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Approximation of Functions by n-Euler Product Means of Fourier Series and its Conjugate

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Abstract:
Our research in this work produced new theorems on approximation of functions by n-Eulers E^q product means of Fourier series and its conjugate Fourier series. This new results generalize the results of [5], [7] and [9].

Key Words:

Approximation of function, Fourier series, Conjugate Fourier series, L^p space, Lebesgue Integral.

1 Introduction

Estimating functions using generalized Fourier series based on trigonometric polynomials has become increasingly important in the development of mathematics and engineering fields in recent years. For instance, a new L^2-bassed technique for creating Finit Impulse Response digital filters to obtain an optimal approximation was devised by Psarakis and Moustakieds [2], utilising the features of approximation of functions.

In the development of digital filters, L^p-space, L^2-space, and L^∞-space are also very important. In the past few years, numerous researchers have grown interested in the inaccuracy of approximation of periodic functions belonging to distinct classes using different summability methods. Several scholars, including Hadish [6], Sonkar and Singh[8], Saxena and Prabhakar [5], Sonkar and Sagwan[7], and Sachin [9], have studied the following topics: $(C,1)(E,q)$, double Euler summability, triple E^1 Euler summability, and triple E^q-euler summability, respectively. We investigated the approximation of functions in this direction using the n-Euler E^q product summability approach. The outcomes of [7], [8], and [9] were generalized by our result.

2 Definitions and Notations

Let g be a Lebesgue integrable function with period 2π on the interval $[0,2\pi]$. The Fourier series of a function g is given by

$$g(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n cosnx + b_n sinnx) \quad (2.1)$$

$$\tilde{g}(x) \approx \sum_{m=1}^{\infty} (b_m cosmx - a_m sinmx) \quad (2.2)$$

with n^{th} partial sum $g_n(x)$.

The $L^p[0,2\pi] -$ space can be defined as

$$L^p[0,2\pi]:=\left\{g:[0,2\pi]\to R:\int_0^{2\pi}|g(x)|^p dx<\infty\right\}, p\geq 1.$$

Let $\sum_{r=0}^{\infty} u_r$ be an infinite series and sequence $\{s_r\}$ is $(r+1)^{th}$ partial sum of given series, then the series $\sum_{r=0}^{\infty} u_r$ is said to be (E,q) summable[1] to s

if

$$t_r^{E^q}=\frac{1}{(1+q)^r}\sum_{j=0}^{r}\binom{r}{j}q^{r-j}s_j\to \quad s \quad as \quad r\to\infty.$$

The series $\sum_{r=0}^{\infty} u_r$ is said to be $(E,q)(E,q)$ summable to s,

if

$$t_r^{E^qE^q}=\frac{1}{(1+q)^r}\sum_{j=0}^{r}\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}q^{j-h}s_h\to \quad s \quad as \quad r\to\infty.$$

The series $\sum_{r=0}^{\infty} u_r$ is said to be $(E,q)(E,q)(E,q)$ summable to s,

if

$$t_r^{E^qE^qE^q}=\frac{1}{(1+q)^r}\sum_{j=0}^{r}\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\sum_{l=0}^{h}\binom{h}{l}q^{h-l}s_l\to \quad s \quad as \quad r\to\infty.$$

Similarly,

the series $\sum_{r=0}^{\infty} u_r$ is said to be $(E,q)(E,q)(E,q)\ldots(E,q)$ summable to s i.e. n-euler E^q summable to s.

if

$$t_r^{E^qE^q\ldots E^q}=\frac{1}{(1+q)^r}\sum_{j=0}^{r}\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\sum_{l=0}^{h}\binom{h}{l}q^{h-l}\ldots\frac{q^{v-i}}{(1+q)^i}\sum_{m=0}^{i}\binom{i}{m}q^{i-m}s_m$$
$$\to \quad s \quad as \quad r\to\infty$$

