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A New Type of $(\epsilon) -$ Lorentzian Para-Sasakian Manifolds

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Abstract:

The current investigation commences by introducing a novel category termed $(\epsilon) -$Lorentzian para-Sasakian manifolds, employing the generalized symmetric metric connection of a specific type(α, β). Several fundamental outcomes concerning with these manifolds are derived. Subsequently, we delve into the examination of conformally flat and Weyl-semi-symmetric $(\epsilon) -$ Lorentzian para-Sasakian manifolds, utilizing the generalized symmetric metric connection of the type(α, β).

Keywords: $(\epsilon) -$Lorentzian para-Sasakian manifolds, generalized symmetric metric connection of the type(α, β), Conformally flat, $\eta -$Einstein manifold Weyl-semisymmetric and quasi-constant curvature.

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1. Introduction

In [2], the authors introduced and studied the notion of special conformally flat space. Bejancu et. al. [1], introduced the concept of (ϵ)-Sasakian manifolds. Also,Xufeng and Xiaoli [4] showed that every (ϵ)-Sasakian manifold must be a real hypersurface of some indefinite Kaehler manifold. T. Takahashi introduced almost contact manifolds equipped with associated indefinite metrics in 1969 and studied Sasakian manifolds equipped with an associated indefinite metric. Since the substantial role that Sasakian manifolds with indefinite metrics play in physics [5], our inclination naturally lies in exploring diverse contact manifolds with indefinite metrics. Recently, in 2009, U.C. De& Sarkar [8], studied(ϵ)-Kenmotsu manifolds. K. Matsumoto [7], introduced the notion of Lorentzian Para-Sasakian manifolds. I. Mihai and R. Rosca [9], defined the same notion independently and several authors [10], [11], [12], [14], [16], [17] also studied different structures.

A linear connection $\overline{\nabla}$ on a Riemannian manifold $M$is suggested to be a generalized symmetric connection if its torsion tensor T is defined as:

$$T(X,Y) = \alpha\{u(Y)X - u(X)Y\} + \beta\{u(Y)\phi X - u(X)\phi Y\} \tag{1.1}$$

For any vector field X and Y on M, where α and β are constant functions on M[13], ϕ can be viewed as tensor of type (1, 1) and u is regarded as a 1-form connected with the vector field which has a non-vanishing smooth non-null unit. A linear metric connection satisfying the equation (1.1) is called generalized symmetric metric connection of type(α, β).

Moreover, the connection $\overline{\nabla}$ is said to be metric connection if $\overline{\nabla} g = 0$, with g as metric tensor [15].

This paper is organized as follows: Section I, is introductory. Section II, is devoted to preliminaries. In section III, we define (ϵ) −Lorentzian para-Sasakian manifold with generalized symmetric metric connection. We also give some basic results of suchtype of manifold in the same section. In Section IV, we have studied conformally flat (ϵ) −Lorentzian para Sasakian manifold with generalized symmetric metric connection. In section V, we consider Weyl-semi-symmetric (ϵ) − Lorentzian para-Sasakian manifold.

2. Preliminaries

An n-dimensional differential manifold is called an (ϵ) − Lorentzian para-Sasakian manifold i.e. (ϵ)-LP Sasakian manifold, if it admits a (1,1) tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfies

$$\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1 \tag{2.1}$$

$$g(\xi, \xi) = -\epsilon, \quad \eta(X) = \epsilon g(X, \xi), \phi\xi = 0, \quad \eta(\phi X) = 0 \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \epsilon\, \eta(X)\, \eta(Y) \tag{2.3}$$

$$(\nabla_X \phi)Y = g(X, Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi \tag{2.4}$$

$$\nabla_X \xi = \epsilon\phi X \tag{2.5}$$

$$(\nabla_X \eta)X = g(\phi X, Y) \tag{2.6}$$

for arbitrary vector field X and Y; where ∇ denotes the operator of covariant differentiation with respect to the metric [7], [8]

On an n- dimensional (ϵ) − Lorentzian para-Sasakian manifold with structure (ϕ, ξ, η, g), the following results hold [8].

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.7}$$

$$R(\xi, X)\xi = \epsilon g(X, Y)\xi - \eta(X)Y \tag{2.8}$$

$$g(R(X, Y)Z, \xi) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \tag{2.9}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y) \tag{2.10}$$

$$S(X, \xi) = (n - 1)\eta(Y) \tag{2.11}$$

$$QX = \epsilon(n - 1)\xi \tag{2.12}$$

for any vector fields X, Y and Z; where R is the Riemannian curvature tensor, S is the Ricci tensor and Q is the Ricci operator given by

$$g(QX, Y) = S(X, Y).$$

We note that, if $\epsilon = 1$and the structure vector field ξ is space like, then an (ϵ)-LP Sasakian manifold is a usual LP-Sasakian manifold.

An (ϵ)-LP Sasakian manifold is said to be Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y) = \lambda g(X,Y)$$

where λ is a constant.

