^{\varsigma_2}(\gamma, q)
\end{aligned}$$

$$\begin{aligned}
&(\mho_{(\Im\cdot\Im_1\cdot\Im_2]})_{\varsigma_1}^{\varsigma_2}(\gamma, q)\\
&= (\mho_{(\Im\cdot\Im_1\cdot\Im_2]}(\gamma, q) \vee \varsigma_2) \wedge \varsigma_1\\
&= \Big[\big[\inf_{\gamma\leq\gamma x_1\gamma x_2\gamma x_3\gamma}\{\mho_\Im(\ell, q) \vee \mho_{\Im_1}(\hbar, q) \vee \mho_{\Im_2}(\flat, q)\} \vee \varsigma_2\big]\Big] \wedge \varsigma_1\\
&= \Big[\inf_{\gamma\leq\gamma x_1\gamma x_2\gamma x_3\gamma}\{\mho_\Im(\ell, q) \vee \mho_{\Im_1}(\hbar, q) \vee \mho_{\Im_2}(\flat, q)\} \vee \varsigma_2 \vee \varsigma_2 \vee \varsigma_2 \vee \varsigma_2\Big] \wedge \varsigma_1\\
&= \Big[\inf_{\gamma\leq\gamma x_1\gamma x_2\gamma x_3\gamma}\{(\mho_\Im(\ell, q) \vee \varsigma_2) \vee (\mho_{\Im_1}(\hbar, q) \vee \varsigma_2) \vee (\mho_{\Im_2}(\flat, q) \vee \varsigma_2)\} \vee \varsigma_2\Big] \wedge \varsigma_1\\
&\leq (\{(\mho_\Im(\gamma, q) \wedge \varsigma_1) \vee (\mho_{\Im_1}(x_1\gamma x_2\gamma x_3) \wedge \varsigma_1) \vee (\mho_{\Im_2}(\gamma, q) \wedge \varsigma_1)\} \vee \varsigma_2) \wedge \varsigma_1\\
&\leq (\{(\mho_\Im(\gamma, q) \wedge \varsigma_1) \vee (\mho_{\Im_1}(\gamma, q) \wedge \varsigma_1) \vee (\mho_{\Im_2}(\gamma, q) \wedge \varsigma_1)\} \vee \varsigma_2) \wedge \varsigma_1\\
&= \{((\mho_\Im(\gamma, q) \vee \mho_{\Im_1}(\gamma, q) \vee \mho_{\Im_2}(\gamma, q)) \wedge \varsigma_1) \vee \varsigma_2\} \wedge \varsigma_1\\
&= \{((\mho_\Im \vee \mho_{\Im_1} \vee \mho_{\Im_2})(\gamma, q) \vee \varsigma_2\} \wedge \varsigma_1\\
&= (\mho_{\Im\cup\Im_1\cup\Im_2})_{\varsigma_1}^{\varsigma_2}(\gamma, q)
\end{aligned}$$

Thus $((\Im \cdot \Im_1 \cdot \Im_2])_{\varsigma_1}^{\varsigma_2} \supseteq ((\Im \cap \Im_1 \cap \Im_2])_{\varsigma_1}^{\varsigma_2}$ and by Theorem 3.17.
Hence $((\Im \cdot \Im_1 \cdot \Im_2])_{\varsigma_1}^{\varsigma_2} = ((\Im \cap \Im_1 \cap \Im_2])_{\varsigma_1}^{\varsigma_2}$.
Conversely assume that $((\Im \cdot \Im_1 \cdot \Im_2])_{\varsigma_1}^{\varsigma_2} = ((\Im \cap \Im_1 \cap \Im_2])_{\varsigma_1}^{\varsigma_2}$. Let $\Im = (\eth_\Im, \mho_\Im)$ be an $(\varsigma_1, \varsigma_2)$IQFRI, $\Im_1 = (\eth_{\Im_1}, \mho_{\Im_1})$ be an $(\varsigma_1, \varsigma_2)$IQFLATI and $\Im_2 = (\eth_{\Im_2}, \Xi_{\Im_2}, \mho_{\Im_2})$ be an