We also write,

$$J_r(t)=\frac{1}{2\pi(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\sum_{l=0}^{h}\binom{h}{l}q^{h-l}\ldots\frac{q^{v-i}}{(1+q)^i}\left\{\sum_{m=0}^{i}\binom{i}{m}q^{i-m}\frac{sin\left(m+\frac{1}{2}\right)t}{sin\frac{t}{2}}\right\}\right]$$

$$\tilde{J}_r(t)=\frac{1}{2\pi(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\sum_{l=0}^{h}\binom{h}{l}q^{h-l}\ldots\frac{q^{v-i}}{(1+q)^i}\left\{\sum_{m=0}^{i}\binom{i}{m}q^{i-m}\frac{cos\left(m+\frac{1}{2}t\right)}{sin\frac{t}{2}}\right\}\right]$$

$$\phi(t)=g(x+t)-2g(t)+g(x-t)$$

and

$$\psi(t)=\frac{g(x+t)-g(x-t)}{2}.$$

3 Main Results

Saxena and Prabhakar[8] obtained novel findings in the domain of function approximation. The triple E^1 Summability approach is introduced by Sonkar and Sangwan[7], who used it to generalise the findings of Saxena and Prabhakar.The triple E^q summability approach was recently introduced by Devaiya and Sriwastva[9], and they used it to generalise the findings of Sonkar and Sagwan[7]. In this work, we establish the more broad setting and generalise the findings of Sachin Devaiya and Shailesh Kumar Srivastava[9].

3.1 Theorem

let p_r be a none increasing positive sequence of real constant such that

$$p_r = \sum_{z=0}^{r} p_z \to \quad s \quad as \quad r \to \infty, \tag{3.1}$$

and given function $\phi(t)$ satisfies,

$$\Phi(t) = \int_0^t |\phi(u)|du = o\left[\frac{t}{\alpha\left(\frac{1}{t}\right)p_t}\right], \quad as \quad t \to 0^+, \tag{3.2}$$

where $\alpha(t)$ is positrive, monotonic and none - increasing function of t.

$$logr = O[\alpha(r).p_r], as \quad r \to \infty \tag{3.3}$$

The approximation of function g at $x = t$ by n- Eular product means of its Fourier series is given by

$$|t_r^{E^qE^qE^q...E^q} - g(x)| = O(1), as \quad r \to \infty.$$

3.2 Theorem

let p_r be a none-increasing positive sequence of real constant such that

$$p_r = \sum_{z=0}^{r} p_z \to \quad s \quad as \quad r \to \infty$$

and given function $\psi(t)$ satisfies,

$$\Psi(t) = \int_0^t |\psi(u)|du = o\left[\frac{t}{\alpha\left(\frac{1}{t}\right)p_t}\right], \quad as \quad t \to 0^+, \tag{3.4}$$

where $\alpha(t)$ is positive, monotonic and none - increasing function of t.

$$logr = O[\alpha(r).p_r], as \quad r \to \infty, \tag{3.5}$$

then approximation of function $\tilde{g}$ at x=t by n- Eular product means of its conjugate fourier series is given by

$$|\tilde{t}_r^{E^qE^qE^q...E^q} - \tilde{g}(x)| = O(1), as \quad r \to \infty.$$

4 lemmas

Here few lemmas are given, Which are useful to prove our theorem

Lemma 4.1 $|J_r(t)| = O(r), for\ 0 \leq t \leq \frac{1}{r}.$

Proof. Using $sinrt \leq rt$ and $sin\frac{t}{2} \geq \frac{t}{\pi}$, we have

$$|J_r(t)| = \frac{1}{2\pi(1+q)^r}\left|\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\cdots\frac{q^{v-i}}{(1+q)^i}\left\{\sum_{m=0}^{i}\binom{i}{m}q^{i-m}\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right]\right|$$

