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A Law of the Iterated Logarithm for the Signum Function

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Abstract

In 2023, S. Ghimire established a one-sided version of a law of the iterated logarithm, denoted as LIL, for summations of signum functions analogous to the LIL proposed by Salem and Zygmund for trigonometric series. In this article, we complete the LIL for these signum functions by establishing the complimentary version of the LIL.

Keywords: Signum functions, q-lacunary series, law of the iterated logarithm, Borel-Cantelli lemma.

1. Introduction

The LIL is a widely recognized theorem in probability that complements two other fundamental theorems: the central limit theorem (CLT) and the law of large numbers (LLN). While the LLN describes the tendencies of the average of independent random variables with increasing sample size, and the CLT outlines the distribution of these sums, the LIL provides insights into the fluctuations of these sums, especially concerning their bounds. The LIL emerged from Khintchine's [1] investigations, in which he sought to ascertain the precise rate of convergence of normal numbers. Kolmogorov [5] later generalized this result to include independent random variables. Since its inception, the LIL has developed into a fundamental theorem with extensive applications spanning various areas of mathematics and statistics. A similar LIL has been developed across different fields, including harmonic functions [7], martingales [10], [12], q-lacunary series [8], random walks, stochastic processes, and more.

In the realm of mathematics, Salem and Zygmund [8] were the first to introduce a LIL for the sums of q-lacunary trigonometric series. Erdos and Gal [6] subsequently obtained a comparable outcome for a particular category of q-lacunary series. In this LIL, only the sum of the first n-terms of the lacunary series was considered as in Kolmogorov's LIL. Subsequently, M. Weiss [4] obtained an LIL for q-lacunary series analogous to Kolmogorov's LIL. Additionally, in the same paper, Salem and Zygmund [8] introduced another LIL for q-lacunary series, as stated below:

Theorem 1 (Salem and Zygmund) Let $\tilde{S}_N = \sum_{i=N}^{\infty}(a_i \cos n_i\theta + b_i \sin n_i\theta)$ with $\frac{n_{i+1}}{n_i} > q > 1$ and $c_i^2 = a_i^2 + b_i^2$ satisfy $\sum_{i=1}^{\infty} c_i^2 < \infty$. Define $\tilde{B}_M = \sum_{i=M}^{\infty} c_i^2$ and $\tilde{C}_M = \max_{i\geq M} |c_i|$. Assume that $\tilde{B}_1 < \infty$ and $\tilde{C}_M^2 \leq K_M \left(\frac{\tilde{B}_M^2}{\ln\ln\frac{1}{\tilde{B}_M}}\right)$ where K_M approaches to 0 and M approaches to infinity. Then

$$\limsup_{M\to\infty} \frac{\tilde{S}_M(\theta)}{\sqrt{2\tilde{B}_M^2 \ln\ln \frac{1}{\tilde{B}_M}}} \le 1$$

for a.e. θ in the unit circle.

In this LIL variant, the focus lies on the sums beyond the initial n-terms, specifically on the tail sums of the series. Consequently, this version is often referred to as the ``tail LIL" because of its emphasis on the tail sum aspect. Salem and Zygmund only derived the upper bound in this tail LIL. Under similar conditions, S. Ghimire and C.N. Moore [9] established the converse of the aforementioned result. Their result is:

Theorem 2. Assuming the same notation and hypotheses as stated in the preceding theorem, we have

$$\limsup_{N\to\infty} \frac{\tilde{S}_N(\theta)}{\sqrt{2\tilde{B}_N^2 \ln\ln \frac{1}{\tilde{B}_N}}} \ge 1$$

for a.e. θ in $[0, 2\pi]$.