Definition 2.1. An (ϵ)-LP Sasakian manifold will be called a manifold of quasi-constant curvature if the curvature tensor $\tilde{R}$ of type (0,4) satisfies the condition

$$\tilde{R}(X,Y,Z,W) = a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + b[T(Y)T(Z)g(X,W)$$
$$-T(X)T(Z)g(Y,W) + T(X)T(W)g(Y,Z) - T(Y)T(W)\}g(X,Z)\} \quad (2.13)$$

where$\tilde{R}(X,Y,Z,W) = g(R(X,Y)Z,W)$

R is the curvature tensor of type (1,3);a, b are scalar functions and ρ is a unit vector field defined by

$$g(X,\rho) = T(X) \quad (2.14)$$

The notion of quasi-constant curvature for Riemannian manifolds was given by Chen and Yano [2].

Definition 2.2. An (ϵ)-LP Sasakian manifold will be called η-Einstein manifold if the Ricci tensor S of type (0, 2) satisfies [2]

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$$

where a and b are scalar functions.

Definition 2.3.A type of Riemannian manifold whose curvature tensor $\tilde{R}$ of type (0, 4) satisfies the condition

$$\tilde{R}(X,Y,Z,W) = F(Y,Z)F(X,W) - F(X,Z)F(Y,W) \quad (2.15)$$

is called a special manifold with the associate symmetric tensor F of type (0,2) and is denoted by $\psi(F)_n$.

In 1956, S. S. Chern [3] studied such type of manifolds. These manifolds are important for the following reasons. Firstly, for possessing some remarkable properties related to curvature and characteristic classes and secondly, for containing a manifold of quasi-constant curvature [2].

Definition 2.4. An (ϵ)-LP Sasakian manifold will be called Weyl-semi-symmetric if it satisfies

$$(R.(X,Y).C)(Y,Z)W = 0$$

where $R(X,Y)$ denotes the curvature operator and $C(Y,Z)W$ is the Weyl-conformal curvature tensor.

3. On (ϵ) −Lorentzian Para-Sasakian Manifold with Parallelized Generalized Symmetric Metric Connection

Theorem 3.1. For an (ϵ) −Lorentzian para-Sasakian manifold, the generalized symmetric metric connection $\overline{\nabla}$ of type (α, β) is given by

$$\overline{\nabla}_X Y = \nabla_X Y + \alpha\{\eta(Y)X - \epsilon g(X,Y)\xi\} + \beta\{\eta(Y)\phi X - \epsilon g(\phi X,Y)\xi\} \quad (3.1)$$

Proof: The relation between a linear connection $\overline{\nabla}$ and Levi-Civita connection ∇ is given by

$$\overline{\nabla}_X Y = \nabla_X Y + H(X, Y) \tag{3.2}$$

for all vector field X and Y. The following equation is such that $\overline{\nabla}$ is a generalized symmetric metric connection of ∇. In which H is viewed as a tensor of type (1, 2), given by

$$H(X, Y) = \frac{1}{2}[T(X, Y) + T'(X, Y) + T'(Y, X) \tag{3.3}$$

where T is viewed as the torsion tensor of $\overline{\nabla}$ and

$$g(T'(X, Y), W) = g(T(W, X), Y) \tag{3.4}$$

Replacing W by ξ, using (1.1), (2.1), (2.2) and (2.3) in (3.4), we have

$$g(T'(X, Y), \xi) = \alpha\eta(X)g(Y, \xi) - \epsilon\alpha g(X, Y)g(\xi, \xi) + \beta\eta(X)g(\phi Y, \xi) - \epsilon\beta g(\phi X, Y)g(\xi, \xi)$$

$$T'(X, Y) = \alpha\eta(X)Y - \epsilon\alpha g(X, Y)\xi + \beta\eta(X)\phi Y - \epsilon\beta g(\phi X, Y)\xi \tag{3.5}$$

$$T'(Y, X) = \alpha\eta(Y)X - \epsilon\alpha g(X, Y)\xi + \beta\eta(Y)\phi X - \epsilon\beta g(X, \phi Y)\xi \tag{3.6}$$

Using equations (1.1), (2.2), (3.5) and (3.6) in (3.3), we obtained

$$H(X, Y) = \alpha\{\eta(Y)X - \epsilon g(X, Y)\xi\} + \beta\{\eta(Y)\phi X - \epsilon g(\phi X, Y)\xi\} \tag{3.7}$$

Using above in (3.2) proves to our assertion.

Now, substituting $Y = \xi$ in equation (3.1), we obtained

$$\overline{\nabla}_X \xi = \nabla_X \xi - \alpha\{X + \eta(X)\xi\} - \beta(\phi X)$$

If the vector field ξ representing a unit of time like is aligned in parallel according to a generalized symmetric metric connection, that is $\overline{\nabla}_X \xi = 0$, we have

$$\nabla_X \xi = \alpha\{X + \eta(X)\xi\} + \beta(\phi X) \tag{3.8}$$

then $\overline{\nabla}$ is called generalized symmetric metric ξ connection.

