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GENERALIZED α-ADMISSIBLE ALMOST z-CONTRACTIONS INVOLVING SIMULATION FUNCTIONS IN A METRIC SPACE

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Abstract. In this paper, we present some fixed point results in complete metric spaces using generalized α-admissible mappings embedded in the simulation function. These results serve to generalize and unify several related fixed point results found in the existing literature. To validate our findings, we provide a specific example illustrating the application of these results.

1. Introduction

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ represents the set of positive integers. As usual $\mathbb{R}$ indicates the set of all real numbers. Furthermore, we set $\mathbb{R}_0^+ := [0, \infty)$. Khojasteh et al.[14] introduced the notion of z-contraction by using a new class of auxiliary function called simulation function. They [14] proved several fixed point theorems and showed that many results in the literature are simple consequences of their obtained results.

Definition 1. [14] *A function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is called a simulation function if ζ satisfies the following conditions:*

(ζ_1) $\zeta(0, 0) = 0$.
(ζ_2) $\zeta(t, s) < s - t$, *for all* $t, s > 0$.
(ζ_3) *If* $\{t_n\}, \{s_n\}$ *are sequences in* $(0, \infty)$ *such that* $\lim_{n\to\infty} t_n = lim_{n\to\infty} s_n = l \in (0, \infty)$, *then* $\lim_{n\to\infty} sup\ \zeta(t_n, s_n) < 0$.

In [14], the following unique fixed point theorem is established.

Theorem 2. [14] *Let (X, d) be a metric space and $T : X \to X$ be a z-contraction with respect to a simulation function ζ, that is*

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0$$

for all $x, y \in X$. Then T has a unique fixed point.

It is worth mentioning that the Banach contraction is an example of z-contractions by defining $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ via $\zeta(t, s) = \lambda s - t$, for all $s, t \in [0, \infty)$, where $\lambda \in [0, 1)$. Argoubi et al. [2] modified Definition (1) as follows.

Definition 3. [2] *A simulation function is a function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ that satisfies the following conditions:*

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(i) $\zeta(t,s) < s - t$, *for all* $s,t > 0$;
(ii) *If* $\{t_n\}$ *and* $\{s_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = lim_{n\to\infty} s_n = l \in (0,\infty)$, *then* $\lim_{n\to\infty} sup\ \zeta(t_n, s_n) < 0$.

It is clear that any simulation function in the sense of Khojasteh et al. [14] (Definition 1) is also a simulation function in the sense of Argoubi et al. [2] (Definition 3). The converse is not true. Very recently many fixed point results by using simulation functions have been provided. We have used some important article related to this paper [1, 4, 9, 10, 12, 15, 16, 17, 19].

In 2012, Samet et al.[20] introduced the concept of α-contraction and α-admissible and established various fixed point results for such class of mapping defined on complete metric space. There after the existence of fixed point of α-admissible contraction type mappings in different metric spaces have been studied by several authors (see [8, 9, 10, 16, 18]) and references cited there in.

Definition 4. [20] *Consider two mappings* $f : X \to X$ *and* $\alpha : X \times X \to [0,\infty)$. *Then* f *is called* α*-admissible mapping if for all* $x, y \in X$ *with* $\alpha(x,y) \geq 1$ *implies* $\alpha(fx, fy) \geq 1$.

In this paper, we introduce the concept of generalized α-admissible almost z-contraction with respect toζ. We also establish the existence of fixed point for this class of mappings in complete metric spaces. The presented theorems extends, generalizes and improve many existing results in the literature, in particular the results [3, 7, 11, 13, 17].

2. Main Results

Here we put forward the notion of Geraghty functions and Geraghty contractions were discussed by Geraghty [11].

Definition 5. [11] *A function* $\beta : [0,\infty) \to (0,1)$ *is called Geraghty function if* $\{r_n\} \subset [0,\infty)$ *and* $\lim_{n\to\infty} \beta(r_n) = 1^-$ *implies* $r_n \to 0^+$ *as* $n \to \infty$.

Definition 6. [11] *A mapping* $T : X \to X$ *is called Geraghty contraction if there exists a Geraghty function* β *such that* $d(Tx, Ty) \leq \beta(d(x,y))d(x,y)$, *for all* $x, y \in X$.

