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Stability Analysis of Two Point Boundary Value Problem on Time Scales

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Abstract:

V. Lakshmi Kantham and M. Rama Mohana Rao contributed a lot for the theory of integro- differential equations of Volterra type and its various properties. In this paper, an attempt is made to explore the existence of a solution for a matrix integro-dynamical equation of Volterra type (MIDT) with two-point boundary value problem (TBVP) in terms of Green's matrix (GM) on time scales (TS). And properties of GM and stability of the system on TS are discussed.

Keywords: Green's matrix, Stability, Time scales.

AMS Subject Classification: 34N05.

1. Introduction

Mathematical models, that describe physical phenomenon are expressed or described in most cases either differential or difference equations with respect to continuous cases or discrete cases respectively. In previous years, so many authors have worked on dynamical systems either in continuous cases or discrete cases. Stefan Hilger observed some interesting patterns in nature. According to Stefan Hilger observation, some plants in poles show growing nature for 6 months and non-growing nature for remaining 6 months. Further, in some species existing in Japan, Mother dies soon after it gives birth to its baby. After having noticed this physical phenomenon, Stefan Hilger drew a conclusion that there exists some systems in real world which contains both discrete and continuous nature. Based on this, Stefan Hilger invented a new technique TS in 1998 as an integral part of his Ph.D thesis which can work with systems involving both cases[5]. In recent years, owing to the unified treatment of analysing both continuous and discrete systems at a time [1, 3, 6], research related to dynamic systems on TS experienced significant significance in diversified fields. The TS applications has a tremendous potential in biology, engineering, economics, physics, neural networks, etc. Burton studied the an integro-differential equations and stability theory for Volterra equations [2]. Making use of asymptotic stability, G. V. S. R. Deekshitulu and G. V. Ramana studied Lyapunov type linear matrix Volterra integro dynamical system on TS [4]. Shah, Zada and Sabrina Streipert are discussed the stability of non-linear Volterra integro dynamic equations on TS [7, 8].

Having seen the potential exploring possibilities in this, an endeavour is made to obtain a solution of BVP concerned with MIDT.

And the obtained solution is tabled in the form of GM. Further, properties of GM and stability are also discussed for the system on TS.

In this paper, the following matrix integro-dynamical equation on TS has been considered.

$$x^{\Delta}(t) = A(t)x(t) + \int_{t_0}^{\sigma(\rho(t))} K(t,s)\, x(s)\Delta s + \mathfrak{z}\big(t, x(t)\big), \quad (1.1)$$

$$w^{a}\, x(a) \ + w^{b}\, x(b) = 0 \quad (1.2)$$

Here $A_{n\times n}$ is rd-continuous in $\mathbb{T}$ and $K(t,s) \in C_{rd}[\mathbb{T}\times\mathbb{T}, \mathbb{R}^{n\times n}]$, w^{a} and $w^{b}$are$(n\times n)$ constant matrices, $\mathfrak{z}\big(t, x(t)\big) \in C_{rd}[\mathbb{T}\times\mathbb{T}, \mathbb{R}^{n}]$and $x(t)$ is a column vector.

2. Preliminary Results

Definition 2.1.[Bohner et al. 2003] The mappings σ and ρ: T → R are defined as

(i) σ(t) = inf {m ∈ T: m > t},

(ii) (ii) ρ(t) = sup {m ∈ T: m < t}.

Definition 2.2.[Bohner et al. 2003] The Graininess function μ: T → [0, ∞] is defined as μ(t) = σ(t) – t and

(i) if T = R, then μ(t) = 0 and

(ii) if T = Z, then μ(t) = 1.

Theorem 2.3.[Bohner et al. 2003] Assume v: T → R and let t ∈ T^{k} and $v^{\triangle}(t)$ exist then

$$v(\sigma(t)) = v(t) + \mu(t)v^{\triangle}(t).$$

Lemma 2.4.[Bohner et al. 2003] Let U, V and W be matrix-valued of order (n × n) and differentiable. Then

(i) $(U + V)^{\triangle} = U^{\triangle} + V^{\triangle}$

(ii) $(\alpha U)^{\triangle} = \alpha U^{\triangle}$ if α is constant

(iii) $(UV)^{\triangle} = U^{\triangle}V^{\sigma} + UV^{\triangle} = U^{\sigma}V^{\triangle} + U^{\triangle}V$

(iv) $(UVW)^{\triangle} = U^{\triangle}VW + U^{\sigma}V^{\triangle}W^{\sigma} + U^{\sigma}VW^{\triangle} = U^{\triangle}VW + U^{\sigma}V^{\triangle}W + U^{\sigma}V^{\sigma}W^{\triangle}$

Definition 2.5. [Bohner et al. 2003] U is a matrix of order (n × n) on a time scale T is said to be regressive if

$I + \mu(t)U(t) \neq 0$ for all t ∈ T^{k}.

Lemma 2.6. [Bohner et al. 2001] Let $u \in C_{rd}(T, R)$, $\xi \in R^{+}$, $\xi \geq 0$, β∈ R. The

$$u(t) \leq \beta + \int_{0}^{t} u(s)\xi(s) \triangle s$$

implies $u(t) \leq \beta e_{\xi}(t, t_0)$, for all t ∈ T.

Lemma 2.7. [Bohner et al. 2003] Let $\xi \in R^{+}$. Then $e_{\xi}(\varrho, \varsigma) \geq 1 + \xi(\varrho - \varsigma)$, for all $\varrho \geq \varsigma$.

