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A New Subclass of Univalent Functions Defined by Raducanu-Orhan Linear Differential Operator

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Abstract:

The univalent function is incredibly exciting, as evidenced by the large number of new studies that have been written about it recently. As injective analytic functions, univalent functions do not take the same value at different points inside their domain. Univalent functions are widely used in several branches of mathematics, physics, and engineering and are particularly significant in complicated analysis. Operators of normalized analytic functions are in considerable demand these days, especially differential and integral operators. There are many mathematical and scientific applications for operators. These operators, which are also employed to explain a wide range of physical processes, can be utilized to solve differential equations. A significant amount of material has been studied and debated by numerous researchers for the operators. In this study, the Raducanu-Orhan differential operator defines the new subclass of univalent functions. Furthermore, the subclass's Fekte-Szego inequality, extreme points, integral means of inequalities, and coefficient inequality have been determined.

Keywords: univalent functions, differential operator, subordination, coefficient inequality.

1. Introduction

A univalent analytic function in the complex plane is a one-to-one function that is also referred to as a univalent function. In many branches of mathematics, such as differential equations and complex analysis, uniform functions play a significant role because of their unique characteristics and uses. For ex ample, univalent functions can be used to express conformal mappings, which maintain angles locally; they are particularly relevant in complex analysis. The detailed study for the class of univalent functions is more important because it has links to many other areas, including the theory of special functions, geo metric function theory, and conformal mapping. To gain a better understanding of these functions' behaviour and applicability in other mathematical domains, researchers frequently look at aspects of these functions such growth requirements, distortion theorems, and coefficient bounds. In recent years, researchers have become popular for defining a new subclass of univalent analytic functions linked with some differential operators [17, 9, 3, 21, 23, 24]. Because it has many applications in mathematics, physics, and engineering. The core challenges in the theory of univalent functions are comprehending limit correspondence in conformal mapping, figuring out univalent requirements and addressing numerous functional theory extreme problems. More precisely defining limits on a range of values for various functions in class.

The area principle was used to generate the first significant findings in the theory of univalent functions. Bieberbach [4] found exact upper and lower bounds for $|f(\zeta)|$ and $|f'(\zeta)|$ for $f \in S$, given $|a_2| \leq 2$ and hypothesized that $|a_n| \leq n$ with the help of the outer area theorem (1916). Additionally, he determined the Koebe [11] constant's precise value. For a long time, mathematicians have been challenged by this conjecture. Louis De Branges [6] found a solution to the conjecture $|a_n| \leq n, (n = 2,3, \ldots.)$ in 1984. Following Loewner [12]'s 1923 proof of $|a_3| \leq 3,$ Fekete-Szego [7] astounded mathematicians with the troublesome inequality $\left|a_3 - \mu a_2^2\right| \leq 1 + 2e^{\left(\frac{-2\mu}{1-\mu}\right)}, 0 \leq \mu \leq 1.$ By then, univalent function theory had acquired its own name.

A univalent function that is analytic in a domain and is also referred to as a one-to-one or injective function in complex analysis is said to be conformal. Locally, angles are preserved by conformal mappings. Formally speaking, if a function $f(\zeta)$ that is defined on a domain $D \subset C$ maintains angles between curves that pass through ζ, then $\zeta \in D$ is conformal at that point. A function is conformal in a domain D if it is both univalent (injective) and analytic in D. In line with conformal maps in complex analysis, this indicates that the mapping maintains the local structure of the domain locally by not distorting the angles between curves. Despite of many practical uses, conformal mapping is an essential technique in complex analysis.

If the function is harmonic that is, it satisfies $\nabla^2 f = 0$ according to Laplace then the conformal mapping transformation of such functions is likewise harmonic. As a result, any field whose equations can be represented by a potential function can be solved using conformal mapping. Laplace's equation $\phi_{xx} + \phi_{yy} = 0$ can be used to formulate various mathematical issues related to the motion of fluids, the field of electrostatic, heat transfer, and many other physical circumstances in a certain region D of the complex plane. For exam ple, it can be used to apply in scattering and diffraction problems, brain surface mapping problem, and the electrostatic potential problems in the shaded region of the ζ plane [31, 32]. It can also be used in stealth technology. Although the concept of conformal mapping is not directly used in stealth technology, the development of effective stealth technologies greatly benefits from an un derstanding of shape optimisation, material science, and electromagnetic wave behaviour [1]. Further, the univalent function help to analysis the frequency analysis problem [16].