The concept of Geraghty contraction mapping has been used in many works for example (see[3, 8, 18]). Berinde [5, 6] extended the class of contractive mappings, introducing the notion of almost contractions as follows.

Definition 7. *Let* (X,d) *be a metric space. A self mapping* T *on* X *is called an almost contraction if there are constants* $\lambda \in (0,1)$ *and* $\theta \geq 0$ *such that* $d(Tx,Ty) \leq \lambda d(x,y) + \theta d(y,Tx)$ *for all* $x, y \in X$.

Berinde [5, 6] *proved that every almost contraction mapping defined in a complete metric space has at least one fixed point. Subsequently, many authors* [7, 9, 13] *demonstrated that almost contractions type mappings have a unique fixed point in different metric spaces.*

By using the concept of Geraghty function (β) *and almost contractions, we introduce the following:*

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Definition 8. *Let (X, d) be a metric space, $f : X \to X$ be a self mapping, there exists $\zeta \in Z$ and $\alpha : X \times X \to [0, \infty)$. Then continuous mapping f is called generalized α-admissible almost z-contraction with respect to ζ and $\beta \in G$ and $L \geq 0$ such that for all $x, y \in X$,*

$$\zeta(\alpha(x, fx)\alpha(y, fy)d(fx, fy), K(x, y)) \geq 0 \tag{2.1}$$

for all distinct $x, y \in X$, where ζ is a simulation function in the sense of Definition 1.

Also

$$K(x, y) = \beta(E(x, y))E(x, y) + LN(x, y), \tag{2.2}$$

where

$$E(x, y) = d(x, y) + |d(x, fx) - d(y, fy)|$$

and

$$N(x, y) = min\{d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}.$$

Now we prove our main result.

Theorem 9. *Let (X, d) be a complete metric space, f is a generalized α-admissible almost z-contraction with respect to ζ furthermore, we suppose for all $x, y \in X$ such that:*

(*i*) *f is α-admissible;*
(*ii*) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*
(*iii*) *for every sequence $\{x_n\} \in X$ such that $\alpha(x_n, fx_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\{x_n\}$ converges to x, then $\alpha(x, fx) \geq 1$;*
(*iv*) *$\alpha(x, fx) \geq 1$ for all $x \in Fix(f)$.*

Then f has a unique fixed point x^* in X.

Proof. On account of (ii), there is a point $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. There exists $x_n \in X$ such that $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$. Since f is α-admissible, we obtain $\alpha(fx_0, fx_1) = \alpha(x_1, x_2) \geq 1$ implies $\alpha(fx_1, fx_2) = \alpha(x_2, x_3) \geq 1$. By induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1, \quad \text{for} \quad \text{all} \quad n \in \mathbb{N} \cup \{0\}. \tag{2.3}$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then $x_n = x_{n+1} = fx_n$ and hence x_n is a fixed point of f.

Therefore, we can assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then we get $d(x_n, x_{n+1}) > 0$, so by (2.1), we have

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_n, fx_n)\alpha(x_{n-1}, fx_{n-1})d(fx_n, fx_{n-1}), K(x_n, x_{n-1})) \\ &= \zeta(\alpha(x_n, x_{n+1})\alpha(x_{n-1}, x_n)d(x_{n+1}, x_n), K(x_n, x_{n-1})) \\ &< K(x_n, x_{n-1}) - \alpha(x_n, x_{n+1})\alpha(x_{n-1}, x_n)d(x_{n+1}, x_n), \end{aligned} \tag{2.4}$$

where $K(x_n, x_{n-1}) = \beta(E(x_n, x_{n-1}))E(x_n, x_{n-1}) + LN(x_n, x_{n-1})$.
Also,

$$\begin{aligned} N(x_n, x_{n-1}) &= min\{d(x_n, fx_n), d(x_{n-1}, fx_{n-1}), d(x_n, fx_{n-1}), \\ &\quad d(x_{n-1}, fx_n)\} \\ &= min\{d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_n), \\ &\quad d(x_{n-1}, x_{n+1})\} = 0, \end{aligned}$$