3. Main Results

Theorem 3.1. Let $A(t), K(t,s), w^a,\ w^b$ and $\mathfrak{z}(t,x(t))$ be previously defined. Then the solution of (1.1) and (1.2) is

$$x(t) = \int_a^b G(t,\sigma(s))\, H(s,\tau)\Delta s \tag{3.1}$$

Where $H(s,\tau) = \int_{t_0}^{\sigma(\rho(s))} K(s,\tau)\, y(\tau)\Delta\tau + \mathfrak{z}(\tau.x(\tau))$ and $G(t,\sigma(s))$ is the GM.

Proof: Consider the non-homogeneous BVP (1.1) and (1.2)

Where $A_{n\times n}$is rd-continuous in $\mathbb{T}$ and $K(t,s) \in C_{rd}[\mathbb{T}\times\mathbb{T},\mathbb{R}^{n\times n}]$, w^a and w^b are $(n\times n)$ constant matrices, $\mathfrak{z}\big(t,x(t)\big) \in C_{rd}[\mathbb{T}\times\mathbb{T},\mathbb{R}^n]$.

Assume that the homogeneous BVP corresponding to given non-homogeneous BVP is incompatible.

Let $\emptyset(t)$ be a fundamental matrix of homogeneous equation (HE) in the form $x(t) = \emptyset(t)C$, where C is a constant matrix.

Let $\overline{x(t)}$ be a particular solution of non-homogeneous equation (NHE). Then any solution of the given NHE takes form

$$x(t) = \emptyset(t)C + \overline{x(t)}. \tag{3.2}$$

Since the particular solution of (1.1) is

$$\overline{x(t)} = \emptyset(t)\int_a^t \emptyset^{-1}(\sigma(s))\left[\int_{t_0}^{\sigma(\rho(s))} K(s,\tau)\, x(\tau)\Delta\tau + \mathfrak{z}(s,x(s))\right]\Delta s \tag{3.3}$$

The general solution of (1.1) is

$$x(t) = \emptyset(t)C + \emptyset(t)\int_a^t \emptyset^{-1}\big(\sigma(s)\big)\left[\int_{t_0}^{\sigma(\rho(s))} K(s,\tau)\, x(\tau)\Delta\tau + \mathfrak{z}\big(s,x(s)\big)\right]\Delta s. \tag{3.4}$$

Now calculate the general solution in the given boundary condition

$$0 = w^{(a)}x(a) + w^{(b)}x(b)$$

$$= w^{(a)}\left[\emptyset(a)C + \emptyset(a)\int_a^a \emptyset^{-1}\big(\sigma(s)\big)\left[\int_{t_0}^{\sigma(\rho(s))} K(s,\tau)\, x(\tau)\Delta\tau + \mathfrak{z}\big(s,x(s)\big)\right]\Delta s\right]$$
$$+ w^{(b)}\left[\emptyset(b)C + \emptyset(b)\int_a^b \emptyset^{-1}\big(\sigma(s)\big)\left[\int_{t_0}^{\sigma(\rho(s))} K(s,\tau)\, x(\tau)\Delta\tau + \mathfrak{z}\big(s,x(s)\big)\right]\Delta s\right]$$

$$= w^{(a)}\emptyset(a)C + w^{(b)}\emptyset(b)C + w^b\emptyset(b)\int_a^b \emptyset^{-1}\big(\sigma(s)\big)\left[\int_{t_0}^{\sigma(\rho(s))} K(s,\tau)\, x(\tau)\Delta\tau + \mathfrak{z}\big(s,x(s)\big)\right]\Delta s$$

$$= [w^{(a)}\emptyset(a) + w^{(b)}\emptyset(b)]C + w^{(b)}\emptyset(b)\int_a^b \emptyset^{-1}\big(\sigma(s)\big)\left[\int_{t_0}^{\sigma(\rho(s))} K(s,\tau)\, x(\tau)\Delta\tau + \mathfrak{z}\big(s,x(s)\big)\right]\Delta s$$

The characteristic matrix $D = [w^{(a)}\emptyset(a) + w^{(b)}\emptyset(b)]$.

Hence the above identity becomes

$$0 = DC + w^{(b)}\emptyset(b)\int_a^b \emptyset^{-1}\big(\sigma(s)\big)\left[\int_{t_0}^{\sigma(\rho(s))} K(s,\tau)\, x(\tau)\Delta\tau + \mathfrak{z}\big(s,x(s)\big)\right]\Delta s$$

Since the homogeneous BVP is incompatible, the index of compatibility is zero and hence D is non-singular.

$$DC = -w^{(b)}\emptyset(b)\int_a^b \emptyset^{-1}\big(\sigma(\mathrm{s})\big)\left[\int_{t_0}^{\sigma(\rho(\mathrm{s}))} K(\mathrm{s},\tau)\,x(\tau)\Delta\tau + \mathfrak{z}\big(\mathrm{s},x(\mathrm{s})\big)\right]\Delta\mathrm{s}$$

$$C = -D^{-1}w^{(b)}\emptyset(b)\int_a^b \emptyset^{-1}\big(\sigma(\mathrm{s})\big)\left[\int_{t_0}^{\sigma(\rho(\mathrm{s}))} K(\mathrm{s},\tau)\,x(\tau)\Delta\tau + \mathfrak{z}\big(\mathrm{s},x(\mathrm{s})\big)\right]\Delta\mathrm{s}$$