$$\leq \frac{1}{2\pi(1+q)^r}\left|\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\cdots\frac{q^{v-i}}{(1+q)^i}\left\{\sum_{m=0}^{i}\binom{i}{m}q^{i-m}\frac{\left(m+\frac{1}{2}\right)t}{\frac{t}{\pi}}\right\}\right]\right|$$

$$\leq \frac{1}{4\pi(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\cdots(2i+1)\frac{q^{v-i}}{(1+q)^i}\left\{\sum_{m=0}^{i}\binom{i}{m}q^{i-m}\right\}\right]$$

$$\leq \frac{1}{4(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\sum_{l=0}^{h}\binom{h}{l}q^{h-l}(2l+1)\right]$$

$$\leq \frac{1}{4(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}q^{j-h}(2h+1)\right]$$

$$\leq \frac{1}{4(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}q^{r-j}(2j+1)\right]$$

$$\leq \frac{1}{4}(2r+1)$$

$$= O(r)$$

Lemma 4.2 $|J_r(t)| = O\left(\frac{1}{t}\right)$ *for* $\frac{1}{r} \leq t \leq \pi$

Proof. Using $sin(rt) \leq 1$ and $sin(\frac{t}{2}) \geq \frac{t}{\pi}$, We have

$$|J_r(t)| = \frac{1}{2\pi(1+q)^r}\left|\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\cdots\frac{q^{v-i}}{(1+q)^i}\left\{\sum_{m=0}^{i}\binom{i}{m}q^{i-m}\frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right]\right|$$

$$\leq \frac{1}{2\pi(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\cdots\frac{q^{v-i}}{(1+q)^i}\left\{\sum_{m=0}^{i}\binom{i}{m}q^{i-m}\frac{\pi}{t}\right\}\right]$$

$$\leq \frac{1}{2t(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\cdots\frac{q^{v-i}}{(1+q)^i}\left\{\sum_{m=0}^{i}\binom{i}{m}q^{i-m}\right\}\right]$$

$$\leq \frac{1}{2t(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}q^{j-h}\right]$$

$$\leq \frac{1}{2t(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}q^{r-j}\right]$$

$$\leq \frac{1}{2t}$$

$$= O\left(\frac{1}{t}\right).$$

Lemma 4.3 $|\tilde{J}(t)| = O\left(\frac{1}{t}\right)$, *for* $0 \leq t \leq \frac{1}{r}$.

Proof. Using $|\cos(rt)| \leq 1$, we have

$$|\tilde{J}_r(t)| = \frac{1}{2\pi(1+q)^r}\left|\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\cdots\frac{q^{v-i}}{(1+q)^i}\left\{\sum_{m=0}^{i}\binom{i}{m}q^{i-m}\frac{\cos\left(m+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right]\right|$$

$$\le \frac{1}{2\pi(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\cdots\frac{q^{v-i}}{(1+q)^i}\left\{\sum_{m=0}^{i}\binom{i}{m}q^{i-m}\frac{\pi}{t}\right\}\right]$$
$$\le \frac{1}{2t(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\cdots\frac{q^{v-i}}{(1+q)^i}\left\{\sum_{m=0}^{i}\binom{i}{m}q^{i-m}\right\}\right]$$
$$\le \frac{1}{2t(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}q^{j-h}\right]$$
$$\le \frac{1}{2t(1+q)^r}\sum_{j=0}^{r}\left[\binom{r}{j}q^{r-j}\right]$$
$$\le \frac{1}{2t}$$
$$= O\left(\frac{1}{t}\right).$$

Lemma 4.4 $|\tilde{J}(t)| = O\left(\frac{l}{t}\right)$, *for* $\frac{l}{r} \le t \le \pi$.