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and

$$\begin{aligned} E(x_n, x_{n-1}) &= d(x_n, x_{n-1}) + |d(x_n, fx_n) - d(x_{n-1}, fx_{n-1})| \\ &= d(x_n, x_{n-1}) + |d(x_n, x_{n+1}) - d(x_{n-1}, x_n)| = d(x_n, x_{n+1}). \end{aligned}$$

Therefore,

$$K(x_n, x_{n-1}) = \beta(d(x_n, x_{n+1}))d(x_n, x_{n+1}),$$

from (2.4), we get

$$0 \leq \beta(d(x_n, x_{n+1}))d(x_n, x_{n+1}) - \alpha(x_n, x_{n+1})\alpha(x_{n-1}, x_n)d(x_{n+1}, x_n)$$

which implies that

$$d(x_n, x_{n+1}) \leq \beta(d(x_n, x_{n+1}))d(x_n, x_{n+1}) < d(x_n, x_{n+1}), \tag{2.5}$$

a contradiction. Consequently, we deduce that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \quad \text{for} \quad \text{each} \quad n \in \mathbb{N}.$$

Thus, we conclude that the sequence $\{d(x_{n-1}, x_n)\}$ is a monotonically decreasing sequence of non-negative reals and bounded from below by zero. So, there is some $r \geq 0$ such that $\lim_{n\to\infty} d(x_{n-1}, x_n) = r$. It is evident that $\lim_{n\to\infty} E(x_{n-1}, x_n) = r$. As a next step, we will show that $\lim_{n\to\infty} d(x_n, x_{n-1}) = 0$. We assert that $r = 0$. Suppose, in contrast that $r \neq 0$, then since f is generalized α-admissible almost z-contraction with respect to $\zeta \in Z$ therefore by (ζ_3) and equation (2.5), and taking limit as $n \to \infty$, we have

$$0 \leq \lim_{n\to\infty} \sup \zeta(\alpha(x_n, x_{n+1})\alpha(x_{n-1}, x_n)d(x_{n+1}, x_n), K(x_n, x_{n-1})) < 0.$$

Therefore

$$\lim_{n\to\infty} \beta(E(x_{n-1}, x_n)) = 1 \Rightarrow \lim_{n\to\infty} E(x_{n-1}, x_n) = 0.$$

Attendantly, $r = 0$ and also

$$r = \lim_{n\to\infty} d(x_n, x_{n-1}) = 0. \tag{2.6}$$

Now, we will show that sequence $\{x_n\}$ is a Cauchy sequence. Assume that $\{x_n\}$ is not a Cauchy sequence, then there exists $\epsilon > 0$ and sequences $\{x_{n_k}\}$, $\{x_{m_k}\} : m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) > \epsilon$ and $d(x_{m_k-1}, x_{n_k}) \leq \epsilon$ for all $m, n, k \in \mathbb{N}$. Therefore, by the triangle inequality, we have that

$$\begin{aligned} \epsilon < d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + \epsilon. \end{aligned} \tag{2.7}$$

Letting $k \to \infty$, using (2.6) and (2.7), we get

$$\lim_{n\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon. \tag{2.8}$$

Since f is a generalized α-admissible almost z-contraction with respect to ζ,

$$0 \leq \zeta(\alpha(x_{m_k-1}, x_{m_k})\alpha(x_{n_k-1}, x_{n_k})d(x_{m_k}, x_{n_k}), K(x_{m_k-1}, x_{n_k-1}))$$

It follows from condition (ζ_2), we get

$$\begin{aligned} 0 &< K(x_{m_k-1}, x_{n_k-1}) - \alpha(x_{m_k-1}, x_{m_k})\alpha(x_{n_k-1}, x_{n_k})d(x_{m_k}, x_{n_k}) \\ d(x_{m_k}, x_{n_k}) &= d(fx_{m_k-1}, fx_{n_k-1}) < K(x_{m_k-1}, x_{n_k-1}). \end{aligned} \tag{2.9}$$