Using equation (2.2) and (2.3) and replacing Y by ϕY in equation (3.1), we have

$$\overline{\nabla}_X \phi Y = \nabla_X \phi Y - \epsilon\alpha g(X, \phi Y)\xi - \epsilon\beta\{g(X, Y) + \epsilon\, \eta(X)\, \eta(Y)\}\xi$$

Using covariant differentiation in above, we have

$$(\overline{\nabla}_X \phi)Y = (\nabla_X \phi)Y - \alpha\{\eta(Y)\phi X + \epsilon g(X, \phi Y)\xi\} - \beta\{\eta(Y)X + \epsilon g(X, Y)\xi + \eta(X)\eta(Y)\xi\} \tag{3.9}$$

Now, as we know that

$$(\overline{\nabla}_X \eta)Y = \overline{\nabla}_X \eta Y + \eta(\overline{\nabla}_X Y)$$

Using equation (3.1) in above, we obtained

$$(\overline{\nabla}_X \eta)Y = (\nabla_X \eta)Y - \alpha\{\epsilon g(X, Y) + \eta(X)\eta(Y)\} - \epsilon\beta g(\phi X, Y) \tag{3.10}$$

for any vector field X and Y on M.

Now, we suppose that

$(\overline{\nabla}_X \phi)Y = 0$ and $(\overline{\nabla}_X \eta)Y = 0$.

Then the equation (3.9) and (3.10) will be as follows

$$(\nabla_X \phi)Y = \alpha\{\eta(Y)\phi X + \epsilon g(X, \phi Y)\xi\} + \beta\{\eta(Y) + \epsilon g(X, Y)\xi + 2\eta(X)\,\eta(Y)\xi\} \quad (3.11)$$

and

$$(\nabla_X \eta)Y = \alpha\{\epsilon g(X, Y) + \eta(X)\eta(Y)\} + \epsilon\beta g(\phi X, Y) \quad (3.12)$$

A linear connection $\overline{\nabla}$ satisfying equation (3.11) and

(3.12) is called ϕ −parallel generalized symmetric connection and η −parallel generalized symmetric connection respectively.

Definition 3.2. Let M be a (ϵ) −Lorentzian para-Sasakianmanifold. If M satisfies the equations (3.8), (3.11) and (3.12), then M is called (ϵ) −Lorentzian para-Sasakian manifold with parallelized generalized symmetric metric connection, this means that the connection $\overline{\nabla}$ is generalized symmetric metric ξ connection, ϕ −parallel generalized symmetric connection and η −parallel generalized symmetric connection

Proposition 3.3. Let M be a (ϵ) −Lorentzian para-Sasakian manifold with parallelized generalized symmetric metric connection then following relation holds

$$(\overline{\nabla}_X \phi)Y = (\epsilon - \beta)\,\eta(Y)X + 2(\epsilon + 1)\eta(X)\,\eta(Y)\xi + (1 - \epsilon\beta)g(X, Y)\xi - \alpha\eta(Y)\phi X - \epsilon\alpha g(X, \phi Y)\xi \quad (3.13)$$

for all vector field X and Y on M.

Proposition 3.4. In a (ϵ) − Lorentzian para-Sasakian manifold with parallelized generalized symmetric metric connection, curvature tensor R and Ricci tensor S and Ricci operator Q has the following relations

$$R(X, Y)\xi = (\alpha^2 + \beta^2)\{\eta(Y)X - \eta(X)Y\} + 2\alpha\beta\{\eta(Y)\phi X - \eta(X)\phi Y\} + (X\beta)\phi Y - (Y\beta)\phi X + (X\alpha)\phi^2 Y - (Y\alpha)\phi^2 X \quad (3.14)$$

$$\eta(R(X, Y)Z = \epsilon(\alpha^2 + \beta^2)\{\eta(X)g(Y, Z) - \eta(Y)g(X, Z)\} \quad (3.15)$$

$$R(\xi, X)Y = (\alpha^2 + \beta^2)\{\epsilon g(X, Y)\xi - \eta(Y)X\} + \{\xi\beta - 2\alpha\beta\eta(Y)\}\phi X + (\xi\alpha)X + \epsilon\alpha\xi g(X, Y)Y - (X\alpha)\xi - (X\alpha)\eta(Y)Y \quad (3.16)$$

$$R(\xi, X)\xi = (\alpha^2 + \beta^2 + \xi\alpha)\phi^2 X + (2\alpha\beta + \xi\beta)\phi X \quad (3.17)$$

$$S(X, \xi) = (n - 1)(\alpha^2 + \beta^2)\eta(X) \quad (3.18)$$

$$Q\xi = \epsilon(n - 1)(\alpha^2 + \beta^2)\xi \quad (3.19)$$

for any $X, Y, Z \in \chi(M)$.