From equation (3.4)

$$x(t) = -\emptyset(t)D^{-1}w^{(b)}\emptyset(b)\int_a^b \emptyset^{-1}\big(\sigma(\mathrm{s})\big)\left[\int_{t_0}^{\sigma(\rho(\mathrm{s}))} K(\mathrm{s},\tau)\,x(\tau)\Delta\tau + \mathfrak{z}\big(\mathrm{s},x(\mathrm{s})\big)\right]\Delta\mathrm{s}$$

$$+\,\emptyset(t)\int_a^t \emptyset^{-1}\big(\sigma(\mathrm{s})\big)\left[\int_{t_0}^{\sigma(\rho(\mathrm{s}))} K(\mathrm{s},\tau)\,x(\tau)\Delta\tau + \mathfrak{z}\big(\mathrm{s},x(\mathrm{s})\big)\right]\Delta\mathrm{s}$$

$$= -\emptyset(t)D^{-1}w^{(b)}\emptyset(b)[\int_a^t + \int_t^b]\,\emptyset^{-1}\big(\sigma(\mathrm{s})\big)\left[\int_{t_0}^{\sigma(\rho(\mathrm{s}))} K(\mathrm{s},\tau)\,x(\tau)\Delta\tau + \mathfrak{z}\big(\mathrm{s},x(\mathrm{s})\big)\right]\Delta\mathrm{s}$$

$$+\,\emptyset(t)\int_a^t \emptyset^{-1}\big(\sigma(\mathrm{s})\big)\left[\int_{t_0}^{\sigma(\rho(\mathrm{s}))} K(\mathrm{s},\tau)\,x(\tau)\Delta\tau + \mathfrak{z}\big(\mathrm{s},x(\mathrm{s})\big)\right]\Delta\mathrm{s}$$

$$= -\emptyset(t)D^{-1}w^{(b)}\emptyset(b)\int_a^t \emptyset^{-1}\big(\sigma(\mathrm{s})\big)\left[\int_{t_0}^{\sigma(\rho(\mathrm{s}))} K(\mathrm{s},\tau)\,x(\tau)\Delta\tau + \mathfrak{z}\big(\mathrm{s},x(\mathrm{s})\big)\right]\Delta\mathrm{s}$$

$$+\,\emptyset(t)\int_a^t \emptyset^{-1}\big(\sigma(\mathrm{s})\big)\left[\int_{t_0}^{\sigma(\rho(\mathrm{s}))} K(\mathrm{s},\tau)\,x(\tau)\Delta\tau + \mathfrak{z}\big(\mathrm{s},x(\mathrm{s})\big)\right]\Delta\mathrm{s}$$

$$-\,\emptyset(t)D^{-1}w^{b}\emptyset(b)\int_t^b \emptyset^{-1}\big(\sigma(\mathrm{s})\big)\left[\int_{t_0}^{\sigma(\rho(\mathrm{s}))} K(\mathrm{s},\tau)\,x(\tau)\Delta\tau + \mathfrak{z}\big(\mathrm{s},x(\mathrm{s})\big)\right]\Delta\mathrm{s}$$

$$= \emptyset(t)[I - D^{-1}w^{(b)}\emptyset(b)]\int_a^t \emptyset^{-1}\big(\sigma(\mathrm{s})\big)\left[\int_{t_0}^{\sigma(\rho(\mathrm{s}))} K(\mathrm{s},\tau)\,x(\tau)\Delta\tau + \mathfrak{z}\big(\mathrm{s},x(\mathrm{s})\big)\right]\Delta\mathrm{s}$$

$$-\,\emptyset(t)D^{-1}w^{(b)}\emptyset(b)\int_t^b \emptyset^{-1}\big(\sigma(\mathrm{s})\big)\left[\int_{t_0}^{\sigma(\rho(\mathrm{s}))} K(\mathrm{s},\tau)\,x(\tau)\Delta\tau + \mathfrak{z}\big(\mathrm{s},x(\mathrm{s})\big)\right]\Delta\mathrm{s}$$

$$= \emptyset(t)D^{-1}[D - w^{(b)}\emptyset(b)]\int_a^t \emptyset^{-1}\big(\sigma(\mathrm{s})\big)\left[\int_{t_0}^{\sigma(\rho(\mathrm{s}))} K(\mathrm{s},\tau)\,x(\tau)\Delta\tau + \mathfrak{z}\big(\mathrm{s},x(\mathrm{s})\big)\right]\Delta\mathrm{s}$$

$$-\,\emptyset(t)D^{-1}w^{(b)}\emptyset(b)\int_t^b \emptyset^{-1}\big(\sigma(\mathrm{s})\big)\left[\int_{t_0}^{\sigma(\rho(\mathrm{s}))} K(\mathrm{s},\tau)\,x(\tau)\Delta\tau + \mathfrak{z}\big(\mathrm{s},x(\mathrm{s})\big)\right]\Delta\mathrm{s}$$

$$= \emptyset(t)D^{-1}w^{(a)}\emptyset(a)]\int_a^t \emptyset^{-1}\big(\sigma(\mathrm{s})\big)\left[\int_{t_0}^{\sigma(\rho(\mathrm{s}))} K(\mathrm{s},\tau)\,x(\tau)\Delta\tau + \mathfrak{z}\big(\mathrm{s},x(\mathrm{s})\big)\right]\Delta\mathrm{s}$$