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Also,

$$K(x_{m_k-1}, x_{n_k-1}) = \beta(E(x_{m_k-1}, x_{n_k-1}))E(x_{m_k-1}, x_{n_k-1}) + LN(x_{m_k-1}, x_{n_k-1}), \quad (2.10)$$

where

$$N(x_{m_k-1}, x_{n_k-1}) = min\{d(x_{m_k-1}, x_{m_k}), d(x_{n_k-1}, x_{n_k}), d(x_{m_k-1}, x_{n_k}), d(x_{n_k-1}, x_{m_k})\}$$

and

$$E(x_{m_k-1}, x_{n_k-1}) = d(x_{m_k-1}, x_{n_k-1}) + |d(x_{m_k-1}, x_{m_k}) - d(x_{n_k-1}, x_{n_k})|.$$

Letting $k \to \infty$, using (2.6) and (2.8), we get

$$\lim_{k\to\infty} K(x_{m_k-1}, x_{n_k-1}) = \epsilon. \quad (2.11)$$

By (2.8), (2.9), (2.11) and the condition (ζ_3), we get

$$0 \leq \lim_{n\to\infty} \sup \zeta((\alpha(x_{m_k-1}, x_{m_k})\alpha(x_{n_k-1}, x_{n_k})d(x_{m_k}, x_{n_k}), K(x_{m_k-1}, x_{n_k-1})) < 0.$$

This is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Thus $\lim_{m,n\to\infty} d(x_n, x_m)$ exists and is equal to zero. Since (X, d) is complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0. \quad (2.12)$$

Now we shall show that $fx^* = x^*$. Since f is continuous, we drive the desired results obviously, that is

$$fx^* = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x^*.$$

Suppose we have (iii),

$$0 = \lim_{m,n\to\infty} d(x_m, x_n) = \lim_{n\to\infty} d(x_n, x^*) = d(x^*, x^*)$$

and $\alpha(x^*, fx^*) \geq 1$. Moreover,

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_n, fx_n)\alpha(x^*, fx^*)d(fx_n, fx^*), K(x_n, x^*)) \\ &= \zeta(\alpha(x_n, x_{n+1})\alpha(x^*, fx^*)d(x_{n+1}, fx^*), K(x_n, x^*)) \\ &< K(x_n, x^*) - \alpha(x_n, x_{n+1})\alpha(x^*, fx^*)d(x_{n+1}, fx^*), \end{aligned} \quad (2.13)$$

where

$$K(x_n, x^*) = \beta(E(x_n, x^*))E(x_n, x^*) + LN(x_n, x^*).$$

Also,

$$\begin{aligned} N(x_n, x^*) &= min\{d(x_n, fx_n), d(x^*, fx^*), d(x_n, fx^*), d(x^*, fx_n)\} \\ &= min\{d(x_n, x_{n+1}), d(x^*, fx^*), d(x_n, fx^*), d(x^*, x_{n+1})\} = 0. \end{aligned} \quad (2.14)$$

And

$$\begin{aligned} E(x_n, x^*) &= d(x_n, x^*) + |d(x_n, fx_n) - d(x^*, fx^*)| \\ &= 0 + |0 - d(x^*, fx^*)| = d(x^*, fx^*), \quad \text{as} \quad n \to \infty. \end{aligned} \quad (2.15)$$

By (2.13), (2.14) and (2.15), we get

$$\begin{aligned} d(x_{n+1}, fx^*) &= d(fx_n, fx^*) \\ &\leq \alpha(x_n, x_{n+1})\alpha(x^*, fx^*)d(fx_n, fx^*) \\ &< d(x^*, fx^*). \end{aligned} \tag{2.16}$$

By letting $n \to \infty$ in (2.13), together with the observation above, we have

$$0 \leq \lim_{n\to\infty} sup\zeta(\alpha(x_n, fx_n)\alpha(x^*, fx^*)d(fx_n, fx^*), K(x_n, x^*)) < 0.$$

This is a contradiction. Hence, therefore x^* is a fixed point of f i.e. $fx^* = x^*$.