Recent years, the new subclasses defined by using the linear differential operators. The differential operator was first introduced by Ruscheweyh [19] in 1975, which is cleared the path. Salagean [20] followed in 1983 with an additional variation of differential and integral operators. Many scholars have examined and debated a wide range of properties related to these two operators. Al-Oboudi [2] generalized the Salagean operator in 2004. In 2010, Raducanu and Orhan [15] generalized the Al-Oboudi differential operator. In this study we define two new subclasses, which is defined by the Raducanu-Orhan differential operator. Also, we have discussed some properties of these subclasses.

Let A be the class of univalent functions consists of the form

$$f(\zeta) = \zeta + \sum_{l=2}^{\infty} a_l \zeta^l, \qquad \zeta \in U := \{\zeta \in C : |\zeta| < 1\}, \tag{1}$$

which is analytic in the unit disk U.

For $f(\zeta) \in A$, the Raducanu-Orhan [15] differential operator is defined as $R^n_{\rho,\mu} f(\zeta)$.

$$R^0_{\rho,\mu} = f = \zeta + \sum_{l=2}^{\infty} a_l \zeta^l,$$

$$R^1_{\rho,\mu} = (1-\rho+\mu) f(\zeta) + (\rho-\mu) f'(\zeta) + (\rho\mu)\zeta^2 f''(\zeta)$$

$$= \zeta + \sum_{l=2}^{\infty} [1 + (l-1)(l\rho\mu + \rho - \mu)] a_l \zeta^l,$$

$$R^2_{\rho,\mu} = R^1_{\rho,\mu} (R^1_{\rho,\mu})$$

Similarly,

$$R^n_{\rho,\mu} = R^1_{\rho,\mu} \left(R^{n-1}_{\rho,\mu}\right) = \zeta + \sum_{l=2}^{\infty} [1 + (l-1)(l\rho\mu + \rho - \mu)]^n a_l \zeta^l. \tag{2}$$

where $n \in N_0 = N \cup 0, N = \{1,2,3,\ldots\}, \mu, \rho \geq 0, \zeta \in U$.

Remark. $R^n_{\rho,0} = D^n$ yields the Al-Oboudi differential operator [2], $R^n_{1,0} = D^n$ gives Salagean differential operator [20].

2. The subclass $S^{m,n}_{b,\tau,\rho,\mu}(\gamma)$

Definition 2.1

Let $S^{m,n}_{b,\tau,\rho,\mu}(\gamma)$ denote the subclass of A consisting of function f which satisfies the inequality

$$Re\left(1 + \frac{1}{b}\left(\frac{R^m_{\rho,\mu} f(\zeta)}{R^n_{\rho,\mu} f(\zeta)} - 1\right)\right) > \tau \left|\frac{R^m_{\rho,\mu} f(\zeta)}{R^n_{\rho,\mu} f(\zeta)} - 1\right| + \gamma. \tag{3}$$

For some $b \in C - \{0\}, m \in N, n \in N_0, \tau, \rho, \mu \geq 0, 0 \leq \gamma < 1$ and all $\zeta \in U$.

For suitable choices of the parameters of the of $S^{m,n}_{b,\tau,\rho,\mu}(\gamma)$ provides several well-known subclasses.

Remark 2.2

$S^{m,n}_{b,0,0,\rho}(\gamma) = S(m,n,\gamma,\tau,\rho,b,l)$ studied by Stalin and Thiruchran [30].

$S^{m,n}_{1,0,0,1}(\gamma) = K_{m,n}(\gamma)$ studied by Sumer Eker and Owa [26].

$S^{m,n}_{1,0,0,\rho}(\gamma) = S_{m,n,\rho}(\gamma)$ studied by Sumer Eker and Ozlem Guney [27].

$S^{n+1,n}_{1,0,0,1}(\gamma) = S_n(\gamma)$ studied by Kadioglu [10].