Proof: As we know that, curvature tensor is

$$R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi \quad (3.20)$$

Using equation (3.8) and (3.12), we obtained

$$\nabla_X\nabla_Y\xi = (X\alpha)(\phi^2Y) + \alpha(\nabla_XY) + \epsilon\alpha^2 g(X,Y + \alpha^2\eta(X)\eta(Y)\xi + \epsilon\alpha\beta g(\phi X,Y)\xi$$

$$+\alpha^2\eta(Y)X + \alpha^2\eta(X)\eta(Y)\xi + \alpha\beta\eta(Y)\phi X + (X\beta)(\phi Y) + \beta(\nabla_X\phi Y) \qquad (3.21)$$

Similarly,

$$\nabla_Y\nabla_X\xi = (Y\alpha)(\phi^2X) + \alpha(\nabla_YX) + \epsilon\alpha^2 g(X,Y)\xi + \alpha^2\eta(X)\eta(Y)\xi + \epsilon\alpha\beta g(X,\phi Y)\xi$$

$$+\alpha^2\eta(X)Y + \alpha^2\eta(X)\eta(Y)\xi + \alpha\beta\eta(X)\phi Y + (Y\beta)(\phi X) + \beta(\nabla_Y\phi X) \qquad (3.22)$$

Using equation (3.8) and (3.11), we can write as

$$\nabla_{[X,Y]}\xi = \alpha\{[X,Y] + \eta([X,Y])\xi\} + \beta\phi[X,Y]$$

$$\nabla_{[X,Y]}\xi = \alpha(\nabla_XY) - \alpha(\nabla_YX) + \beta\nabla_X\phi Y - \beta\nabla_Y\phi X - \alpha\beta\eta(Y)\phi X + \alpha\beta\eta(X)\phi Y$$

$$+\beta^2\eta(X)Y - \beta^2\eta(Y)X \qquad (3.23)$$

Using equation (3.20), (3.21), (3.22) and (3.23), we obtained

$$R(X,Y)\xi = (\alpha^2+\beta^2)\{\eta(Y)X - \eta(X)Y + 2\alpha\beta\{\eta(Y)\phi X - \eta(X)\phi Y\} + (X\beta)(\phi Y)$$

$$-(Y\beta)(\phi X) + (X\alpha)(\phi^2Y) - (Y\alpha)(\phi^2X) \qquad (3.24)$$

Interchanging Z and ξ in (3.14), we have

$$g(R(X,Y)Z,\xi) = (\alpha^2+\beta^2)\{\epsilon g(Y,Z)g(X,\xi) - \epsilon g(X,Z)g(Y,\xi)\}$$

$$+2\alpha\beta\{\epsilon g(Y,Z)g(\phi X,\xi) - \epsilon g(X,Z)g(\phi Y,\xi)\} + (X\beta)g(\phi Y,\xi)$$

$$-(Y\beta)g(\phi X,\xi) + (X\alpha)g(\phi^2Y,\xi) - (Y\alpha)g(\phi^2X,\xi)$$

$$\eta(R(X,Y)Z) = \epsilon(\alpha^2+\beta^2)\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\} \qquad (3.25)$$

From equation (3.14), we have

$$R(\xi,X)Y = (\alpha^2+\beta^2)\{\epsilon g(X,Y)\xi - \eta(Y)X\} + \{\xi\beta - 2\alpha\beta\eta(Y)\}\phi X + (\xi\alpha)X$$

$$+\epsilon\alpha\xi g(X,Y)Y - (X\alpha)\xi - (X\alpha)\eta(Y)Y \qquad (3.26)$$

Putting Y for ξ in above equation, we have

$$R(\xi,X)\xi = (\alpha^2+\beta^2)\{X + \eta(X)\xi\} + \{\xi\beta + 2\alpha\beta\}\phi X + \xi\alpha\{X + \eta(X)\xi\}$$

$$R(\xi,X)\xi = (\alpha^2+\beta^2+\xi\alpha)\phi^2X + (\xi\beta + 2\alpha\beta)\phi X \qquad (3.27)$$

Now, from equation (3.15), we have

$$g(R(X,Y)Z,\xi) = (\alpha^2+\beta^2)\{\epsilon g(X,\xi)g(Y,Z) - \epsilon g(Y,\xi)g(X,Z)\}$$

Putting $Y = Z = e_i$, where e_i is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i and $i = 1,2,\ldots\ldots,n$ then, we get

$$g(R(X,e_i)e_i,\xi) = (\alpha^2+\beta^2)\{\epsilon g(X,\xi)g(e_i,e_i) - \epsilon g(e_i,\xi)g(X,e_i)\}$$

$$S(X,\xi) = (n-1)(\alpha^2+\beta^2)\eta(X) \qquad (3.28)$$

Using equation (3.28), we have

$$Q\xi = \epsilon(n-1)(\alpha^2+\beta^2)\xi \tag{3.29}$$

In a 3-dimensional manifold, the curvature tensor is given by

$$R(X,Y)Z = S(Y,Z)X - g(X,Z)QY + g(Y,Z)Q - S(X,Z)Y - \frac{r}{2}\{g(Y,Z)X - g\{X,Z)Y\} \tag{3.30}$$

Theorem 3.5: In a 3 dimensional $(\epsilon)-$Lorentzian para-Sasakian manifold with parallelized generalized symmetric metric connection, the Ricci operator Q is given by

$$QX = \{\frac{r}{2} - 2\epsilon(\alpha^2+\beta^2) + \epsilon\xi\alpha\}X + \{\frac{r}{2} - 4\epsilon(\alpha^2+\beta^2) + \epsilon\xi\alpha\}\eta(X)\xi + \epsilon\{\xi\beta + 2\alpha\beta\} \tag{3.31}$$

for any $X, Y, Z \in \chi(M)$.