$$-\,\emptyset(t)D^{-1}w^{(b)}\emptyset(b)\int_t^b \emptyset^{-1}\big(\sigma(\mathrm{s})\big)\left[\int_{t_0}^{\sigma(\rho(\mathrm{s}))} K(\mathrm{s},\tau)\,x(\tau)\Delta\tau + \mathfrak{z}\big(\mathrm{s},x(\mathrm{s})\big)\right]\Delta\mathrm{s}$$

$$x(t) = \int_a^b G\big(t,\sigma(\mathrm{s})\big)\left[\int_{t_0}^{\sigma(\rho(\mathrm{s}))} K(\mathrm{s},\tau)\,x(\tau)\Delta\tau + \mathfrak{z}\big(\mathrm{s},x(\mathrm{s})\big)\right]\Delta\mathrm{s}$$

$= \int_a^b G\big(t,\sigma(\mathrm{s})\big)\,H(\mathrm{s},\tau)\Delta\mathrm{s}.$

Where $H(s,\tau) = \int_{t_0}^{\sigma(\rho(\text{s}))} K(\text{s},\tau)\, x(\tau)\Delta\tau + \mathfrak{z}(\text{s}, x(\text{s}))$ and $G(t,\sigma(\text{s}))$ is the Greens matrix and is given by

$$G(t,\sigma(\text{s})) = \begin{cases} \emptyset(t)D^{-1}w^{(a)}\emptyset(a)\,\emptyset^{-1}(\sigma(\text{s})), & a \le \text{s} \le t \le b \\ -\emptyset(t)D^{-1}w^{(b)}\emptyset(b)\,\emptyset^{-1}(\sigma(\text{s})), & a \le t \le \text{s} \le b. \end{cases}$$

Next, we discuss the two extreme cases of $\mathbb{T}$ and obtain some interesting results.

Corollary 3.2. If $\mathbb{T} = \mathbb{R}^+$, we have $\sigma(t) = t$ and $\emptyset^{-1}(\sigma(\text{s})) = \emptyset^{-1}(\text{s})$ in Theorem 3.1 the state equation is

$$\left.\begin{aligned} x'(t) = A(t)x(t) + \int_{t_0}^{t} K(t,\text{s})\, x(\text{s})\Delta\text{s} + \mathfrak{z}(t, x(t)), \\ 0 = w^{(a)}x(a) + w^{(b)}x(b) \end{aligned}\right\} \quad (3.5)$$

Where $A_{n\times n}$ is a continuous matrix on $\mathbb{R}^+$, $K(t,\text{s})_{n\times n}$ is a continuous matrix for

$0 \le \text{s} \le t \le \infty$ and $F \in C[\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^n]$, then the solution of (3.5) is

$$x(t) = \int_a^b G(t,\text{s})\, H(\text{s},\tau)\Delta\text{s}$$

Where $H(\text{s},\tau) = \int_{t_0}^{\text{s}} K(\text{s},\tau)\, x(\tau)d\tau + \mathfrak{z}(\tau, x(\tau))$ and $G(t,\text{s})$ is the GM and is given by

$$G(t,\text{s}) = \begin{cases} \emptyset(t)D^{-1}w^{(a)}\emptyset(a)\,\emptyset^{-1}(\text{s}), & a \le \text{s} \le t \le b \\ -\emptyset(t)D^{-1}w^{(b)}\emptyset(b)\,\emptyset^{-1}(\text{s}), & a \le t \le \text{s} \le b. \end{cases}$$

Where $\emptyset(t)$ is a fundamental matrix and $D = [w^{(a)}\emptyset(a) + w^{(b)}\emptyset(b)]$.

Corollary 3.3. If $\mathbb{T} = \mathbb{N}$, we have $\sigma(\xi) = \xi + 1$ and $\emptyset^{-1}(\sigma(\xi)) = \emptyset^{-1}(\xi + 1)$ in Theorem 3.1 the state equation is

$$\left.\begin{aligned} \Delta x(\xi) = A(\xi)x(\xi) + \sum_{\text{s}=\xi_0}^{\xi-1} K(\xi,\text{s})\, x(\text{s}) + \mathfrak{z}(\xi, x(\xi)), \\ 0 = w^{(a)}x(a) + w^{(b)}x(b) \end{aligned}\right\} \quad (3.6)$$

Where $A(\xi)$ and $K(\xi,\text{s})$ are $\eta \times \eta$ matrices for each$\xi,\text{s} \in \mathbb{N}$ and

$F: \mathbb{N}_{\xi_0}^+ \times \mathbb{N}_{\xi_0}^+ \to \mathbb{R}^d, \mathbb{N}_{\xi_0}^+ = \{\xi_0, \xi_0 + 1 \ldots \xi_0 + k, \ldots\}, \xi_0,\ k \in \mathbb{N}$ then the solution of (3.6) is

$x(\xi) = \sum_{\text{s}=a}^{b-1} G(\xi,\text{s}+1)\, H(\text{s},\tau)$.