Now Theorem 2, when combined with Theorem 1 give the conclusion

$$\limsup_{N\to\infty} \frac{\tilde{S}_N(\theta)}{\sqrt{2\tilde{B}_N^2 \ln\ln \frac{1}{\tilde{B}_N}}} = 1$$

for a.e. θ in $[0, 2\pi]$. A similar LIL for summation of signum functions was recently obtained by S. Ghimire [11] showing that the convergence rate of summation of the functions is controlled by the tail sums of the square function as in the LIL introduced by Salem and Zygmund. The one-sided version of S. Ghimire's LIL is as follows:

Theorem 3. Suppose $\{u_i\}$ is a sequence of signum functions defined by $u_i(t) = sgn\,(\sin 2^i \pi t)$ and $\{b_i\}_{i=1}^{\infty}$ where $b_i \in \mathbb{R}$ satisfies $\sum_{i=1}^{\infty} b_i^2 < \infty$. Then

$$\limsup_{n\to\infty} \frac{|\sum_{i=n+1}^{\infty} b_i u_i(t)|}{\sqrt{2\sum_{i=n+1}^{\infty} b_i^2 \,\ln\ln\left(\frac{1}{\sum_{i=n+1}^{\infty} b_i^2}\right)}} \le 1$$

for a.e. $t \in [0, 1)$.

Here, we obtain the lower limit version of the LIL for the summation of signum functions. Our main result is:

Theorem 4. Suppose $\{u_i\}$ is a sequence of signum functions defined by $u_i(t) = sgn\,(\sin 2^i \pi t)$ and $\{b_i\}_{i=1}^{\infty}$ where $\{b_i\}$ is a square integrable real-valued sequence with $B_n = \sum_{i=n}^{\infty} b_i^2$ and assume $\lim_{n\to\infty} \frac{b_n^2}{B_n} = 0$. Then

$$\limsup_{n\to\infty} \frac{|\sum_{i=n+1}^{\infty} b_i u_i(t)|}{\sqrt{2B_n \ln\ln \frac{1}{B_n}}} \geq 1$$

for a.e. $t \in [0,1)$.

Note that if the sequence $\{|b_i|\}$ is non increasing, or more generally, if $\lim_{i\to\infty} \left|\frac{b_i}{b_{i+1}}\right| \geq 1$, then for large n, we have

$$\frac{b_n^2}{B_n^2} \leq \frac{b_n^2}{\sum_{i=m}^{\infty} b_i^2} \leq \frac{1}{n+1}$$

and assumption in the theorem is satisfied. The proof consists of stopping time argument with application of certain estimates. To prove our main result, we first utilize sub-Gaussian type estimates for the summation of signum functions followed by the application of both versions of Borelli lemma. In what follows, we use measure space $(I = (0,1), \mathcal{B}, \mu)$ and $|.|$ stands for probability measure μ restricted on I. To establish our main result, we begin by introducing some definitions and obtaining estimates.

2. Preliminaries

Let's revisit the definition of a general signum function:

$$sgn(t) = \begin{cases} 1 \; if \; t \geq 0; \\ -1 \; if \; t < 0. \end{cases}$$

In constructing a sequence, we define $u_i(t) = sgn(\sin 2^i \pi t)$ on the interval $(0,1)$. We say A_n happens infinitely often, abbreviated as $A_n \; i.o.$, if for all n there is $m \geq n$ such that A_m is true. We now state Borelli lemma of both versions. Please see [3] for the proof.

Lemma 5 (Borel-Cantelli 1) If $\{A_k\}$ satisfies $\sum_{k=1}^{\infty} |A_k| < \infty$, then $|\{A_k \; i.o.\}| = 0$.

Lemma 6 (Borel-Cantelli 2) If independent events $\{A_k\}$ satisfies $\sum_{k=1}^{\infty} |A_k| = \infty$, then $|\{A_k \; i.o.\}| = 1$.

Next, we state a result on exponential estimate for independent random variables which will be used in the proof of our main result. For the proof, please see [2].

Theorem 7. Suppose $\{Y_k\}$ is a sequence of random variables on sample spac $(I = (0,1), \mathcal{B}, \mu)$, with zero mean and variance σ_k^2. Let

$$S_n = \sum_{k=1}^{n} Y_k, \quad s_n^2 = \sum_{k=1}^{n} \sigma_k^2 \; and \; Z_n = \frac{\max_{k\leq n} |Y_k|}{s_n}$$

Then, for given $\beta > 0$, if $Z_n(\beta)$ is very small and $\gamma = \gamma(\beta)$ is very large, then

$$\left| \left\{ t \in I \, \frac{|S_n(t)|}{s_n} > \gamma \right\} \right| > \exp\left(-\frac{\gamma^2}{2}(1+\beta) \right).$$

Following, we present a sub-Gaussian type estimate crucial to proving our main result. For a detailed proof, refer to [11]. We sketch the proof.