Proof. Appling $\sin\left(\frac{t}{2}\right) \ge \frac{t}{\pi}$, we have

$$|\tilde{J}_r(t)| = \frac{1}{2\pi(1+q)^r}\left|\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\cdots\frac{q^{v-i}}{(1+q)^i}\left\{\sum_{m=0}^{i}\binom{i}{m}q^{i-m}\frac{\cos\left(m+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right]\right|$$

$$\le \frac{1}{2\pi(1+q)^r}\left|\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\cdots\frac{q^{v-i}}{(1+q)^i}\left\{\sum_{m=0}^{i}\binom{i}{m}q^{i-m}\frac{\cos\left(m+\frac{1}{2}\right)t}{\frac{t}{\pi}}\right\}\right]\right|$$
$$\le \frac{1}{2t(1+q)^r}\left|\sum_{j=0}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\cdots\frac{q^{v-i}}{(1+q)^i}Re\left\{\sum_{m=0}^{i}\binom{i}{m}q^{i-m}e^{imt}\right\}\right]\right|$$

$$\le \frac{1}{2t(1+q)^r}\left|\sum_{j=0}^{\tau-1}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\cdots\frac{q^{v-i}}{(1+q)^i}Re\left\{\sum_{m=0}^{i}\binom{i}{m}q^{i-m}e^{imt}\right\}\right]\right|$$
$$+\frac{1}{2t(1+q)^r}\left|\sum_{j=\tau}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\cdots\frac{q^{v-i}}{(1+q)^i}Re\left\{\sum_{m=0}^{i}\binom{i}{m}q^{i-m}e^{imt}\right\}\right]\right|$$

$$\le \frac{1}{2t(1+q)^r}\sum_{j=0}^{\tau-1}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\cdots\frac{q^{v-i}}{(1+q)^i}Re\left\{\sum_{m=0}^{i}\binom{i}{m}q^{i-m}\right\}\right]|e^{imt}|$$

$$+\frac{1}{2t(1+q)^r}\sum_{j=\tau}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\cdots\frac{q^{v-i}}{(1+q)^i}max_{0\le m\le i}\sum_{m=0}^{i}\binom{i}{m}q^{i-m}e^{imt}\right]$$

$$\leq \frac{1}{2t(1+q)^r}\sum_{j=0}^{\tau-1}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\cdots q^{v-i}\right]$$

$+\frac{1}{2t(1+q)^r}\sum_{j=\tau}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\frac{q^{j-h}}{(1+q)^h}\cdots q^{v-i}\right]$

$\leq \frac{1}{2t(1+q)^r}\sum_{j=0}^{\tau-1}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}q^{j-h}\right]$

$+\frac{1}{2t(1+q)^r}\sum_{j=\tau}^{r}\left[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}q^{j-h}\right]$

$=\frac{1}{2t(1+q)^r}\sum_{j=0}^{\tau-1}\binom{r}{j}q^{r-j}+\frac{1}{2t(1+q)^r}\sum_{j=\tau}^{r}\binom{r}{j}q^{r-j}$

$=\frac{1}{2t(1+q)^r}\sum_{j=0}^{r}\binom{r}{j}q^{r-j}$

Hence $|\tilde{J}_r(t)| = O(1/t)$.

5 Proof of Theorem 3.1

Proof. We have

$$s_r(g;x)-g(x)=\frac{1}{2\pi}\int_0^{\pi}\frac{\phi(t)sin\left(r+\frac{1}{2}\right)t}{sin\left(\frac{t}{2}\right)}dt,$$

and

$t_r^{E^qE^qE^q\ldots E^q}(g;x)-g(x)=\frac{1}{2\pi(1+q)^r}|\sum_{j=0}^{r}[\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\cdots\frac{q^{v-i}}{(1+q)^i}$

$\int_0^{\pi}\phi(t)\left\{\sum_{m=0}^{i}\binom{i}{m}q^{i-m}\frac{sin\left(m+\frac{1}{2}\right)t}{sin\frac{t}{2}}\right\}]|$

$=\int_0^{\pi}\phi(t)J_r(t)dt$

$=\left[\int_0^{\frac{1}{r}}\phi(t)+\int_{\frac{1}{r}}^{\gamma}\phi(t)+\int_{\gamma}^{\pi}\phi(t)\right]J_r(t)dt$