Proof: In a 3-dimensional $(\epsilon)-$Lorentzian para-Sasakian manifold with parallelized generalized symmetric metric connection, the curvature tensor is

$$R(X,Y)\xi = S(Y,\xi)X - S(X,\xi)Y - g(X,\xi)QY + g(Y,\xi)QX - \frac{r}{2}\{g(Y,\xi)X - g(X,\xi)Y\}$$

Using equation (3.14) and (3.18) in above equation, then we have

$$\left\{\frac{r}{2\epsilon} - 2\left(\alpha^2+\beta^2\right)\right\}\{\eta(Y)X - \eta(X)Y\} + 2\alpha\beta\{\eta(Y)\phi X - \eta(X)\phi Y\}$$
$$+(X\beta)\phi Y - (Y\beta)\phi X + (X\alpha)\phi^2 Y - (Y\alpha)\phi^2 X = \frac{1}{\epsilon}\eta(Y)QX - \frac{1}{\epsilon}\eta(X)QY$$

Putting $Y = \xi$ in above equation, we obtained

$$QX = \{\frac{r}{2} - 2\epsilon(\alpha^2+\beta^2) + \epsilon\xi\alpha\}\{X + \eta(X)\xi\} + \epsilon(2\alpha\beta + \xi\beta))\phi X - 2\epsilon(\alpha^2+\beta^2)\eta(X)\xi$$

Thus, from above we have result (3.31).

Theorem 3.6: In a 3 dimensional $(\epsilon)-$Lorentzian Para-Sasakian manifold with parallelized generalized symmetric metric connection, scalar and Ricci curvature are given by the following expressions

$$r = 2\left(1 + \frac{8}{\epsilon-1}\right)(\alpha^2+\beta^2) \tag{3.32}$$

$$S(Y,Z) = 2(\alpha^2+\beta^2)\left\{\left(1 + \frac{4}{\epsilon-1}\right)g(Y,Z) + 2\eta(Y)\eta(Z)\right\} \tag{3.33}$$

for any $X, Y, Z \in \chi(M)$.

Proof: On putting ξ in place of X in equation (3.30), we have the equation as

$$R(\xi,Y)Z = S(Y,Z)\xi - g(\xi,Z)QY + g(Y,Z)Q\xi - S(\xi,Z)Y - \frac{r}{2}\{g(Y,Z)\xi - g\{\xi,Z)Y\}$$

$$\eta(R(\xi,Y)Z) = -S(Y,Z) - \eta(Z)S(Y,\xi) + \epsilon S(\xi,\xi)g(Y,Z) - \eta(Y)S(\xi,Z) + \frac{r}{2}\{g(Y,Z)$$
$$+ \epsilon\eta(Y)\eta(Z)\}$$

Using equation (3.15), (3.18) in above, we have

$$\epsilon(\alpha^2+\beta^2)\{\eta(X)g(Y,Z)-\eta(Y)g(X,Z)\} = -S(Y,Z)-\eta(Z)\{2(\alpha^2+\beta^2)\eta(Y)\}$$
$$+\epsilon\{2\epsilon(\alpha^2+\beta^2)g(\xi,\xi)\}g(Y,Z)$$
$$-\eta(Y)\{2(\alpha^2+\beta^2)\eta(Y)\}+\frac{r}{2}\{g(Y,Z)+\epsilon\eta(Y)\eta(Z)\}$$

$$S(Y,Z)=\left\{\frac{r}{2}-\epsilon(\alpha^2+\beta^2)+\Lambda\right\}\{g(Y,Z)+\epsilon\eta(Y)\eta(Z)\}-\Lambda g(Y,Z)\ -2\epsilon\Lambda\eta(Y)\eta(Y) \quad (3.34)$$

where $\Lambda=2\epsilon(\alpha^2+\beta^2)$, now taking an orthonormal frame filed in the above equation over Y and Z, we have the scalar curvature.

Using equations (3.32) and (3.34), the expression of the Ricci tensor is obtained.

4. Conformally Flat (ϵ) −Lorentzian Para-Sasakian Manifold with Parallelized Generalized Symmetric Metric Connection

The Weyl conformal curvature tensor C of type (1, 3) of an n-dimensional Riemannian manifold is given by

$$C(X,Y)Z\ =\ R(X,Y)Z\ -\frac{1}{(n-2)}[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY]$$
$$+\frac{r}{(n-1)(n-2)}[g(Y,Z)X-g(X,Z)Y] \quad (4.1)$$

where Q is the Ricci operator defined by $g(QX,Y)=S(X,Y)$and r is the scalar curvature.