Theorem 2.3

Let $f \in A$ satisfies

$$\sum_{l=2}^{\infty} \phi(\gamma,\tau)|a_l| \leq 2(1-\gamma)|b|. \tag{4}$$

For some $b \in C-\{0\}, m \in N, n \in N_0, \tau,\rho,\mu \geq 0, 0 \leq \gamma < 1$ and all $\zeta \in U$, then $f \in S^{m,n}_{b,\tau,\rho,\mu}(\gamma)$, where $\phi(\gamma,\tau) = |(1+(l-1)(l\rho\mu+\rho-\mu))^m - (1+\gamma b)(1+(l-1)(l\rho\mu+\rho-\mu))^n|$
$+(1+(l-1)(l\rho\mu+\rho-\mu))^m + ((2-\gamma)b-1)(1+(l-1)(l\rho\mu+\rho-\mu))^n$
$+2b\tau|(1+(l-1)(l\rho\mu+\rho-\mu))^m - (1+(l-1)(l\rho\mu+\rho-\mu))^n|$.

Proof. Suppose that $\sum_{l=2}^{\infty} \phi(\gamma,\tau)|a_l| \leq 2(1-\gamma)|b|$ is true. For some $b \in C-\{0\}, m \in N, n \in N_0, \tau,\rho,\mu \geq 0, 0 \leq \gamma < 1$ and all $\zeta \in U$, then it is sufficient to prove that $\left|\frac{F(\zeta)-1}{F(\zeta)+1}\right| < 1$.

For $f \in A$, then define the function F by

$$F(\zeta) = 1 + \frac{1}{b}\left(\frac{R^m_{\rho,\mu}f(\zeta)}{R^n_{\rho,\mu}f(\zeta)} - 1\right) - \tau\left|\frac{R^m_{\rho,\mu}f(\zeta)}{R^n_{\rho,\mu}f(\zeta)} - 1\right| - \gamma$$

$$F(\zeta) - 1 = \frac{R^m_{\rho,\mu}f(\zeta) - (1+\gamma b)R^n_{\rho,\mu}f(\zeta) - b\tau\left|R^m_{\rho,\mu}f(\zeta) - R^n_{\rho,\mu}f(\zeta)\right|}{bR^n_{\rho,\mu}f(\zeta)}$$

and

$$F(\zeta) - 1 = \frac{R^m_{\rho,\mu}f(\zeta) - (1+(\gamma-2)b)R^n_{\rho,\mu}f(\zeta) - b\tau\left|R^m_{\rho,\mu}f(\zeta) - R^n_{\rho,\mu}f(\zeta)\right|}{bR^n_{\rho,\mu}f(\zeta)}$$

Therefore,

$$\left|\frac{F(\zeta)-1}{F(\zeta)+1}\right| = \left|\frac{R^m_{\rho,\mu}f(\zeta) - (1+\gamma b)R^n_{\rho,\mu}f(\zeta) - b\tau\left|R^m_{\rho,\mu}f(\zeta) - R^n_{\rho,\mu}f(\zeta)\right|}{R^m_{\rho,\mu}f(\zeta) - (1+(\gamma-2)b)R^n_{\rho,\mu}f(\zeta) - b\tau\left|R^m_{\rho,\mu}f(\zeta) - R^n_{\rho,\mu}f(\zeta)\right|}\right| < 1.$$

$$\Rightarrow \sum_{l=2}^{\infty}\left\{\begin{matrix} |(1+(l-1)(l\rho\mu+\rho-\mu))^m - (1+\gamma b)(1+(l-1)(l\rho\mu+\rho-\mu))^n| \\ +(1+(l-1)(l\rho\mu+\rho-\mu))^m + ((2-\gamma)b-1)(1+(l-1)(l\rho\mu+\rho-\mu))^n \\ +2b\tau|(1+(l-1)(l\rho\mu+\rho-\mu))^m - (1+(l-1)(l\rho\mu+\rho-\mu))^n| \end{matrix}\right\}|a_l|$$
$$\leq 2(1-\gamma)|b|.$$

$$\therefore \sum_{l=2}^{\infty} \phi(\gamma,\tau)|a_l| \leq 2(1-\gamma)|b|.$$

Hence, equation (4) holds.

Put $\tau = 0$ and $\mu = 0$ in $S^{m,n}_{b,\tau,\rho,\mu}(\gamma)$ then this class reduces as $S(m,n,\gamma,\tau,\rho,b)$, which was studied by Thirucheran and Stalin [30].