$=R_1+R_2+R_3$ (5.1)

Applying Lemma 4.1, condion 3.2 and 3.3 and second mean value theorem is applying for second term integral , we have

$|R_1|\leq\int_0^{\frac{1}{r}}|\phi(t)J_r(t)|dt$

$=O(r)\left[\int_o^{\frac{1}{r}}|\phi(t)|dt\right]$

$=O(r)\left[O\left(\frac{\frac{1}{r}}{\alpha(r).P_r}\right)\right]$

$=O\left(\frac{1}{logr}\right)=O(1)\quad as\quad r\to\infty$ (5.2)

Apply Lemma 4.2, condition 3.2 and 3.3 and second mean value theorem is appling for second term integral, We have

$$|R_2| \leq \int_{\frac{1}{r}}^{\gamma} |\phi(t)J_r(t)|dt$$

$$= O\left[\int_{\frac{1}{r}}^{\gamma} |\phi(t)|\left(\frac{1}{t}\right)\right]$$

$$= O\left[\left\{\frac{1}{t}\phi(t)\right\}_{\frac{1}{r}}^{\gamma} + \int_{\frac{1}{r}}^{\gamma} o\left\{\frac{\phi(t)}{t^2}\right\}dt\right]$$

$$= O\left[o\left\{\frac{1}{\alpha\left(\frac{1}{t}\right).P_t}\right\}_{\frac{1}{r}}^{\gamma} + \int_{\frac{1}{r}}^{r} o\left(\frac{\frac{1}{t}}{\alpha\left(\frac{1}{t}\right).P_t}\right)dt\right]$$

$$= O\left[o\left\{\frac{1}{\alpha(r)P_r}\right\} + \int_{\frac{1}{\gamma}}^{r} o\left(\frac{\frac{1}{u}}{\alpha(u)p_u}\right)du\right]$$

$$= O\left(\frac{1}{\alpha(r)P_r}\right) + O\left(\frac{\frac{1}{r}}{\alpha(r)p_r}\right)\int_{\frac{1}{r}}^{r} 1du$$

$$= O\left(\frac{1}{logr}\right) + O\left(\frac{1}{logr}\right) = O(1), \quad as \quad r \to \infty \tag{5.3}$$

Applying Riemann-Lebesgue theorem and regularity condition of summability, we have

$$|R_3| \leq \int_{\frac{1}{r}}^{\gamma} |\phi(t)J_r(t)|dt$$

$$= O(1), \quad as \quad r \to \infty. \tag{5.4}$$

Collecting (5.1)-(5.4), we get

$$|t_r^{E^qE^qE^q\ldots Eq}(g;x) - g(x)| = O(1), as \quad r \to \infty.$$

Hence complete the proof.

6 Proof of Theorem 3.2

Proof. We have

$$\tilde{s}_r(g;x) - \tilde{g}(x) = \frac{1}{2\pi}\int_0^{\pi} \frac{\psi(t)cos\left(r+\frac{1}{2}\right)t}{sin\left(\frac{t}{2}\right)}dt$$

and

$$\tilde{t}_r^{E^qE^q\ldots E^q}(g;x) - \tilde{g}(x) = \frac{1}{2\pi(1+q)^r}\sum_{j=0}^{r} [\binom{r}{j}\frac{q^{r-j}}{(1+q)^j}\sum_{h=0}^{j}\binom{j}{h}\cdots\frac{q^{v-i}}{(1+q)^i}$$

$$\int_0^{\pi} \psi(t)\left\{\sum_{m=0}^{i}\binom{i}{m}q^{i-m}\frac{cos\left(m+\frac{1}{2}\right)t}{sin\frac{t}{2}}\right\}dt]$$

$$= \int_0^{\pi} \psi(t)\tilde{J}_r(t)dt$$

$$= \left[\int_0^{\frac{1}{r}} \psi(t) + \int_{\frac{1}{r}}^{\gamma} \psi(t) + \int_{\gamma}^{\pi} \psi(t)\right] \tilde{J}_r(t)dt$$

$$= \tilde{R}_1 + \tilde{R}_2 + \tilde{R}_3, \text{say.}$$

(6.1)