Lemma 8. Let $\{b_i\}$ where $b_i \in \mathbb{R}$ and $\{u_i\}$ be a sequence of signum functions defined by $u_i(t) = sgn\,(\sin 2^i \pi t)$. Then for all $\alpha > 0$ and for a fixed number n, we have

$$\left|\left\{t \in I: \sup_{m\geq n}\left|\sum_{i=m+1}^{\infty} b_i u_i(t)\right| > \alpha\ \right\}\right| \leq 12 \exp\left(\frac{-\alpha^2}{2\ \sum_{i=n+1}^{\infty} b_i^2}\right).$$

Proof: Let $M \gg n$ and we write $g_i(t) = \sum_{k=1}^{i} b_k u_k(t)$. Then using Levy's inequality, we get

(2.1) $$\left|\left\{t \in I: \max_{M\geq m\geq n}|g_m(t) - g_n(t)| > \alpha\ \right\}\right| \leq |\{t \in I: |g_M(t) - g_n(t)| > \alpha\}|$$

Using Lemma 1 in [11], we get

(2.2) $$\left|\left\{t \in I: \max_{m\geq n}|g_m(t) - g_n(t)| > \alpha\ \right\}\right| \leq 6 \exp\left(\frac{-\alpha^2}{2\ \sum_{i=n+1}^{\infty} b_i^2}\right)$$

Using (2.1) in (2.2), we get

$$\left|\left\{t \in I: \sup_{M\geq m\geq n}|g_M(t) - g_m(t)| > \alpha\ \right\}\right| \leq 12 \exp\left(\frac{-\alpha^2}{2\ \sum_{i=n+1}^{\infty} b_i^2}\right).$$

Then continuity property gives

$$\left|\left\{t \in I: \sup_{m\geq n}|g(t) - g_m(t)| > \alpha\ \right\}\right| \leq \lim_{M\to\infty}\left|\left\{t \in I: \sup_{M\geq m\geq n}|g_M(t) - g_m(t)| > \alpha\ \right\}\right|$$
$$\leq 12 \exp\left(\frac{-\alpha^2}{2\ \sum_{i=n+1}^{\infty} b_i^2}\right).$$

Thus we have

$$\left|\left\{t \in I: \sup_{m\geq n}\left|\sum_{i=m+1}^{\infty} b_i u_i(t)\right| > \alpha\ \right\}\right| \leq 12 \exp\left(\frac{-\alpha^2}{2\ \sum_{i=n+1}^{\infty} b_i^2}\right).$$

3. Main Result: Proof of Theorem 4

Let θ be very large and $0 << \epsilon < 1$. We next choose $0 < \alpha < 2$ in such a way that $(1 - \epsilon^2)(1 + \alpha) > 1$. Define stopping times by

$$n_j = \min\left(n: \sum_{i=n+1}^{\infty} b_i^2\ < \frac{1}{\theta^j}\right).$$

We have

$$\sum_{i=n_j}^{\infty} b_i^2\ = b_{n_j}^2 + \sum_{i=n_j+1}^{\infty} b_i^2$$

Then for sufficiently large n_j, we have

(3.1) $$(1 - \epsilon^2)\sum_{i=n_j}^{\infty} b_i^2\ < \sum_{i=n_j+1}^{\infty} b_i^2\ < \frac{1}{\theta^j}$$

By definition of n_j,

(3.2) $$(1-\epsilon^2)\frac{1}{\theta^j} < (1-\epsilon^2)\sum_{i=n_j}^{\infty} b_i^2$$

So from (3.1) and (3.2), we get,

$$(1-\epsilon^2)\frac{1}{\theta^j} < \sum_{i=n_j+1}^{\infty} b_i^2 < \frac{1}{\theta^j}$$