Suppose that x^* and u^* be two fixed points of f and hence $x^*, u^* \in Fix(f)$ which is a generalized α-admissible almost z-contraction self mappings of a metric space (X, d). By (2.1), we have that

$$0 \leq \zeta(\alpha(x^*, fx^*)\alpha(u^*, fu^*)d(fx^*, fu^*), K(x^*, u^*)), \tag{2.17}$$

where

$$K(x^*, u^*) = \beta(E(x^*, u^*))E(x^*, u^*) + LN(x^*, u^*). \tag{2.18}$$

Also

$$E(x^*, u^*) = d(x^*, u^*) + |d(x^*, fx^*) - d(u^*, fu^*)| = d(x^*, u^*) \tag{2.19}$$

and

$$N(x^*, u^*) = min\{d(x^*, fx^*), d(u^*, fu^*), d(x^*, fu^*), d(u^*, fx^*)\} = 0. \tag{2.20}$$

Therefore, from (2.17), (2.18), (2.19) and (2.20) we get that

$$\begin{aligned} 0 &\leq \zeta(\alpha(x^*, fx^*)\alpha(u^*, fu^*)d(fx^*, fu^*), d(x^*, u^*)) \\ &= \zeta(\alpha(x^*, x^*)\alpha(u^*, u^*)d(x^*, u^*), d(x^*, u^*)). \end{aligned}$$

This is a contradiction. Thus, we have $x^* = u^*$. Hence f is a unique fixed point. □

Theorem 10. *Let (X, d) be a complete metric space, f is a generalized α-admissible almost z-contraction with respect to ζ. Assume that*

- (i) *f is a α-admissible,*
- (ii) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$,*
- (iii) *X is a regular and for every sequence $\{x_n\}$ in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and we have $\alpha(x_m, x_n) \geq 1$ for all $m, n \in \mathbb{N}$ with $m < n$,*
- (iv) *$\alpha(x, y) \geq 1$, for all $x, y \in Fix(f)$.*

Then f has a unique fixed point x^* in X.

Proof. By (ii), let $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$. There exist $x_n \in X$ such that $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$. We have by Theorem 9, $\{x_n\}$ is a Cauchy sequence such that $\lim_{n\to\infty} d(x_n x_{n+1}) = 0$. Thus $\lim_{m,n\to\infty} d(x_n, x_m)$ exists and is equal to 0. Since (X, d) is complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0, \tag{2.21}$$

then

$$\lim_{m,n\to\infty} d(x_m, x_n) = \lim_{n\to\infty} d(x_n, x^*) = d(x^*, x^*) = 0.$$

Since X is regular, therefore there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq 1$ for all $k \in \mathbb{N}$. Therefore

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_{n_k}, fx_{n_k})\alpha(x^*, fx^*)d(fx_{n_k}, fx^*), K(x_{n_k}, x^*)) \\ &= \zeta(\alpha(x_{n_k}, x_{n_{k+1}})\alpha(x^*, fx^*)d(x_{n_{k+1}}, fx^*), K(x_{n_k}, x^*)) \\ &< K(x_{n_k}, x^*) - \alpha(x_{n_k}, x_{n_{k+1}})\alpha(x^*, fx^*)d(x_{n_{k+1}}, fx^*), \end{aligned} \tag{2.22}$$

where

$$K(x_{n_k}, x^*) = \beta(E(x_{n_k}, x^*))E(x_{n_k}, x^*) + LN(x_{n_k}, x^*). \tag{2.23}$$

Also

$$\begin{aligned} E(x_{n_k}, x^*) &= d(x_{n_k}, x^*) + |d(x_{n_k}, fx_{n_k}) - d(x^*, fx^*)| \\ &= d(x_{n_k}, x^*) + |d(x_{n_k}, x_{n_{k+1}}) - d(x^*, fx^*)| \\ &= d(x^*, fx^*) \quad \text{for} \quad large \quad k, \end{aligned} \tag{2.24}$$

and

$$N(x_{n_k}, x^*) = min\{d(x_{n_k}, fx_{n_k}), d(x^*, fx^*), d(x_{n_k}, fx^*), d(x^*, fx_{n_k})\}$$

$$= min\{d(x_{n_k}, x_{n_{k+1}}), d(x^*, fx^*), d(x_{n_k}, fx^*), d(x^*, x_{n_{k+1}})\} = 0. \tag{2.25}$$

Therefore $K(x_{n_k}, x^*) = d(x^*, fx^*)$. Consequently, we have

$$d(x_{n_k+1}, fx^*) = d(fx_{n_k}, fx^*) \leq \alpha(x_{n_k}, fx_{n_k})\alpha(x^*, fx^*)d(fx_{n_k}, fx^*)$$

$$< d(x^*, fx^*) \quad \text{for} \quad \text{all} \quad k \in \mathbb{N}. \tag{2.26}$$