Corollary 2.4

Let $f \in A$ satisfies

$$\sum_{l=2}^{\infty} \phi(m,n,\gamma,\tau,\rho,b,l) \ \ |a_l| \leq 2(1-\gamma)b.$$

For some $(0 \leq \gamma < 1), \tau \geq 0, m \in N, n \in N_0, \ \rho(\rho \geq 0)$, and all $\zeta \in U$, then $f \in S(m,n,\gamma,\tau,\rho,b)$, Where $\phi(m,n,\gamma,\tau,\rho,b,l) = |(1 + (l-1)\rho)^m - (1+\gamma b)(1+(l-1)\rho)^n|$

$$+(1+(l-1)\rho)^m + \big((2-\gamma)b-1\big)(1+(l-1)\rho)^n$$

$$+2b\tau\,|(1+(l-1)\rho)^m - (1+(l-1)\rho)^n|.$$

If $b = 1$ and $\tau = 0$, then the class $S_{b,\tau,\rho,\mu}^{m,n}(\gamma)$ reduces to $Re\left(\frac{R_\rho^m f(\zeta)}{R_\rho^m f(\zeta)}\right) > \gamma$ which analogues to the class $S_{m,n,\rho}(\gamma)$ introduced by Sevtap Sumer Eker and Ozlem Guney[27].

Corollary 2.5

Let $f \in A$ satisfies the inequality $\sum_{l=2}^{\infty}\phi(\gamma,m,n,\rho,l)\ |a_l| \le 2(1-\gamma)$, for some $b \in C - \{0\}$, $m \in N, n \in N0, \rho,\mu \ge 0, 0 \le \gamma < 1$, then $f \in S_{m,n,\rho}(\gamma)$, where

$$\phi(\gamma,m,n,\rho,l) = |(1+(l-1)\rho)^m - (1+\gamma)(1+(l-1)\rho)^n|$$

$$+(1+(l-1)\rho)^m + (1-\gamma)(1+(l-1)\rho)^n$$

If $b = 1, \tau = 0$ and $\rho = 1$, then the class $S_{b,\tau,\rho,\mu}^{m,n}(\gamma)$ given the class $S_{m,n}(\gamma)$, which is discussed by Sevtap Sumer Eker and Owa [26].

Corollary 2.6

Let $f \in A$ satisfies $\sum_{l=2}^{\infty}\phi(\gamma,m,n,l)\,|a_l| \le 2(1-\gamma)$, for some $\gamma(0 \le \gamma < 1), m \in N, n \in N_0$, then $f \in S_{m,n}(\gamma)$, where $\phi(\gamma,m,n,l) = |(l)^m - (1+\gamma)(l)^n| + (l)^m + (1-\gamma)(l)^n$.

We define the subclass $\widetilde{S_{b,\tau,\rho,\mu}^{m,n}}(\gamma) \subset S_{b,\tau,\rho,\mu}^{m,n}(\gamma)$, and determine the extreme points of the subclass for the subclass $\widetilde{S_{b,\tau,\rho,\mu}^{m,n}}(\gamma)$.

Theorem 2.7

Let $f_1(\zeta) = \zeta$, and $f_l(\zeta) = \zeta + \sum_{l=2}^{\infty}\eta_l\frac{2(1-\gamma)b}{\phi(\gamma,\tau)}\zeta^l,\ \ (l = 2,3,4,\dots)$, then $f \in \widetilde{S_{b,\tau,\rho,\mu}^{m,n}}(\gamma)$ if it is able to represented as $f = \sum_{l=1}^{\infty}\eta_l f_l(\zeta),\ \eta_l > 0$ and $\sum_{l=1}^{\infty}\eta_l = 1$.

Proof. Suppose that

$$f = \sum_{l=1}^{\infty}\eta_l f_l(\zeta)$$

$$= \zeta + \sum_{l=2}^{\infty}\eta_l\frac{2(1-\gamma)b}{\phi(\gamma,\tau)}\zeta^l$$

$$= 2(1-\gamma)b\sum_{l=2}^{\infty}\eta_l$$

$$= 2(1-\gamma)b(1-\eta_1)$$

$$< 2(1-\gamma)b.$$

Which shows that $f \in \widetilde{S^{m,n}_{b,\tau,\rho,\mu}}(\gamma)$.

Conversely, suppose that $f \in S^{m,n}_{b,\tau,\rho,\mu}(\gamma)$

Since, $a_l \leq \frac{2(1-\gamma)b}{\phi(\gamma,\tau)}$

Let

$$\eta_l \leq \frac{\phi(\gamma,\tau)}{2(1-\gamma)b} a_l$$

and

$$\eta_1 = 1 - \sum_{l=2}^{\infty} \eta_l,$$

then we obtain

$$f = \sum_{l=1}^{\infty} \eta_l f_l(\zeta).$$

Let $\mu = 0$, then the class $S^{m,n}_{b,\tau,\rho,\mu}(\gamma)$ reduces and analogues to the class, which is examined by Thirucheran and Stalin [30].