Thus we have

(3.3) $$\frac{\sum_{i=n_j+1}^{\infty} b_i^2}{\sum_{i=n_{j+1}+1}^{\infty} b_i^2} \geq (1-\epsilon^2)\theta$$

This gives

$$\left|\left\{t \in I: \sup_{n\geq n_{j+1}} \left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right| > \sqrt{\frac{2(1+\alpha)}{\theta}\sum_{i=n_j+1}^{\infty} b_i^2 \ \ln\ln\left(\frac{1}{\sum_{i=n_j+1}^{\infty} b_i^2}\right)}\right\}\right|$$

$$= \left|\left\{t \in I: \sup_{n\geq n_{j+1}} \frac{\left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right|}{\sqrt{\sum_{i=n_{j+1}+1}^{\infty} b_i^2}} > \sqrt{\frac{2(1+\alpha)}{\theta}\frac{\sum_{i=n_j+1}^{\infty} b_i^2}{\sum_{i=n_{j+1}+1}^{\infty} b_i^2}\ln\ln\left(\frac{1}{\sum_{i=n_j+1}^{\infty} b_i^2}\right)}\right\}\right|$$

$$\leq \left|\left\{t \in I: \sup_{n\geq n_{j+1}} \frac{\left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right|}{\sqrt{\sum_{i=n_{j+1}+1}^{\infty} b_i^2}} \ \sqrt{\frac{2(1+\alpha)}{\theta}\ \theta(1-\epsilon^2)\ln\ln\left(\frac{1}{\sum_{i=n_j+1}^{\infty} b_i^2}\right)}\right\}\right|$$

$$= \left|\left\{t \in I: \sup_{n\geq n_{j+1}} \left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right| > \sqrt{2(1+\alpha)(1-\epsilon^2)\sum_{i=n_{j+1}+1}^{\infty} b_i^2 \ \ln\ln\left(\frac{1}{\sum_{i=n_j+1}^{\infty} b_i^2}\right)}\right\}\right|$$

$$= \left|\left\{t \in I: \sup_{n\geq n_{j+1}} \left|\sum_{i=1}^{\infty} b_i u_i(t) - \sum_{i=1}^{n} b_i u_i(t)\right| > \sqrt{2(1+\alpha)(1-\epsilon^2)\sum_{i=n_{j+1}+1}^{\infty} b_i^2 \ \ln\ln\left(\frac{1}{\sum_{i=n_j+1}^{\infty} b_i^2}\right)}\right\}\right|$$

Using Lemma 8, we have

$$\left|\left\{t \in I\colon \sup_{n\geq n_{j+1}} \left|\sum_{i=1}^{\infty} b_i u_i(t) - \sum_{i=1}^{n} b_i u_i(t)\right| > \sqrt{2(1+\alpha)(1-\epsilon^2)\sum_{i=n_{j+1}+1}^{\infty} b_i^2 \ln\ln\left(\frac{1}{\sum_{i=n_j+1}^{\infty} b_i^2}\right)}\right\}\right|$$

$$\leq 24\exp\left(\frac{2(1+\alpha)(1-\epsilon^2)\sum_{i=n_{j+1}+1}^{\infty} b_i^2 \ln\ln\left(\frac{1}{\sum_{i=n_j+1}^{\infty} b_i^2}\right)}{2\sum_{i=n_{j+1}+1}^{\infty} b_i^2}\right)$$

$$= 24\left(\ln\left(\frac{1}{\sum_{i=n_j+1}^{\infty} b_i^2}\right)\right)^{-(1+\alpha)(1-\epsilon^2)}$$

$$< 24\left(\frac{1}{\ln\theta^j}\right)^{(1+\alpha)(1-\epsilon^2)}$$

Thus,

$$\left|\left\{t \in I\colon \sup_{n\geq n_{j+1}} \left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right| > \sqrt{\frac{2(1+\alpha)}{\theta}\sum_{i=n_j+1}^{\infty} b_i^2 \ln\ln\left(\frac{1}{\sum_{i=n_j+1}^{\infty} b_i^2}\right)}\right\}\right| < 24\left(\frac{1}{\ln\theta^j}\right)^{(1+\alpha)(1-\epsilon^2)}$$