Applying 4.3, condition 3.4 and 3.5 and second mean value theorem is applying for second term integral , we have

$$|\tilde{R}_1| \le \int_0^{\frac{1}{r}} |\psi(t)\tilde{J}_r(t)|dt$$

$$= \left[\int_o^{\frac{1}{r}} |\psi(t)| \frac{1}{t} dt\right]$$

$$= O(r)\left[\int_o^{\frac{1}{r}} |\psi(t)|dt\right]$$

$$= O(r)\left[o\left(\frac{\frac{1}{r}}{\alpha(r).P_r}\right)\right]$$

$$= O\left(\frac{1}{logr}\right) = O(1), \text{as} \quad r \to \infty.$$

(6.2)

Apply Lemma 4.4, condition 3.4 and 3.5 and second mean value theorem is applying for second term integral, we have

$$|\tilde{R}_2| \le \int_{\frac{1}{r}}^{\gamma} |\psi(t)\tilde{J}_r(t)|dt$$

$$= O\left[\int_{\frac{1}{r}}^{\gamma} |\psi(t)| \left(\frac{1}{t}\right) dt\right]$$

$$= O\left[\left\{\frac{1}{t}\Psi(t)\right\}_{\frac{1}{r}}^{\gamma} + \int_{\frac{1}{r}}^{\gamma} o\left\{\frac{\psi(t)}{t^2}\right\} dt\right]$$

$$= O\left[o\left\{\frac{1}{\alpha(r)}.p_r\right\} + \int_{\frac{1}{r}}^{\gamma} o\left(\frac{\frac{1}{t}}{\alpha\left(\frac{1}{t}\right).p_t}\right) dt\right]$$

$$= O\left[o\left\{\frac{1}{\alpha(r)P_r}\right\} + \int_{\frac{1}{\gamma}}^{r} o\left(\frac{\frac{1}{u}}{\alpha(u)p_u}\right) du\right]$$

$$= O\left(\frac{1}{\alpha(r)P_r}\right) + O\left(\frac{\frac{1}{r}}{\alpha(r)p_r}\right) \int_{\frac{1}{r}}^{r} 1du$$

$$= O\left(\frac{1}{logr}\right) + O\left(\frac{1}{logr}\right) = O(1), \quad as \quad r \to \infty. \qquad (6.3)$$

Applying Riemann-Lebesgue theorem and regularity condition of summability, we have

$$|\tilde{R}_3| \le \int_{\frac{1}{r}}^{\gamma} |\psi(t)\tilde{J}_r(t)|dt$$

$$= O(1), \quad as \quad r \to \infty.$$

(6.4)

Collecting (6.1)-(6.4), we get

$$\left|\tilde{t}_r^{E^qE^qE^q\ldots Eq}(g;x)-\tilde{g}(x)\right|=O(1), as \quad r\to\infty.$$

Hence complete the proof.

7 Corollaries

Below some corollaries are given, which are derived from our theorems 3.1 and 3.2

7.1 Corollary

If we take q=1 in theorem 3.1 then n- Eular product summability (E,q) $(E,q)(E,q)$...(E,q) reduce to $(E,1)$ $(E,1)$ $(E,1)$...$(E,1)$, then

$$|t_r^{E^1E^1E^1\ldots E1}(g;x)-g(x)|=O(1), as \quad r\to\infty.$$

7.2 Corollary

If we take q=1 in theorem 3.2, then n eular product summaility

(E,q) $(E,q)(E,q)$...(E,q) reduce to $(E,1)$ $(E,1)$ $(E,1)$...$(E,1)$, then

$$|\tilde{t}_r^{E^1E^1E^1\ldots E1}(g;x)-\tilde{g}(x)|=O(1), as \quad r\to\infty.$$