$(\varsigma_1, \varsigma_2)$IQFLI of $\mathcal{M}$. Then by Theorem 3.11, $\chi_{\Im}$ is a $(\varsigma_1, \varsigma_2)$IQFRI, $\chi_{\Im_1}$ is a $(\varsigma_1, \varsigma_2)$IQFLATI and $\chi_{\Im_2}$ be a $(\varsigma_1, \varsigma_2)$IQFLI of $\mathcal{M}$. By Lemma 3.14 and Theorem 3.15, $(\chi_{(\Im\cap\Im_1\cap\Im_2]})_{\varsigma_1}^{\varsigma_2}=$ $(\chi_{\Im} \cap \chi_{\Im_1} \cap \chi_{\Im_2})_{\varsigma_1}^{\varsigma_2} = (\chi_{\Im} \cdot \chi_{\Im_1} \cdot \chi_{\Im_2})_{\varsigma_1}^{\varsigma_2} = (\chi_{(\Im\cdot\Im_1\cdot\Im_2]})_{\varsigma_1}^{\varsigma_2}$. This implies $(\Im \cap \Im_1 \cap \Im_2]_{\varsigma_1}^{\varsigma_2} = ((\Im \cdot \Im_1 \cdot \Im_2])_{\varsigma_1}^{\varsigma_2}$. Hence by Corollary 2.8, $\mathcal{M}$ is regular.

Theorem 3.19. *An ordered ternary semigroup $\mathcal{M}$ is regular, $\Im$ be an $(\varsigma_1, \varsigma_2)$IQFBI, $\Im_1$ be an $(\varsigma_1, \varsigma_2)$IQFLATI and $\Im_2$ be an $(\varsigma_1, \varsigma_2)$IQFLI of $\mathcal{M}$ if and only if $((\Im \cdot \Im_1 \cdot \Im_2])_{\varsigma_1}^{\varsigma_2} = ((\Im \cap \Im_1 \cap \Im_2])_{\varsigma_1}^{\varsigma_2}$.*

Proof. Let $\mathcal{M}$ be an ordered ternary regular semigroup and $\Im$ be an $(\varsigma_1, \varsigma_2)$IQFBI and $\Im_2$ be an $(\varsigma_1, \varsigma_2)$IQFLI of $\mathcal{M}$. Let $(\ell, \flat) \in X_\gamma$. If $X_\gamma \neq \emptyset$, then $\gamma \leq \ell\hbar\flat$. Thus $\eth_{\Im}(\gamma, q) \geq \eth_{\Im}(\ell\hbar\flat, q) \geq \eth_{\Im}(\ell, q)$ and $\mho_{\Im}(\gamma, q) \leq \mho_{\Im}(\ell\hbar\flat, q) \leq \mho_{\Im}(\ell, q)$.
Similarly $\eth_{\Im_1}(\gamma, q) \geq \eth_{\Im_1}(\ell\hbar\flat, q) \geq \eth_{\Im_1}(\hbar, q)$ and $\mho_{\Im_1}(\gamma, q) \leq \mho_{\Im_1}(\ell\hbar\flat, q) \leq \mho_{\Im_1}(\hbar, q)$.
Similarly, $\eth_{\Im_2}(\gamma, q) \geq \eth_{\Im_2}(\ell\hbar\flat, q) \geq \eth_{\Im_2}(\flat, q)$ and $\mho_{\Im_2}(\gamma, q) \leq \mho_{\Im_2}(\ell\hbar\flat, q) \leq \mho_{\Im_2}(\flat, q)$.
For $\gamma \in \mathcal{M}$, there exists $x \in \mathcal{M}$ such that $\gamma \leq \gamma x_1\gamma x_2\gamma x_3\gamma x_4\gamma x_5\gamma$.
Then $\gamma \leq (\gamma x_1\gamma x_2\gamma), (x_3\gamma x_4\gamma x_5), \gamma \in X_\gamma$. We have