Where $H(\text{s},\tau) = \sum_{\tau=\xi_0}^{\text{s}-1} K(\text{s},\tau)\, x(\tau) + \mathfrak{z}(\xi, x(\xi))$ and $G(\xi,\text{s}+1)$ is the GM and is given by

$$G(\xi,\text{s}+1) = \begin{cases} \emptyset(\xi)D^{-1}w^{(a)}\emptyset(a)\,\emptyset^{-1}(\text{s}+1), & a \le \text{s} \le \xi \le b-1 \\ -\emptyset(\xi)D^{-1}w^{(b)}\emptyset(b)\,\emptyset^{-1}(\text{s}+1), & a \le \xi \le \text{s} \le b-1. \end{cases}$$

Where $\emptyset(\xi)$ is a fundamental matrix and $D = [w^{(a)}\emptyset(a) + w^{(b)}\emptyset(b)]$.

Theorem 3.4. GM exhibits the below mentioned characteristics:

i. The components of $G(t,\sigma(\text{s}))$ are functions of t withfixeds, possess rd-continuous delta differentiable on the intervals $[a,\text{s})$and $(\text{s},b]$. At the point $t = \text{s}$, G exhibits an upward jump of magnitude $\emptyset(a)\,\emptyset^{-1}(\sigma(\text{s}))$.

i.e.,$G\big(\mathrm{s}^{+},\sigma(\mathrm{s})\big)-G\big(\mathrm{s}^{-},\sigma(\mathrm{s})\big)=\emptyset(a)\,\emptyset^{-1}\big(\sigma(\mathrm{s})\big)$.

ii. $G\big(t,\sigma(\mathrm{s})\big)$is a formal solution of the homogeneous BVP

$$\left.\begin{aligned} x^{\Delta}(t) &= A(t)x(t)\\ 0 &= w^{(a)}\emptyset(a)+w^{(b)}\emptyset(b)\end{aligned}\right\} \tag{3.7}$$

G, ceases to be a true solution solely due to its jump condition at $t\ =\mathrm{s}$.

iii. The Green's matrix $G\big(t,\sigma(\mathrm{s})\big)$ having the properties (i) and (ii) is unique.

Proof:(i). The GM for the BVP

$$x^{\Delta}(t)=A(t)x(t)$$
$$0=w^{(a)}\emptyset(a)+w^{(b)}\emptyset(b).$$

is

$$G\big(t,\sigma(\mathrm{s})\big)=\begin{cases}\emptyset(t)D^{-1}w^{(a)}\emptyset(a)\,\emptyset^{-1}\big(\sigma(\mathrm{s})\big), & a\le \mathrm{s}\le t\le b\\ -\emptyset(t)D^{-1}w^{(b)}\emptyset(b)\,\emptyset^{-1}\big(\sigma(\mathrm{s})\big), & a\le t\le \mathrm{s}\le b\end{cases}$$

Where G is considered as a function of t with fixed s, the GM can be written as

$$G\big(t,\sigma(\mathrm{s})\big)=\begin{cases}\emptyset(t)H_{+}, & \mathrm{s}\le t\\ \emptyset(t)H_{-}, & t\le \mathrm{s}.\end{cases}$$

Where the matrices

$$H_{+}=\ D^{-1}w^{(a)}\emptyset(a)\,\emptyset^{-1}\big(\sigma(\mathrm{s})\big)$$
$$H_{-}=\ -D^{-1}w^{(b)}\emptyset(b)\,\emptyset^{-1}\big(\sigma(\mathrm{s})\big)$$

They are independent of t.Consequently, the components of $G\big(t,\sigma(\mathrm{s})\big)$ possess rd-continuous first delta derivatives with respect to$t$on $[a,\mathrm{s})$ and $(\mathrm{s},b]$. Further

$$\begin{aligned} H_{+}-H_{-} &=\ D^{-1}w^{(a)}\emptyset(a)\,\emptyset^{-1}\big(\sigma(\mathrm{s})\big)+D^{-1}w^{(b)}\emptyset(b)\,\emptyset^{-1}\big(\sigma(\mathrm{s})\big)\\ &=\ D^{-1}[w^{(a)}\emptyset(a)+w^{(b)}\emptyset(b)]\,\emptyset^{-1}\big(\sigma(\mathrm{s})\big)\\ &=\ D^{-1}D\,\emptyset^{-1}\big(\sigma(\mathrm{s})\big)\\ &=\ \emptyset^{-1}\big(\sigma(\mathrm{s})\big).\end{aligned}$$

Therefore

$$\begin{aligned} G\big(\mathrm{s}^{+},\sigma(\mathrm{s})\big)-G\big(\mathrm{s}^{-},\sigma(\mathrm{s})\big) &=\emptyset(\mathrm{s})H_{+}-\emptyset(\mathrm{s})H_{-}\\ &=\emptyset(\mathrm{s})[H_{+}-H_{-}]\\ &=\emptyset(\mathrm{s})\emptyset^{-1}\big(\sigma(\mathrm{s})\big).\end{aligned}$$