Define $A = \left\{t \in I\colon \sup_{n\geq n_{j+1}} \left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right| > \sqrt{\frac{2(1+\alpha)}{\theta}\sum_{i=n_j+1}^{\infty} b_i^2 \ln\ln\left(\frac{1}{\sum_{i=n_j+1}^{\infty} b_i^2}\right)}\right\}$

Hence

$$|A| < \frac{24}{(\ln\theta)^{(1+\alpha)(1-\epsilon^2)}}\frac{1}{j^{(1+\alpha)(1-\epsilon^2)}}.$$

Set

$$S_n = \sum_{i=n+1}^{\infty} b_i u_i, \quad s_n^2 = \sum_{i=m}^{n} b_i^2 \quad and \quad Z_n = \frac{\max_{k\leq n}|b_k u_k|}{s_n}$$

Fix $\beta > 0$ and choose $Z_n(\beta)$ and $\gamma(\beta)$ accordingly. Suppose n_j is sufficiently large. Then for this n_j, Theorem 7 gives

$$\left|\left\{t \in I : \frac{\left|\sum_{i=n+1}^{n} b_i u_i(t)\right|}{\sqrt{\sum_{i=n_j+1}^{n} b_i^2}} > \gamma\right\}\right| > \exp\left(\frac{-\gamma^2(1+\beta)}{2}\right).$$

Choose $\gamma = \sqrt{\frac{(2-\alpha)}{(1+\beta)}\ln\ln\left(\frac{1}{\sqrt{\sum_{i=n_j+1}^{\infty} b_i^2}}\right)}$ where $\alpha > 0$. Clearly for large n_j, γ is large as needed in Theorem 7. Thus,

$$\left|\left\{t \in I : \frac{\left|\sum_{i=n_j+1}^{n} b_i u_i(t)\right|}{\sqrt{\sum_{i=n_j+1}^{n} b_i^2}} > \sqrt{\frac{(2-\alpha)}{(1+\beta)}\ln\ln\left(\frac{1}{\sqrt{\sum_{i=n_j+1}^{\infty} b_i^2}}\right)}\right\}\right|$$

$$> \exp\left(\frac{-(2-\alpha)}{(1+\beta)}\ln\ln\left(\frac{1}{\sqrt{\sum_{i=n_j+1}^{\infty} b_i^2}}\right)\frac{(1+\beta)}{2}\right)$$

$$\geq \frac{1}{(j\ln\theta + \ln(1-\epsilon^2))^{\frac{2-\alpha}{2}}}$$

Therefore for large n_j, we have

$$\left|\left\{t \in I : \frac{\left|\sum_{i=n_j+1}^{n} b_i u_i(t)\right|}{\sqrt{\sum_{i=n_j+1}^{n} b_i^2\ \frac{(2-\alpha)}{(1+\beta)}\ln\ln\left(\frac{1}{\sqrt{\sum_{i=n_j+1}^{\infty} b_i^2}}\right)}} > 1\right\}\right| > \frac{1}{2}\frac{1}{(j\ln\theta)^{\frac{2-\alpha}{2}}}$$

This gives

$$\left|\left\{t \in I : \frac{\left|\sum_{i=n_j+1}^{\infty} b_i u_i(t) - \sum_{i=n+1}^{\infty} b_i u_i(t)\right|}{\sqrt{2\sum_{i=n_j+1}^{n} b_i^2\ \ln\ln\left(\frac{1}{\sqrt{\sum_{i=n_j+1}^{\infty} b_i^2}}\right)}} > \sqrt{\frac{(2-\alpha)}{2(1+\beta)}}\right\}\right| > \frac{1}{2}\frac{1}{(j\ln\theta)^{\frac{2-\alpha}{2}}}$$