Let us suppose that the manifold is conformally flat. Then from the above equation, we have

$$g(R(X,Y)Z,W)\ =\frac{1}{(n-2)}[S(Y,Z)g(X,W)-S(X,Z)g(Y,W)+g(Y,Z)S(X,W)$$
$$-g(X,Z)S(Y,W)]\ -\frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)] \quad (4.2)$$

Putting $W=\xi$ in (4.2), we get

$$g(R(X,Y)Z,\xi)=\frac{1}{(n-2)}[S(Y,Z)g(X,\xi)-S(X,Z)g(Y,\xi)+g(Y,Z)S(X,\xi)-g(X,Z)S(Y,\xi)]$$
$$-\frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,\xi)-g(X,Z)g(Y,\xi)]$$

$$\epsilon\eta(R(X,Y)Z)\ =\frac{1}{(n-2)}[\epsilon S(Y,Z)\eta(X)-\epsilon S(X,Z)\eta(Y)+g(Y,Z)S(X,\xi)-g(X,Z)S(Y,\xi)]$$
$$-\frac{r}{(n-1)(n-2)}[\epsilon g(Y,Z)\eta(X)-\epsilon g(X,Z)\eta(Y)] \quad (4.3)$$

Using equation (3.15), (3.18) and (4.3), we obtained

$$(\alpha^2+\beta^2)\{\eta(X)g(Y,Z)-\eta(Y)g(X,Z)\}$$
$$=\frac{\epsilon}{(n-2)}[S(Y,Z)\eta(X)-S(X,Z)\eta(Y)+(n-1)(\alpha^2+\beta^2)g(Y,Z)\eta(X)$$

$$-(n-1)(\alpha^2+\beta^2)g(X,Z)\eta(Y)] - \frac{\epsilon r}{(n-1)(n-2)}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]$$

$$S(Y,Z)\eta(X) = S(X,Z)\eta(Y) + \left\{\frac{r}{(n-1)} - \epsilon(\alpha^2+\beta^2)\right\}\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \quad (4.4)$$

Using equation (3.18) and replacing $X = \xi$ in (4.4), we obtained

$$S(Y,Z) = \left\{\frac{r}{(n-1)} - \epsilon(\alpha^2+\beta^2)\right\}g(Y,Z) + \left\{\frac{r}{(n-1)} - \epsilon n(\alpha^2+\beta^2)\right\}\eta(Y)\eta(Z) \quad (4.5)$$

Hence, we can state the following using definition (2.2).

Theorem 4.1. An n −dimensional$(n > 1)$ conformally flat (ϵ) − Lorentzian para-Sasakian manifold with parallelized generalized symmetric metric connection is an η −Einstein manifold.

Using equation (4.5) in (4.2), we get

$$\begin{aligned} g(R(X,Y)Z,W) = \frac{1}{(n-2)}[\{&\left\{\frac{r}{(n-1)} - \epsilon(\alpha^2+\beta^2)\right\}g(Y,Z) \\ &+ \left\{\frac{r}{(n-1)} - \epsilon n(\alpha^2+\beta^2)\right\}\eta(Y)\eta(Z)\}g(X,W) \\ &-\{\left\{\frac{r}{(n-1)} - \epsilon(\alpha^2+\beta^2)\right\}g(X,Z) \\ &+ \left\{\frac{r}{(n-1)} - \epsilon n(\alpha^2+\beta^2)\right\}\eta(X)\eta(Z)\}g(Y,W) \\ &+\{\left\{\frac{r}{(n-1)} - \epsilon(\alpha^2+\beta^2)\right\}g(X,W) \\ &+ \left\{\frac{r}{(n-1)} - \epsilon n(\alpha^2+\beta^2)\right\}\eta(X)\eta(W)\}g(Y,Z) \\ &\left\{\frac{r}{(n-1)} - \epsilon(\alpha^2+\beta^2)\right\}g(Y,W) \\ &+ \left\{\frac{r}{(n-1)} - \epsilon n(\alpha^2+\beta^2)\right\}\eta(Y)\eta(W)\}g(X,Z)] \\ &- \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \end{aligned}$$

$$\begin{aligned} g(R(X,Y)Z,W) = &\left\{\frac{r - 2\epsilon(n-1)(\alpha^2+\beta^2)}{(n-1)(n-2)}\right\}\left\{\begin{matrix} g(Y,Z)g(X,W) \\ - g(X,Z)g(Y,W) \end{matrix}\right\} \\ &+ \left\{\frac{r - \epsilon n(n-1)(\alpha^2+\beta^2)}{(n-1)(n-2)}\right\}\left\{\begin{matrix} \eta(Y)\eta(Z)g(X,W) \\ -\eta(X)\eta(Z)g(Y,W) \\ +\eta(X)\eta(W)g(Y,Z) \\ -\eta(Y)\eta(W)\}g(X,Z) \end{matrix}\right\} \end{aligned}$$

In view of definition (2.1) and above relation, we have the following.

Theorem 4.2. An n −dimensional $(n > 1)$ conformally flat $(\epsilon) -$ Lorentzian para-Sasakian manifold with parallelized generalized symmetric metric connection is of quasi-constant curvature.

It is already proved that $\psi(F)_n$ contains a manifold of quasi-constant curvature as a subclass.