(ii). We want to show that $G\big(t,\sigma(\mathrm{s})\big)$ is a matrix solution of homogeneous BVP. The representation of $G\big(t,\sigma(\mathrm{s})\big)$is the solution of (3.7) on $[a,\mathrm{s})$ and $(\mathrm{s},b]$. Now to show that $G\big(t,\sigma(\mathrm{s})\big)$ satisfies the given boundary condition, we obtain

$$w^{(a)}G\big(a,\sigma(\mathrm{s})\big)+w^{(b)}G\big(b,\sigma(\mathrm{s})\big)=w^{(a)}\emptyset(a)H_{-}+w^{(b)}\emptyset(b)H_{+}$$
$$=w^{(a)}\emptyset(a)H_{-}+w^{(b)}\emptyset(b)H_{+}+w^{(b)}\emptyset(b)H_{-}-w^{(b)}\emptyset(b)H_{-}$$

$$= w^{(a)}\emptyset(a)H_- + w^{(b)}\emptyset(b)H_- + w^{(b)}\emptyset(b)[H_+ - H_-]$$
$$= w^{(a)}[\emptyset(a) + w^{(b)}\emptyset(b)]H_- + w^{(b)}\emptyset(b)\ \emptyset^{-1}(\sigma(\mathrm{s}))$$
$$= DH_- + w^{(b)}\emptyset(b)\ \emptyset^{-1}(\sigma(\mathrm{s}))$$
$$= D\left[-D^{-1}w^{(b)}\emptyset(b)\ \emptyset^{-1}(\sigma(\mathrm{s}))\right] + w^{(b)}\emptyset(b)\emptyset^{-1}(\sigma(\mathrm{s}))$$
$$= -w^{(b)}\emptyset(b)\ \emptyset^{-1}(\sigma(\mathrm{s})) + w^{(b)}\emptyset(b)\ \emptyset^{-1}(\sigma(\mathrm{s}))$$
$$= 0$$

Thus G is a solution of the homogeneous BVP.

(iii). Now, let us prove the uniqueness of G, let $G_1(t,\sigma(\mathrm{s}))$ and $G_2(t,\sigma(\mathrm{s}))$ be rd-continuous matrices with (i) and (ii). Write

$L(t,\mathrm{s}) = G_1(t,\sigma(\mathrm{s})) - G_2(t,\sigma(\mathrm{s}))$. Clearly L satisfies the following at$t = \mathrm{s}$

$$L(\mathrm{s}^+,\mathrm{s}) - L(\mathrm{s}^-,\mathrm{s}) = [G_1(\mathrm{s}^+,\sigma(\mathrm{s})) - G_2(\mathrm{s}^+,\sigma(\mathrm{s}))] - [G_1(\mathrm{s}^-,\sigma(\mathrm{s})) - G_2(\mathrm{s}^-,\sigma(\mathrm{s}))]$$
$$= [G_1(\mathrm{s}^+,\sigma(\mathrm{s})) - G_1(\mathrm{s}^-,\sigma(\mathrm{s}))] - [G_2(\mathrm{s}^+,\sigma(\mathrm{s})) - G_2(\mathrm{s}^-,\sigma(\mathrm{s}))]$$
$$= \emptyset(\mathrm{s})\ \emptyset^{-1}(\sigma(\mathrm{s})) - \emptyset(\mathrm{s})\ \emptyset^{-1}(\sigma(\mathrm{s}))$$
$$= 0.$$

Thus, L exhibits a removable jump condition at $t = \mathrm{s}$ and

$$w^{(a)}L(a,\mathrm{s}) + w^{(b)}L(b,\mathrm{s}) = w^{(a)}[G_1(a,\sigma(\mathrm{s})) - G_2(a,\sigma(\mathrm{s}))] + w^{(b)}[G_1(b,\sigma(\mathrm{s})) - G_2(b,\sigma(\mathrm{s}))]$$
$$= [w^{(a)}G_1(a,\sigma(\mathrm{s})) + w^{(b)}G_1(b,\sigma(\mathrm{s}))] - [w^{(a)}G_2(a,\sigma(\mathrm{s})) + w^{(b)}G_2(b,\sigma(\mathrm{s}))]$$
$$= 0.$$

Since L is a solution of homogeneous BVP and $L(t,\mathrm{s}) = 0$. i.e., $G_1(t,\sigma(\mathrm{s})) - G_2(t,\sigma(\mathrm{s})) = 0$ implies $G_1(t,\sigma(\mathrm{s})) = G_2(t,\sigma(\mathrm{s}))$. Thus G is unique.

Theorem 3.5. Assume that

(a) $|\emptyset(t)| \leq R$,

(b) $\left|\emptyset(t)\ \emptyset^{-1}(\sigma(t))\right| \leq S$,

(c) $\sup\int_{t_0}^{\sigma(\rho(\mathrm{s}))}|K(\mathrm{s},\tau)||x(\tau)|\Delta\tau + \mathfrak{z}(\mathrm{s},\ x(\mathrm{s})) \leq T|x(t)|$,

Where $R, S, T > 0$. Then solution of (1.1) is uniformly stable.

Proof: For any $\varepsilon > 0$, let $\delta(\varepsilon) < \frac{\varepsilon}{Re_{ST}(t,a)}$ and $|C| \leq \delta_\varepsilon$. Suppose $\exists$, $t_1 \geq a$ such that $|x(t_1)| = \varepsilon$ and $|x(t)| < \varepsilon$ on $[a, t_1)$.