Corollary 2.8

Let $f_1(\zeta) = \zeta$, and $f_l(\zeta) = \zeta + \sum_{l=2}^{\infty} \eta_l \frac{2(1-\gamma)b}{\phi(m,n,\gamma,\tau,\rho,b,l)} \zeta^l$, $(l = 2,3,4, \dots)$, then $f \in \widetilde{S^{m,n}_{b,\tau,\rho}}(\gamma)$ if it is able to represented as $f = \sum_{l=1}^{\infty} \eta_l f_l(\zeta)$, $\eta_l > 0$ and $\sum_{l=1}^{\infty} \eta_l = 1$.

If $b = 1$ and $\tau = 0$, we get the result of the class $S_{m,n,\rho}(\gamma)$ introduced by Sevtap Sumer Eker and Ozlem Guney [27].

Corollary 2.9

Let $f_1(\zeta) = \zeta$, and $f_l(\zeta) = \zeta + \sum_{l=2}^{\infty} \eta_l \frac{2(1-\gamma)b}{\phi(\gamma,m,n,\rho,l)} \zeta^l$, $(l = 2,3,4, \dots)$, then $f \in \tilde{S}_{m,n,\rho}$ if it is able to represented as $f = \sum_{l=1}^{\infty} \eta_l f_l(\zeta)$, $\eta_l > 0$ and $\sum_{l=1}^{\infty} \eta_l = 1$.

Theorem 2.10

Let $f \in S^{m,n}_{b,\tau,\rho,\mu}(\gamma)$ and suppose that f is defined by

$$\zeta + \frac{2(1-\gamma)b\varepsilon_l}{\phi(\gamma,\tau)} \zeta^l, (l = 2,3,4, \dots), |\varepsilon_l| = 1.$$

If an analytic function is present $w(\zeta)$ given by

$$\{w(\zeta)\}^{l-1} = \frac{\phi(\gamma,\tau)}{2(1-\gamma)b\varepsilon_l} \sum_{l=2}^{\infty} a_l \zeta^{l-1}, \ \left(\zeta = re^{i\theta}, 0 < r < 1\right),$$

then

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\mu} d\theta \leq \int_0^{2\pi} \left|1 + \frac{2(1-\gamma)b\varepsilon_l}{\phi(\gamma,\tau)}\zeta^{l-1}\right|^{\mu} d\theta.$$

Proof. We show that

$$\int_0^{2\pi} \left|1 + \sum_{l=2}^{\infty} a_l \zeta^{l-1}\right|^{\mu} d\theta \leq \int_0^{2\pi} \left|1 + \frac{2(1-\gamma)b\varepsilon_l}{\phi(\gamma,\tau)}\zeta^{l-1}\right|^{\mu} d\theta.$$

By the help of little wood subordination theorem [13], it is sufficient to show that

$$1 + \sum_{l=2}^{\infty} a_l \zeta^{l-1} \prec 1 + \frac{2(1-\gamma)b\varepsilon_l}{\phi(\gamma,\tau)}\zeta^{l-1}.$$

Let

$$1 + \sum_{l=2}^{\infty} a_l \zeta^{l-1} = 1 + \frac{2(1-\gamma)b\varepsilon_l}{\phi(\gamma,\tau)}(w(\zeta))^{l-1}.$$

$$\therefore (w(\zeta))^{l-1} = \frac{\phi(\gamma,\tau)}{2(1-\gamma)b\varepsilon_l}\sum_{l=2}^{\infty} a_l\zeta^{l-1}.$$

Which readily yields $w(0) = 0$.

Further, we prove that the analytic function $w(\zeta)$ satisfies $|w(\zeta)| < 1$ using schwarz lemma. We know that

$$\left|(w(\zeta))^{l-1}\right| = \left|\frac{\phi(\gamma,\tau)}{2(1-\gamma)b\varepsilon_l}\sum_{l=2}^{\infty} a_l\zeta^{l-1}.\right| \leq |\zeta| < 1.$$

Let $\mu = 0$, then the class $S_{b,\tau,\rho,\mu}^{m,n}(\gamma)$ reduces and analogues to the class, which is examined by Thirucheran and Stalin [30].