Let us suppose

$$F(X,Y) = pg(X,Y) + q\,(X)\eta(Y) \tag{4.6}$$

where,

$$p = \sqrt{\left\{\frac{r - 2\epsilon(n-1)(\alpha^2 + \beta^2)}{(n-1)(n-2)}\right\}}$$

and

$$q = \left\{\frac{r - \epsilon n(n-1)(\alpha^2 + \beta^2)}{(n-1)(n-2)}\right\} \sqrt{\left\{\frac{(n-1)(n-2)}{r - 2\epsilon(n-1)(\alpha^2 + \beta^2)}\right\}}$$

Now, from the equation (2.13), we have

$$\widetilde{R}(X,Y,Z,W) = F(Y,Z)F(X,W) - F(X,Z)F(Y,W)$$

Therefore, the manifold of quasi-contact curvature is a $\psi(F)_n$.

From the above equation & Theorem 4.2, we have the following result

Theorem4.3. A conformally flat $(\epsilon) -$ Lorentzian para-Sasakian manifold with generalized symmetric metric connection is a $\psi(F)_n$.

5. Weyl-Semi-symmetric $(\epsilon) -$ Lorentzian Para-Sasakian Manifold with Parallelized generalized Symmetric Metric Connection

An $(\epsilon) -$ Lorentzian para-Sasakian manifold is said to be Weyl-semi-symmetric if

$$R.C = 0 \tag{5.1}$$

From (4.1), we have

$$\eta(C(X,Y)Z) = \frac{1}{(n-2)}\left[\left\{\frac{r}{(n-1)} - \epsilon(n-1)(\alpha^2 + \beta^2)\right\}\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y) - S(Y,Z)\eta(X) + S(X,Z)\eta(Y)\}\right] \tag{5.2}$$

Putting $Z = \xi$ in above equation, we get

$$\eta(C(X,Y)\xi) = 0 \tag{5.3}$$

Again putting $X = \xi$ in equation (5.2), we get

$$\eta(C(\xi,Y)Z) = \frac{1}{(n-2)}\left[\left\{\frac{r}{(n-1)} - \epsilon(n-1)(\alpha^2 + \beta^2)\right\} + S(Y,Z) + (n-1)(\alpha^2 + \beta^2)\eta(X)\eta(Y) - g(X,Z) - \epsilon\eta(X)\eta(Y)\}\right] \tag{5.4}$$

If the manifold is Weyl-semi-symmetric then, we have

$$g[R(\xi,Y)C(U,V)W,\xi] - g[C(R(\xi,Y)U,V)W,\xi] - g[C(U,R(\xi,Y)W,\xi]$$
$$- g[C(U,V)R(\xi,Y)W,\xi] = 0 \quad (5.5)$$

From equation (3.16), we obtained

$$g(R(\xi,X)Y,\xi) = -(\alpha^2+\beta^2)\{g(X,Y)+\epsilon\eta(X)\eta(Y)\} + \epsilon(\xi\alpha)\eta(X) + (\alpha\xi)g(X,Y)\eta(Y)$$
$$+\epsilon(X\alpha) - \epsilon(X\alpha)\eta(Y)\eta(Y) \quad (5.6)$$

Using equation (5.5) and (5.6), we obtained

$$(\alpha^2+\beta^2)\{C'(U,V,W,Y)+\epsilon\eta(Y)\eta(C(U,V)W)\} - \epsilon(\xi\alpha)\eta(Y) - (\alpha\xi)C'(U,V,W,U)\eta(C(U,V)W)$$
$$- \epsilon(Y\alpha) + \epsilon(Y\alpha)\eta(C(U,V)W)\eta(C(U,V)W) + g[C((\alpha^2+\beta^2)(\epsilon g(Y,U)\xi$$
$$- \eta(U)Y) + (\xi\beta - 2\alpha\beta\eta(U))\phi Y + (\xi\alpha)Y + \epsilon\alpha\xi g(Y,U)U - (Y\alpha)\xi$$
$$- (Y\alpha)\eta(U)U,V)W,\xi] + g[C(U,(\alpha^2+\beta^2)(\epsilon g(Y,V)\xi - \eta(V)Y) + (\xi\beta$$
$$- 2\alpha\beta\eta(V))\phi Y + (\xi\alpha)Y + \epsilon\alpha\xi g(Y,V)V - (Y\alpha)\xi - (Y\alpha)\eta(V)V,W),\xi]$$
$$+ g[C(U,V)((\alpha^2+\beta^2)(\epsilon g(Y,W)\xi - \eta(W)Y) + (\xi\beta - 2\alpha\beta\eta(W))\phi Y + (\xi\alpha)Y$$
$$+ \epsilon\alpha\xi g(Y,W)W$$
$$-(Y\alpha)\xi - (Y\alpha)\eta(W)W),\xi] = 0 \quad (5.7)$$