$$\begin{aligned}
&(\eth_{(\Im\cdot\Im_1\cdot\Im_2]})_{\varsigma_1}^{\varsigma_2}(\gamma, q)\\
&= (\eth_{(\Im\cdot\Im_1\cdot\Im_2]}(\gamma, q) \wedge \varsigma_2) \vee \varsigma_1\\
&= \Big[\big[\sup_{\gamma\leq\gamma x_1\gamma x_2\gamma x_3\gamma x_4\gamma x_5\gamma}\{\eth_{\Im}(\ell, q) \wedge \eth_{\Im_1}(\hbar, q) \wedge \eth_{\Im_2}(\flat, q)\} \wedge \varsigma_2\big]\Big] \vee \varsigma_1\\
&= \Big[\sup_{\gamma\leq\gamma x_1\gamma x_2\gamma x_3\gamma x_4\gamma x_5\gamma}\{\eth_{\Im}(\ell, q) \wedge \eth_{\Im_1}(\hbar, q) \wedge \eth_{\Im_2}(\flat, q)\} \wedge \varsigma_2 \wedge \varsigma_2 \wedge \varsigma_2 \wedge \varsigma_2\Big] \vee \varsigma_1\\
&= \Big[\sup_{\gamma\leq\gamma x_1\gamma x_2\gamma x_3\gamma x_4\gamma x_5\gamma}\{(\eth_{\Im}(\ell, q) \wedge \varsigma_2) \wedge (\eth_{\Im_1}(\hbar, q) \wedge \varsigma_2) \wedge (\eth_{\Im_2}(\flat, q) \wedge \varsigma_2)\} \wedge \varsigma_2\Big] \vee \varsigma_1\\
&\geq (\{(\eth_{\Im}(\gamma x_1\gamma x_2\gamma, q) \vee \varsigma_1) \wedge (\eth_{\Im_1}(x_3\gamma x_4\gamma x_5) \vee \varsigma_1) \wedge (\eth_{\Im_2}(\gamma, q) \vee \varsigma_1)\} \wedge \varsigma_2) \vee \varsigma_1\\
&\geq (\{(\eth_{\Im}(\gamma, q) \vee \varsigma_1) \wedge (\eth_{\Im_1}(\gamma, q) \vee \varsigma_1) \wedge (\eth_{\Im_2}(\gamma, q) \vee \varsigma_1)\} \wedge \varsigma_2) \vee \varsigma_1\\
&= \{((\eth_{\Im}(\gamma, q) \wedge \eth_{\Im_1}(\gamma, q) \wedge \eth_{\Im_2}(\gamma, q)) \vee \varsigma_1) \wedge \varsigma_2\} \vee \varsigma_1\\
&= \{((\eth_{\Im} \wedge \eth_{\Im_1} \wedge \eth_{\Im_2})(\gamma, q) \wedge \varsigma_2\} \vee \varsigma_1\\
&= (\eth_{\Im\cap\Im_1\cap\Im_2})_{\varsigma_1}^{\varsigma_2}(\gamma, q)
\end{aligned}$$