Corollary 2.11

Let $f \in S(m,n,\gamma,\tau,\rho,b)$ and suppose that f is defined by

$$\zeta + \frac{2(1-\gamma)b\varepsilon_l}{\phi(m,n,\gamma,\tau,\rho,b,l)}\zeta^l, (l = 2,3,4,\dots), |\varepsilon_l| = 1.$$

If an analytic function is present $w(\zeta)$ given by

$$\{w(\zeta)\}^{l-1} = \frac{\phi(m,n,\gamma,\tau,\rho,b,l)}{2(1-\gamma)b\varepsilon_l}\sum_{l=2}^{\infty} a_l\zeta^{l-1}, \ \left(\zeta = re^{i\theta}, 0 < r < 1\right),$$

then

$$\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^{\mu} d\theta \le \int_0^{2\pi} \left|g\left(re^{i\theta}\right)\right|^{\mu} d\theta, \ \ \mu > 0.$$

3. The Fekete-Szego inequality for the subclass $S_{b,\tau,\rho,\mu}^{m,n}(\gamma)$

In 1933, Fekete-Szego [7] obtained the maximum value of $|a_3 - \mu a_2^2|$ as a function of the real parameter μ, for the function of class A. Since then, the various authors were investigated and obtained the Fekete-Szego inequalities for different subclasses of the class A [5] [7],[8],[18],[22],[25], [28], [29]. In this article, we introduced two new subclasses of univalent functions which are defined by using Raducanu-Ohran differential operator in the open unit disc. For these subclasses, we obtain the Fekete-Szego inequality $|a_3 - \mu a_2^2|$.

If replacing special values for the subclass, we obtained Several well-known subclasses.

Remark

$S_{1,0,0,1}^{1,0}(\gamma) = S^*(\gamma)$ studied by Ma and Minda [14].

$S_{b,0,0,1}^{1,0}(\gamma) = S_b^*(\gamma)$ studied by Ravichandran et.al. [18].

$S_{b,0,0,\rho}^{2,0}(\gamma) = M_{a,b}(\phi)$ studied by Suchitra et.al. [25].

$S_{b,0,0,\rho}^{2,1}(\gamma) = M_{\gamma}(\phi)$studied by Shanmugam and Sivasubramanian [22].

Lemma 3.2

If $P(\zeta) = 1 + c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + \cdots$ is a function with positive real part in U and μ is a complex number, then $\left|c_2 - \mu c_1^2\right| \le 2\, Max\{1, |2\mu - 1|\}$. The result is sharp for the function is given by

$P(\zeta) = \frac{1+\zeta^2}{1-\zeta^2}$ and $P(\zeta) = \frac{1+\zeta}{1-\zeta}$.

Theorem 3.3

Let $\phi(\zeta) = 1 + B_1\zeta + B_2\zeta^2 + B_3\zeta^3 + \cdots$, with $B_1 = 0$. If $f \in S_{b,\tau,\rho,\mu}^{m,n}(\gamma)$ satisfies the inequality

$$Re\left(1 + \frac{1}{b}\left(\frac{R_{\rho,\mu}^{m} f(\zeta)}{R_{\rho,\mu}^{n} f(\zeta)} - 1\right)\right) - \tau\left|\frac{R_{\rho,\mu}^{m} f(\zeta)}{R_{\rho,\mu}^{n} f(\zeta)} - 1\right| \prec \phi(\zeta).$$

Then

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1|b}{|\,Y_2 - b\tau|Y_2|}\max\left\{1, \left|\frac{B_2}{B_1} + \left[\frac{[Y_3 - b\tau|Y_3|] - \mu[[Y_2] - b\tau|Y_2|]}{[Y_1 - b\tau[Y_1]]^2}\right] bB_1\right|\right\}, \qquad (5)$$

where

$Y_1 = ((1 + (\rho - \mu + 2\rho\mu)))^m - ((1 + (\rho - \mu + 2\rho\mu)))^n,$

$Y_2 = ((1 + 2(\rho - \mu + 3\rho\mu)))^m - ((1 + 2(\rho - \mu + 3\rho\mu)))^n$

and

$Y_3 = ((1 + (\rho - \mu + 2\rho\mu)))^{m+n} - ((1 + (\rho - \mu + 2\rho\mu)))^{2n}$.