From (3.4),

$$x(t) = \emptyset(t)C + \emptyset(t)\int_a^t \emptyset^{-1}(\sigma(\mathrm{s}))[\int_{t_0}^{\sigma(\rho(\mathrm{s}))} K(\mathrm{s},\tau)x(\tau)\Delta\tau + \mathfrak{z}(\mathrm{s},x(\mathrm{s})]\Delta\mathrm{s} \quad \text{on } [a, t_1]$$

$$|x(t)| \leq |\emptyset(t)||C| + |\emptyset(t)| \int_a^t \left| \emptyset^{-1}(\sigma(\mathrm{s}))\right| [\int_{t_0}^{\sigma(\rho(\mathrm{s}))} |K(\mathrm{s},\tau)||x(\tau)|\Delta\tau + |\mathfrak{Z}(\mathrm{s},x(\mathrm{s})|]\Delta\mathrm{s}$$

$$\leq R\delta_\varepsilon + ST\int_a^t |x(t)|\Delta t$$

By Lemma 2.6,

$$|x(t)| \leq R\delta_\varepsilon e_{ST}(t,a)$$
$$\leq Re_{ST}(t,a)\frac{\varepsilon}{Re_{ST}(t,a)}$$
$$= \varepsilon$$

Therefore $|x(t_1)| < \varepsilon$, which contradicts. Thus, the solution of (1.1) is uniformly stable.

Theorem 3.6. Assume

(a) $|\emptyset(t)||C| \leq Re_P(t_0,t)$

(b) $\left|\emptyset(t)\ \emptyset^{-1}(\sigma(t))\right| \leq Se_P(0,t)$,

(c) $\sup\int_{t_0}^{\sigma(\rho(\mathrm{s}))} |K(\mathrm{s},\tau)||x(\tau)|\Delta\tau + \mathfrak{Z}(\mathrm{s},\ x(\mathrm{s})) \leq Le_P(\mathrm{s},0)x(\mathrm{s})$,

where S, T, L > 0 and (P ⊖ LT) > 0. Then each solution U(t) of (3.1) tends to 0, as t → ∞.

Proof: From (3.4), we have

$x(t) = \emptyset(t)C + \emptyset(t)\int_a^t \emptyset^{-1}(\sigma(\mathrm{s}))[\int_{t_0}^{\sigma(\rho(\mathrm{s}))} K(\mathrm{s},\tau)x(\tau)\Delta\tau + \mathfrak{Z}(\mathrm{s},x(\mathrm{s})]\Delta\mathrm{s}$ on $[a,t_1]$

$$|x(t)| \leq |\emptyset(t)||C| + |\emptyset(t)| \int_a^t \left| \emptyset^{-1}(\sigma(\mathrm{s}))\right| [\int_{t_0}^{\sigma(\rho(\mathrm{s}))} |K(\mathrm{s},\tau)||x(\tau)|\Delta\tau + |\mathfrak{Z}(\mathrm{s},x(\mathrm{s})|]\Delta\mathrm{s}$$

$$= \mathrm{Re_P(t_0,t) + SLe_P(0,t)\int_{t_0}^{t} e_P(s,0)|x(s)| \triangle s}$$

$$= \mathrm{Re_P(t_0,0)e_P(0,t) + SLe_P(0,t)\int_{t_0}^{t} e_P(s,0)|x(s)| \triangle s}$$

$$\mathrm{|U(t)|e_P(t,0) \leq Re_P(t_0,0) + SL\int_{t_0}^{t} e_P(s,0)|x(s)| \triangle s}$$

By Lemma 2.6,

$\mathrm{|x(t)|e_P(t,0)} \leq R\mathrm{e_P(t_0,0)e_{SL}(t,t_0)}$

$$\mathrm{|x(t)| \leq Re_P(t_0,0)e_P(0,t)e_{SL}(t,t_0)}$$

$$= R\mathrm{e_P(t_0,0)e_P(0,t)e_{\ominus SL}(t_0,t)}$$

$$= R\mathrm{e_P(t_0,0)e_P(0,t)e_{\ominus SL}(t_0,0)e_{\ominus SL}(0,t)}$$

$$= R\mathrm{e_{P\ominus SL}(t_0,0)e_{P\ominus SL}(0,t)}$$

By Lemma 2.7,

$e_{P\ominus SL}(0,t) \le \frac{1}{1+(P\ominus SL)t}$, so we obtain

$$|x(t)| \le R e_{P\ominus SL}(t_0,0)\frac{1}{1+(P\ominus SL)t}$$

Since $(P\ominus SL) > 0$. Hence, we obtain desired result.

4. Example

Example 4.1. Consider non-homogeneous boundary value problem on time scales

$$\left.\begin{aligned} x^{\Delta}(t) &= \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}x(t) + \int_0^{\sigma(\rho(t))}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x(s)\,\Delta s + \begin{bmatrix}t\\ t\end{bmatrix},\\ 0 &= \begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}x(0) + \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}x(10),\end{aligned}\right\} \tag{4.1}$$

for $0 \le s \le t < \infty$.

Case(a): If $\mathbb{T} = \mathbb{R}^+$ then (4.1) is

$$\left.\begin{aligned} x'(t) &= \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}x(t) + \int_0^{t}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x(s)\,ds + \begin{bmatrix}t\\ t\end{bmatrix},\\ 0 &= \begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}x(0) + \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}x(10).\end{aligned}\right\} \tag{4.2}$$

The fundamental matrix of $x'(t) = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}x(t)$ is $\emptyset(t) = \begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}$.