where $C'(U,V,W,Y) = g(C(U,V)W,Y)$

Putting $Y = U$ in (5.7), we have

$$(\alpha^2+\beta^2)\{C'(U,V,W,U)+\epsilon\eta(U)\eta(C(U,V)W)\} - \epsilon(\xi\alpha)\eta(U)$$
$$-(\alpha\xi)C'(U,V,W,U)\eta(C(U,V)W) - \epsilon(U\alpha)$$
$$+ \epsilon(U\alpha)\eta(C(U,V)W)\eta(C(U,V)W) \qquad + g[C((\alpha^2+\beta^2)(\epsilon g(U,U)\xi$$
$$- \eta(U)U) + \big(\xi\beta - 2\alpha\beta\eta(U)\big)\phi U + (\xi\alpha)U$$
$$+\epsilon\alpha\xi g(U,U)U - (U\alpha)\xi - (U\alpha)\eta(U)U,V)W,\xi]$$
$$+g[C(U,(\alpha^2+\beta^2)(\epsilon g(U,V)\xi - \eta(V)U)$$
$$+\big(\xi\beta - 2\alpha\beta\eta(V)\big)\phi U + (\xi\alpha)U + \epsilon\alpha\xi g(U,V)V - (U\alpha)\xi - (U\alpha)\eta(V)V,W),\xi]$$
$$+g\Big[C(U,V)\Big((\alpha^2+\beta^2)(\epsilon g(U,W)\xi - \eta(W)U) + \big(\xi\beta - 2\alpha\beta\eta(W)\big)\phi U + (\xi\alpha)U +$$
$$\epsilon\alpha\xi g(U,W)W - (U\alpha)\xi - (U\alpha)\eta(W)W\Big),\xi\Big] = 0 \quad (5.8)$$

Taking an orthogonal frame in the equation (5.8) over U, we obtained

$$\sum_{i=1}^{n} C'(e_i,V,W,e_i) = 0 \quad (5.9)$$

and using equation (5.3) in (5.8), we have

$$\eta(C(\xi,V)W) = 0 \quad (5.10)$$

Using equation (5.3) and (5.8) we have

$$C'(U,V,W,Y) + \eta(Y)\eta(C(U,V)W) + \epsilon\eta(U)\eta(C(Y,V)W) + \epsilon\eta(V)\eta(C(U,Y)W)$$

$$+\epsilon\eta(W)\eta(C(U,V)Y) = 0 \quad (5.11)$$

Using equation (5.2) and (5.11), we have

$$C'(U,V,W,Y) + \eta(Y)\frac{1}{(n-2)}\left[\left\{\frac{r}{(n-1)} - \epsilon(n-1)(\alpha^2+\beta^2)\right\}\{g(V,W)\eta(U) - g(U,W)\eta(V)\right.$$
$$\left. - S(V,W)\eta(U) + S(U,W)\eta(V)\}\right]$$
$$+ \epsilon\eta(U)\frac{1}{(n-2)}\left[\left\{\frac{r}{(n-1)} - \epsilon(n-1)(\alpha^2+\beta^2)\right\}\{g(V,W)\eta(Y) - g(Y,W)\eta(V)\right.$$
$$\left. - S(V,W)\eta(Y) + S(Y,W)\eta(V)\}\right]$$
$$+ \epsilon\eta(V)\frac{1}{(n-2)}\left[\left\{\frac{r}{(n-1)} - \epsilon(n-1)(\alpha^2+\beta^2)\right\}\{g(V,W)\eta(U) - g(U,W)\eta(V)\right.$$
$$\left. - S(V,W)\eta(U) + S(U,W)\eta(V)\}\right]$$
$$+ \epsilon\eta(W)\frac{1}{(n-2)}\left[\left\{\frac{r}{(n-1)} - \epsilon(n-1)(\alpha^2+\beta^2)\right\}\{g(V,Y)\eta(U) - g(U,Y)\eta(V)\right.$$
$$\left. - S(V,Y)\eta(U) + S(U,Y)\eta(V)\}\right] = 0 \quad (5.12)$$

From equation (5.4) and (5.10), we have

$$S(Y,Z) = \left\{\frac{r}{(n-1)} - \epsilon(\alpha^2+\beta^2)\right\}g(Y,Z) + \left\{\frac{r}{(n-1)} - \epsilon n(\alpha^2+\beta^2)\right\}\eta(Y)\eta(Z) \quad (5.13)$$

Using equation (5.12) and (5.11), we have

$$C'(U,V,W,Y) = 0 \quad (5.14)$$

From the above equation we can see that $R.C = 0$, this implies that $C = 0$.

Hence this condition with the help of Theorem 4.2, gives the following results:

Theorem 5.1. An $n-$dimensional Weyl-semi-symmetric $(\epsilon)-$ Lorentzian para-Sasakian manifolds with parallelized generalized symmetric metric connection is of quasi-constant curvature.

Theorem 4.3 and equation (5.14) leads to the following result:

Corollary 5.2. An $n-$dimensional Weyl-semi-symmetric $(\epsilon)-$ Lorentzian para-Sasakian manifolds with parallelized generalized symmetric metric connection is a $\psi(F)_n$.

Application. $(\epsilon)-$ Lorentzian para-Sasakian manifolds with parallelized generalized symmetric metric connection are used in the Newtons law of gravitational field and theory of Relativity.

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