Using (3.1) for $n \geq n_{j+1}$, we have

$$\sum_{i=n_j+1}^{n} b_i^2 = \sum_{i=n_j+1}^{\infty} b_i^2 - \sum_{i=n+1}^{\infty} b_i^2 \geq (1-\epsilon^2)\frac{1}{\theta^j} - \frac{1}{\theta^{j+1}} \geq \sum_{i=n_j+1}^{\infty} b_i^2 \left(1 - \epsilon^2 - \frac{1}{\theta}\right)$$

Then this gives

$$\left|\left\{t \in I : \frac{\left|\sum_{i=n_j+1}^{\infty} b_i u_i(t) - \sum_{i=n+1}^{\infty} b_i u_i(t)\right|}{\sqrt{2\sum_{i=n_j+1}^{\infty} b_i^2 \left(1-\epsilon^2-\frac{1}{\theta}\right) \ln\ln\left(\frac{1}{\sqrt{\sum_{i=n_j+1}^{\infty} b_i^2}}\right)}} > \sqrt{\frac{(2-\alpha)}{2(1+\beta)}}\right\}\right| > \frac{1}{2}\frac{1}{(j\ln\theta)^{\frac{2-\alpha}{2}}}$$

i.e.

$$\left|\left\{t \in I : \frac{\left|\sum_{i=n_j+1}^{\infty} b_i u_i(t) - \sum_{i=n+1}^{\infty} b_i u_i(t)\right|}{\sqrt{2\sum_{i=n_j+1}^{\infty} b_i^2 \ \ln\ln\left(\frac{1}{\sqrt{\sum_{i=n_j+1}^{\infty} b_i^2}}\right)}} > \sqrt{\frac{(2-\alpha)\left(1-\epsilon^2-\frac{1}{\theta}\right)}{2(1+\beta)}}\right\}\right| > \frac{1}{2}\frac{1}{(j\ln\theta)^{\frac{2-\alpha}{2}}}$$

Define

$$B := \left\{t \in I : \frac{\left|\sum_{i=n_j+1}^{\infty} b_i u_i(t) - \sum_{i=n+1}^{\infty} b_i u_i(t)\right|}{\sqrt{2\sum_{i=n_j+1}^{\infty} b_i^2 \ \ln\ln\left(\frac{1}{\sqrt{\sum_{i=n_j+1}^{\infty} b_i^2}}\right)}} > \sqrt{\frac{(2-\alpha)\left(1-\epsilon^2-\frac{1}{\theta}\right)}{2(1+\beta)}}\right\}$$

Consequently, $|B| \geq \frac{1}{2}\frac{1}{(j\ln\theta)^{\frac{2-\alpha}{2}}}$. Next define

$$C := \left\{t \in I : \frac{\left|\sum_{i=n_{j+1}+1}^{\infty} b_i u_i(t) - \sum_{i=n_j+1}^{\infty} b_i u_i(t)\right|}{\sqrt{2\sum_{i=n_j+1}^{\infty} b_i^2 \ln\ln\left(\frac{1}{\sqrt{\sum_{i=n_j+1}^{\infty} b_i^2}}\right)}} > \sqrt{\frac{(2-\alpha)\left(1-\epsilon^2-\frac{1}{\theta}\right)}{2(1+\beta)}} - 2\sqrt{\frac{(1-\epsilon^2)}{\theta}(1+\alpha)}\right\}$$

Using triangle inequality, we have $B \cap A^c \subset C$. So we have $|B - A| \leq |C|$. Thus, we have

$$|C| \geq \frac{1}{2(j\ln\theta)^{\frac{2-\alpha}{2}}} - \frac{24}{(j\ln\theta)^{(1-\epsilon^2)(1+\alpha)}}.$$

Since $\alpha \in (0,2)$ and $(1-\epsilon^2)(1+\alpha) > 1$, for large j,

$$\frac{1}{3(j\ln\theta)^{\frac{(2-\alpha)}{2}}} \geq \frac{24}{(j\ln\theta)^{(1-\epsilon^2)(1+\alpha)}}$$