Then the result is sharp.

Proof. If $f \in S^{m,n}_{b,\tau,\rho,\mu}(\gamma)$, then there is a Schwarz function $w(\zeta)$, analytic in U with $w(0) = 0$ and $|w(\zeta)| < 1$ in such that

$$1 + \frac{1}{b}\left(\frac{R^m_{\rho,\mu}f(\zeta)}{R^n_{\rho,\mu}f(\zeta)} - 1\right) - \tau\left|\frac{R^m_{\rho,\mu}f(\zeta)}{R^n_{\rho,\mu}f(\zeta)} - 1\right| \prec \phi(w(\zeta)).$$

Define $P(\zeta)$ by $P(\zeta) = \frac{1+w(\zeta)}{1-w(\zeta)} = 1 + c_1\zeta + c_2\zeta^2 + c_3\zeta^3 + \cdots$

Since $w(\zeta)$ is a Schwarz function, it is clear that $ReP(\zeta) > 0$ and $P(0) = 1$.

$$\therefore \phi(\zeta) = \phi\left(\frac{P(\zeta)-1}{P(\zeta)+1}\right) = 1 + \frac{B_1c_1}{2}\zeta + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2c_1^2}{4}\right]\zeta^2 + \ldots$$

Now,

$$1 + \frac{1}{b}\left(\frac{R^m_{\rho,\mu}f(\zeta)}{R^n_{\rho,\mu}f(\zeta)} - 1\right) - \tau\left|\frac{R^m_{\rho,\mu}f(\zeta)}{R^n_{\rho,\mu}f(\zeta)} - 1\right| = 1 + \frac{B_1c_1}{2}\zeta + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2c_1^2}{4}\right]\zeta^2 + \cdots$$

$$\therefore a_2 = \frac{bB_1c_1}{2[Y_1 - b\tau|Y_1|]}$$

And

$$a_3 = \frac{bB_1c_2}{2[Y_2 - b\tau|Y_2|]} + \frac{bB_1c_1^2}{4[Y_2 - b\tau|Y_2|]}\left[\frac{[Y_1 - b\tau|Y_1|]bB_1}{[Y_1 - b\tau|Y_1|]^2} - \left(1 - \frac{B_2}{B_1}\right)\right]$$

$$\therefore a_3 - \mu a_2^2 = \frac{bB_1}{2[Y_2 - b\tau|Y_2|]}\{c_2 - vc_1^2\},$$

Where $v = \frac{1}{2}\left(1 - \frac{B_2}{B_1} + \frac{\mu bB_{1[Y_2 - b\tau|Y_2|]}}{[Y_1 - b\tau|Y_1|]^2} - \frac{bB_{1[Y_3 - b\tau|Y_3|]}}{[Y_1 - b\tau|Y_1|]^2}\right)$.

Hence,

$$|a_3 - \mu a_2^2| \leq \frac{B_1|b}{|\,Y_2 - b\tau|Y_2|}\max\left\{1, \left|\frac{B_2}{B_1} + \left[\frac{[Y_3 - b\tau|Y_3|] - \mu[[Y_2] - b\tau|Y_2|]}{[Y_1 - b\tau[Y_1]]^2}\right]bB_1\right|\right\}.$$

Therefore, the result (22) is sharp for the function defined by

$$1 + \frac{1}{b}\left(\frac{R^m_{\rho,\mu}f(\zeta)}{R^n_{\rho,\mu}f(\zeta)} - 1\right) - \tau\left|\frac{R^m_{\rho,\mu}f(\zeta)}{R^n_{\rho,\mu}f(\zeta)} - 1\right| = \phi(\zeta^2)$$

and

$$1 + \frac{1}{b}\left(\frac{R^m_{\rho,\mu}f(\zeta)}{R^n_{\rho,\mu}f(\zeta)} - 1\right) - \tau\left|\frac{R^m_{\rho,\mu}f(\zeta)}{R^n_{\rho,\mu}f(\zeta)} - 1\right| = \phi(\zeta).$$

4. Conclusion

This present work, we got some results that were obtained from the new sub class of normalized analytic univalent function. We also acquired and examined a few fundamental characteristics of univalent functions, comparing them to earlier findings. Also, sharp boundaries were obtained for the subclass. The researchers are motivated to improve the findings of this subclass in future by using bi-univalent, multivalent, q-analoque, and meromorpihic functions with positive and negative coefficients.

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