7.3 Corollary

If we take n=2 in our results 3.1 then $n-$ Euler product summability $(E,q)(E,q)(E,q)\ldots(E,q)$ reduces to $(E,q)(E,q)$ double Euler summability,

then

$$|t_r^{E^qE^q}(g;x)-g(x)|=O(1), as \quad r\to\infty.$$

7.4 Corollary

If we take n=2 in our results 3.2 then $n-$ Euler product summability $(E,q)(E,q)(E,q)\ldots(E,q)$ reduces to $(E,q)(E,q)$ double Euler summability,

then

$$|\tilde{t}_r^{E^qE^q}(g;x)-\tilde{g}(x)|=O(1), as \quad r\to\infty$$

7.5 Corollary

If we take n=2 and q=1 in our result 3.1, $n-$ Euler product summability $(E,q)(E,q)(E,q)\ldots(E,q)$ reduces to $(E,1)(E,1)$

then

$$|t_r^{E^1E^1}(g;x)-g(x)|=O(1), as \quad r\to\infty$$

7.6 Corollary

If we take n=2 and q=1 in our result 3.2, $n-$ Euler product summability $(E,q)(E,q)(E,q)\ldots(E,q)$ reduces to $(E,1)(E,1)$

then

$$|\tilde{t}_r^{E^1E^1}(g;x)-\tilde{g}(x)|=O(1), as \quad r\to\infty.$$

7.7 Corollary

If we consider n=3 in our theorem 3.1, $n-$ Euler product summability $(E,q)(E,q)(E,q)\ldots(E,q)$ reduces to $(E,q)(E,q)(E,q)$ i.e. triple E^q summability

then

$$|t_r^{E^qE^qE^q}(g;x)-g(x)|=O(1), as \quad r\to\infty.$$

7.8 Corollary

If we consider n= 3 in our theorem 3.2, $n-$ Euler product summability $(E,q)(E,q)(E,q)\ldots(E,q)$ reduces to $(E,q)(E,q)(E,q)$ i.e. triple E^q summability

then

$$|\tilde{t}_r^{E^qE^qE^q}(g;x)-\tilde{g}(x)|=O(1), as \quad r\to\infty.$$

7.9 Corollary

If we take n=3 and q=1 in our result 3.1, $n-$ Euler product summability $(E,q)(E,q)(E,q)\ldots(E,q)$ reduces to $(E,1)(E,1)(E,1)$ i.e. triple E^1 summability

then

$$|t_r^{E^1E^1E^1}(g;x)-g(x)|=O(1), as \quad r\to\infty.$$

7.10 Corollary

If we take n=3 and q=1 in our result 3.2, $n-$ Euler product summability $(E,q)(E,q)(E,q)\ldots(E,q)$ reduces to $(E,1)(E,1)(E,1)$ i.e. triple E^1 summability

then

$$|\tilde{t}_r^{E^1E^1E^1}(g;x)-\tilde{g}(x)|=O(1), as \quad r\to\infty$$

8 Particular case

8.1

In view of Corollary 7.3 and 7.4, theorem 1 and 2 [5] are particular cases of our Theorem 3.1 and 3.2 respectively.

8.2

In view of Corollary 7.5 and 7.6, theorem 1 and 2 [4] are particular cases of our Theorem 3.1 and 3.2 respectively.

8.3

In view of Corollary 7.7 and 7.8, theorem 3.1 and 3.2 [3] are particular cases of our Theorem 3.1 and 3.2 respectively.

8.4

In view of Corollary 7.9 and 7.10, theorem 3.1 and 3.2 [7] are particular cases of our Theorem 3.1 and 3.2 respectively.

9 Conclusion

The results of the paper are aimed to formulate the problem of approximation of function g and their conjugates $\tilde{g}$ by iterates of Euler sum of their Fourier series and conjugate Fourier series respectively.

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