This gives

$$|C| \geq \frac{1}{6(j\ln\theta)^{\frac{(2-\alpha)}{2}}}.$$

Now summing over all j we have,

$$\sum_{j=1}^{\infty}\left|\left\{t \in I : \frac{\left|\sum_{i=n_{j+1}+1}^{\infty} b_i u_i(t) - \sum_{i=n_j+1}^{\infty} b_i u_i(t)\right|}{\sqrt{2\sum_{i=n_j+1}^{\infty} b_i^2 \ln\ln\left(\frac{1}{\sqrt{\sum_{i=n_j+1}^{\infty} b_i^2}}\right)}} > \sqrt{\frac{(2-\alpha)\left(1-\epsilon^2-\frac{1}{\theta}\right)}{2(1+\beta)}} - 2\sqrt{\frac{(1-\epsilon^2)}{\theta}(1+\alpha)}\right\}\right|$$

$$\geq \sum_{j=1}^{\infty}\frac{1}{6(j\ln\theta)^{\frac{(2-\alpha)}{2}}} = \frac{1}{6(\ln\theta)^{\frac{(2-\alpha)}{2}}}\sum_{j=1}^{\infty}\frac{1}{j^{\frac{(2-\alpha)}{2}}} = \infty.$$

Here we note that $\left\{\sum_{i=n_j+1}^{\infty} b_i u_i(t) - \sum_{i=n_{j+1}+1}^{\infty} b_i u_i(t)\right\}_{i=1}^{\infty}$ is a sequence of independent random variables. Apply Lemma 5 for a.e. t, there exists an infinite sequence $n_1 < n_2 < n_3 < \cdots$ such that,

$$\frac{\left|\sum_{i=n_{j+1}+1}^{\infty} b_i u_i(t) - \sum_{i=n_j+1}^{\infty} b_i u_i(t)\right|}{\sqrt{2\sum_{i=n_j+1}^{\infty} b_i^2 \ln\ln\left(\frac{1}{\sqrt{\sum_{i=n_j+1}^{\infty} b_i^2}}\right)}} > \sqrt{\frac{(2-\alpha)\left(1-\epsilon^2-\frac{1}{\theta}\right)}{2(1+\beta)}} - 2\sqrt{\frac{(1-\epsilon^2)}{\theta}(1+\alpha)}$$

By triangle inequality, we have

$$\frac{\left|\sum_{i=n_{j+1}+1}^{\infty} b_i u_i(t)\right|}{\sqrt{2\sum_{i=n_j+1}^{\infty} b_i^2 \ln\ln\left(\frac{1}{\sqrt{\sum_{i=n_j+1}^{\infty} b_i^2}}\right)}} + \frac{\left|\sum_{i=n_j+1}^{\infty} b_i u_i(t)\right|}{\sqrt{2\sum_{i=n_j+1}^{\infty} b_i^2 \ln\ln\left(\frac{1}{\sqrt{\sum_{i=n_j+1}^{\infty} b_i^2}}\right)}}$$

$$> \sqrt{\frac{(2-\alpha)\left(1-\epsilon^2-\frac{1}{\theta}\right)}{2(1+\beta)}} - 2\sqrt{\frac{(1-\epsilon^2)}{\theta}(1+\alpha)}. \tag{3.4}$$

We have

$$|A| = \left|\left\{t \in I: \sup_{n \geq n_{j+1}} \left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right| > \sqrt{\frac{2(1+\alpha)}{\theta} \sum_{i=n_j+1}^{\infty} b_i^2 \,(1-\epsilon^2)\, \ln\ln\left(\frac{1}{\sum_{i=n_j+1}^{\infty} b_i^2}\right)}\right\}\right|$$
$$< \frac{24}{(\ln\theta)^{(1+\alpha)(1-\epsilon^2)}} \frac{1}{j^{(1+\alpha)(1-\epsilon^2)}}.$$