Hence the solution of (4.2) is

$$\begin{aligned} x(t) &= \frac{1}{12}\int_0^t \begin{bmatrix}11-t & -11-s(11-t)+t\\ -1 & s+1\end{bmatrix}\begin{bmatrix}e^s-1\\ e^s-1\end{bmatrix} ds\\ &\quad -\frac{1}{12}\int_t^{10}\begin{bmatrix}1+t & 1+t+10(1+t)-s(1+t)\\ 1 & 11-s\end{bmatrix}\begin{bmatrix}e^s-1\\ e^s-1\end{bmatrix} ds\\ &= \frac{1}{12}\begin{bmatrix}59+24e^t+59t-6t^2-3e^{10}(1+t)\\ 71-3e^{10}+12e^t-12t\end{bmatrix}.\end{aligned}$$

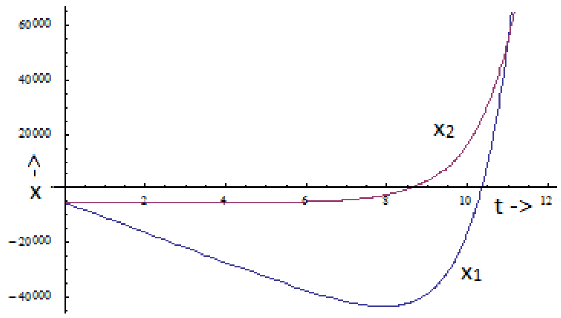

Figure 1. Continuous solution graph

Case(b): If $\mathbb{T} = \mathbb{N}$ then (4.1) is

$$\left.\begin{aligned}\Delta x(\xi) &= \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} x(\xi) + \sum_{s=0}^{\xi-1}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} x(s) + \begin{bmatrix}\xi\\ \xi\end{bmatrix},\\ 0 &= \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix} x(0) + \begin{bmatrix}0 & 0\\1 & 1\end{bmatrix} x(10),\end{aligned}\right\} \tag{4.3}$$

The fundamental matrix of $\Delta x(\xi) = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} x(\xi)$ is $\emptyset(\xi) = \begin{bmatrix}1 & \xi\\0 & 1\end{bmatrix}$.

Using Laplace transformation on discrete case then the solution of (4.3) is

$$x(\xi) = \frac{1}{12}\sum_{s=0}^{\xi-1}\begin{bmatrix}11-\xi & -11+\xi+(11-\xi)(-1-s)\\ -1 & s+2\end{bmatrix}\begin{bmatrix}2^s-1\\2^s-1\end{bmatrix}$$
$$-\frac{1}{12}\sum_{s=\xi}^{9}\begin{bmatrix}1+\xi & 1+\xi+10(1+\xi)+(1+\xi)(-1-s)\\ 1 & 10-s\end{bmatrix}\begin{bmatrix}2^s-1\\2^s-1\end{bmatrix}$$
$$= \frac{1}{12}\begin{bmatrix}6(-503+2^{2+\xi}-502\xi-\xi^2)\\ 6(-501+2^{1+\xi}-2\xi)\end{bmatrix}$$

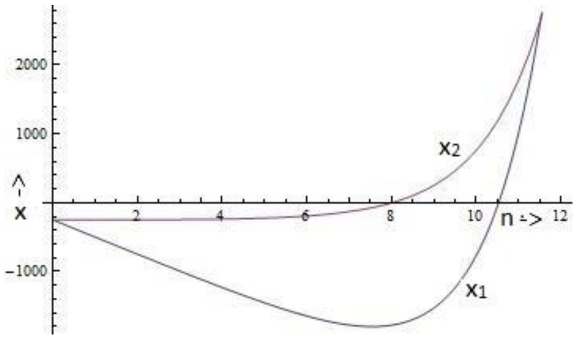

Figure 2. Discrete solution graph (take n = ξ)

Using Z- transformation on discrete case then the solution of (4.3) is

$$x(\xi) = \frac{1}{12}\sum_{s=0}^{\xi-1}\begin{bmatrix}11-\xi & -11+\xi+(11-\xi)(-1-s)\\ -1 & s+2\end{bmatrix}\begin{bmatrix}-1\\-1\end{bmatrix}$$
$$-\frac{1}{12}\sum_{s=\xi}^{9}\begin{bmatrix}1+\xi & 1+\xi+10(1+\xi)+(1+\xi)(-1-s)\\ 1 & 10-s\end{bmatrix}\begin{bmatrix}-1\\-1\end{bmatrix}$$
$$= \frac{1}{12}\begin{bmatrix}65+59\xi-6\xi^2\\ 65-12\xi\end{bmatrix}$$

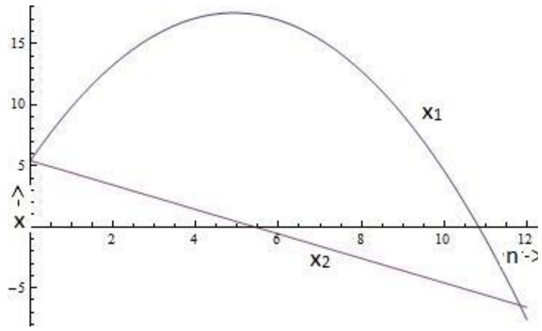

Figure 3. Discrete solution graph (take n = ξ)

5. Conclusion

In this work, a solution is obtained for integro-dynamical system of Volterra type in the form of greens matrix. Further, properties of obtained green matrix and stability analysis for the said system are discussed. Most importantly, even the system is not continuous on some finite piecewise continuous intervals, still the solution is obtained with the help of green matrix. This work is authenticated with numerical examples. Graphs are also provided to explore the existence of solution and analyse stability of system.

Future work: This work can be extended onto the systems of lyapunov type, impulsive and fractional order. This work may make better some processes concerned with robotics, image processing, system trajectories and various other fields.

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