$$\begin{aligned}
&(\mho_{(\Im\cdot\Im_1\cdot\Im_2]})_{\varsigma_1}^{\varsigma_2}(\gamma, q)\\
&= (\mho_{(\Im\cdot\Im_1\cdot\Im_2]}(\gamma, q) \vee \varsigma_2) \wedge \varsigma_1\\
&= \Big[\big[\inf_{\gamma\leq\gamma x_1\gamma x_2\gamma x_3\gamma x_4\gamma x_5\gamma}\{\mho_{\Im}(\ell, q) \vee \mho_{\Im_1}(\hbar, q) \vee \mho_{\Im_2}(\flat, q)\} \vee \varsigma_2\big]\Big] \wedge \varsigma_1\\
&= \Big[\inf_{\gamma\leq\gamma x_1\gamma x_2\gamma x_3\gamma x_4\gamma x_5\gamma}\{\mho_{\Im}(\ell, q) \vee \mho_{\Im_1}(\hbar, q) \vee \mho_{\Im_2}(\flat, q)\} \vee \varsigma_2 \vee \varsigma_2 \vee \varsigma_2 \vee \varsigma_2\Big] \wedge \varsigma_1\\
&= \Big[\inf_{\gamma\leq\gamma x_1\gamma x_2\gamma x_3\gamma x_4\gamma x_5\gamma}\{(\mho_{\Im}(\ell, q) \vee \varsigma_2) \vee (\mho_{\Im_1}(\hbar, q) \vee \varsigma_2) \vee (\mho_{\Im_2}(\flat, q) \vee \varsigma_2)\} \vee \varsigma_2\Big] \wedge \varsigma_1\\
&\leq (\{(\mho_{\Im}(\gamma x_1\gamma x_2\gamma, q) \wedge \varsigma_1) \vee (\mho_{\Im_1}(x_3\gamma x_4\gamma x_5) \wedge \varsigma_1) \vee (\mho_{\Im_2}(\gamma, q) \wedge \varsigma_1)\} \vee \varsigma_2) \wedge \varsigma_1\\
&\leq (\{(\mho_{\Im}(\gamma, q) \wedge \varsigma_1) \vee (\mho_{\Im_1}(\gamma, q) \wedge \varsigma_1) \vee (\mho_{\Im_2}(\gamma, q) \wedge \varsigma_1)\} \vee \varsigma_2) \wedge \varsigma_1\\
&= \{((\mho_{\Im}(\gamma, q) \vee \mho_{\Im_1}(\gamma, q) \vee \mho_{\Im_2}(\gamma, q)) \wedge \varsigma_1) \vee \varsigma_2\} \wedge \varsigma_1\\
&= \{((\mho_{\Im} \vee \mho_{\Im_1} \vee \mho_{\Im_2})(\gamma, q) \vee \varsigma_2\} \wedge \varsigma_1\\
&= (\mho_{\Im\cup\Im_1\cup\Im_2})_{\varsigma_1}^{\varsigma_2}(\gamma, q)
\end{aligned}$$

Thus, $((\Im\cdot\Im_1\cdot\Im_2])_{\varsigma_1}^{\varsigma_2} \supseteq ((\Im\cap\Im_1\cap\Im_2])_{\varsigma_1}^{\varsigma_2}$ and by Theorem 3.17 and hence $((\Im\cdot\Im_1\cdot\Im_2])_{\varsigma_1}^{\varsigma_2} = ((\Im\cap\Im_1\cap\Im_2])_{\varsigma_1}^{\varsigma_2}$.
Conversely assume that $((\Im\cdot\Im_1\cdot\Im_2])_{\varsigma_1}^{\varsigma_2} = ((\Im\cap\Im_1\cap\Im_2])_{\varsigma_1}^{\varsigma_2}$. Let $\Im = (\eth_{\Im}, \mho_{\Im})$ be an $(\varsigma_1, \varsigma_2)$IQFBI, $\Im_1 = (\eth_{\Im_1}, \Xi_{\Im_1}, \mho_{\Im_1})$ be an $(\varsigma_1, \varsigma_2)$IQFLATI and $\Im_2 = (\eth_{\Im_2}, \mho_{\Im_2})$ be an $(\varsigma_1, \varsigma_2)$IQFLI of $\mathcal{M}$. Then by Theorem 3.11, $\chi_{\Im}$ is a $(\varsigma_1, \varsigma_2)$IQFBI, $\chi_{\Im_1}$ is a $(\varsigma_1, \varsigma_2)$IQFLATI and $\chi_{\Im_2}$ be a $(\varsigma_1, \varsigma_2)$IQFLI of $\mathcal{M}$. By Lemma 3.14 and Theorem 3.15, $(\chi_{(\Im\cap\Im_1\cap\Im_2]})_{\varsigma_1}^{\varsigma_2} = (\chi_{\Im}\cap\chi_{\Im_1}\cap\chi_{\Im_2})_{\varsigma_1}^{\varsigma_2} = (\chi_{\Im}\cdot\chi_{\Im_1}\cdot\chi_{\Im_2})_{\varsigma_1}^{\varsigma_2} = (\chi_{(\Im\cdot\Im_1\cdot\Im_2]})_{\varsigma_1}^{\varsigma_2}$. This implies $(\Im\cap\Im_1\cap\Im_2]_{\varsigma_1}^{\varsigma_2} = ((\Im\cdot\Im_1\cdot\Im_2])_{\varsigma_1}^{\varsigma_2}$. Hence by Corollary 2.8, $\mathcal{M}$ is regular.

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