By (2.22), (2.26) and the condition (ζ_3), we get

$$0 \leq lim_{n\to\infty} sup \zeta(\alpha(x_n, fx_n)\alpha(x^*, fx^*)d(fx_n, fx^*), K(x_n, x^*)) < 0.$$

This is a contradiction. Hence, therefore x^* is a fixed point of f. Suppose that x^* and u^* be two fixed points of f and hence $x^*, u^* \in Fix(f)$ which is a generalized α-admissible almost z-contraction self -mappings of a metric space (X, d). By (2.1), we have that

$$0 \leq \zeta(\alpha(x^*, fx^*)\alpha(u^*, fu^*)d(fx^*, fu^*), K(x^*, u^*)), \tag{2.27}$$

where $K(x^*, u^*) = d(x^*, u^*)$, by using (2.18), (2.19) and (2.20). This together with (2.27) shows that

$$\begin{aligned} 0 &\leq \zeta(\alpha(x^*, fx^*)\alpha(u^*, fu^*)d(fx^*, fu^*), K(x^*, u^*)) \\ &= \zeta(\alpha(x^*, x^*)\alpha(u^*, u^*)d(x^*, u^*), d(x^*, u^*)). \end{aligned}$$

This is a contradiction. Thus, we have $x^* = u^*$. Hence f has a unique fixed point. □

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Corollary 11. *Let (X, d) be a complete metric space, $f : X \to X$ be a self-mapping, there exists $\zeta \in Z$ and $\alpha : X \times X \to [0, \infty)$ be a function with $\alpha(x, y) = 1$ for all $x, y \in X$ such that*

$$\zeta(d(fx, fy), K(x, y)) \geq 0$$

for all distinct $x, y \in X$, where

$$K(x, y) = \beta(E(x, y))E(x, y) + LN(x, y)$$

and

$$E(x, y) = d(x, y) + |d(x, fx) - d(y, fy)|$$

and

$$N(x, y) = min\{d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}.$$

Then f has a unique fixed point x^* in X.

Example 12. *Let $X = [0, 1]$ endowed with metric $d(x, y) = |x - y|$ for all $x, y \in X$. Let $\zeta(t, s) = s - t$ and considering $\beta : [0, \infty) \to [0, 1)$ as $\beta(t) = \frac{1}{1+t}$ for all $t \geq 0$ and $L \geq 0$. Let $f : X \to X$ be defined by $f(x) = \frac{x}{3}$ for all $x \in [0, 1]$ and $\alpha : X \times X \to [0, \infty)$ be defined by*

$$\alpha(x, y) = \begin{cases} 1, & \textit{if}, \quad x, y \in [0, 1]; \\ 0, & \textit{otherwise}. \end{cases}$$

Note that f is an α-admissible if $\alpha(x, fx) \geq 1$ implies $\alpha(fx, f^2x) \geq 1$. Now by definition of α and $x, y \in [0, 1]$, we have $\alpha(x, fx) = \alpha(x, \frac{x}{3}) = 1$. Similarly $\alpha(y, fy) = 1$ for all $x, y \in X$.

From above, it is clear that f is a generalized α-admissible mapping. Now

$$\begin{aligned} \zeta(d(fx, fy), K(x, y)) &= K(x, y) - d(fx, fy) \\ &= \beta(E(x, y))E(x, y) + LN(x, y) - \frac{1}{3}|x - y| \\ &= \frac{E(x, y)}{1 + E(x, y)} + LN(x, y) - \frac{1}{3}|x - y| \\ &\leq \frac{\frac{5}{3}d(x, y)}{1 + \frac{5}{3}d(x, y)} + LN(x, y) - \frac{1}{3}|x - y| \\ &= \frac{\frac{5}{3}|x - y|}{1 + \frac{5}{3}|x - y|} + LN(x, y) - \frac{1}{3}|x - y| \geq 0. \end{aligned}$$

Therefore, f is generalized α-admissible almost z-contraction with respect to $\zeta \in Z$. Hence all the assumptions of Theorem 9 and Corollary 11 are satisfied and hence f has a unique fixed point.

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