So

$$\sum_{j=1}^{\infty} \left|\left\{t \in I: \sup_{n \geq n_{j+1}} \left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right| > \sqrt{\frac{2(1+\alpha)}{\theta} \sum_{i=n_j+1}^{\infty} b_i^2 (1-\epsilon^2)\, \ln\ln\left(\frac{1}{\sum_{i=n_j+1}^{\infty} b_i^2}\right)}\right\}\right|$$
$$< \sum_{j=1}^{\infty} \frac{24}{(\ln\theta)^{(1+\alpha)(1-\epsilon^2)}} \frac{1}{j^{(1+\alpha)(1-\epsilon^2)}} = \frac{24}{(\ln\theta)^{(1+\alpha)(1-\epsilon^2)}} \sum_{j=1}^{\infty} \frac{1}{j^{(1+\alpha)(1-\epsilon^2)}} < \infty.$$

Applying Lemma 6, for a.e. t, we get

$$\sup_{n \geq n_{j+1}} \left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right| \leq \sqrt{\frac{2(1+\alpha)}{\theta} \sum_{i=n_j+1}^{\infty} b_i^2 (1-\epsilon^2)\, \ln\ln\left(\frac{1}{\sum_{i=n_j+1}^{\infty} b_i^2}\right)}$$

i.e.

$$\sup_{n \geq n_{j+1}} \frac{\left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right|}{\sqrt{\sum_{i=n_j+1}^{\infty} b_i^2 \;\, 2\ln\ln\left(\frac{1}{\sum_{i=n_j+1}^{\infty} b_i^2}\right)}} \leq \sqrt{\frac{(1-\epsilon^2)(1+\alpha)}{\theta}} \tag{3.5}$$

for sufficiently large $j \geq N$ (say). Thus from (3.4) and (3.5), for a.e. t we get $n_1 < n_2 < n_3 < \cdots$ such that,

$$\frac{\left|\sum_{i=n_j+1}^{\infty} b_i u_i(t)\right|}{\sqrt{2\sum_{i=n_j+1}^{\infty} b_i^2 \ln\ln\left(\frac{1}{\sqrt{\sum_{i=n_j+1}^{\infty} b_i^2}}\right)}}$$
$$> \sqrt{\frac{(2-\alpha)\left(1-\epsilon^2-\frac{1}{\theta}\right)}{2(1+\beta)}} - 3\sqrt{\frac{(1-\epsilon^2)}{\theta}(1+\alpha)} \qquad .$$

Consequently, for a.e. t

$$\frac{\left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right|}{\sqrt{2\sum_{i=n+1}^{\infty} b_i^2 \ln\ln\left(\frac{1}{\sqrt{\sum_{i=n+1}^{\infty} b_i^2}}\right)}} > \sqrt{\frac{(2-\alpha)\left(1-\epsilon^2-\frac{1}{\theta}\right)}{2(1+\beta)}} - 3\sqrt{\frac{(1-\epsilon^2)}{\theta}(1+\alpha)} \ .$$

Letting $\theta \nearrow \infty,\ \epsilon, \alpha, \beta \searrow 0$, we get

$$\frac{\left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right|}{\sqrt{2\sum_{i=n+1}^{\infty} b_i^2 \ln\ln\left(\frac{1}{\sqrt{\sum_{i=n+1}^{\infty} b_i^2}}\right)}} \geq 1.$$

Thus,

$$\limsup_{n\to\infty} \frac{\left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right|}{\sqrt{2\sum_{i=n+1}^{\infty} b_i^2 \ln\ln\left(\frac{1}{\sqrt{\sum_{i=n+1}^{\infty} b_i^2}}\right)}} \geq 1.$$

This gives

$$\limsup_{n\to\infty} \frac{\left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right|}{\sqrt{2\, B_n \ln\ln\frac{1}{B_n}}} \geq 1$$

for a.e. $t \in (0, 1)$. This completes the proof of the main theorem.

Conclusion

When we combine the result of Theorem 3 with result obtained in Theorem 4, we have

$$\limsup_{n\to\infty} \frac{\left|\sum_{i=n+1}^{\infty} b_i u_i(t)\right|}{\sqrt{2\, B_n \ln\ln\frac{1}{B_n}}} = 1$$

for a. e. $t \in (0, 1)$. This completes the law of the iterated logarithm for the summation of the signum functions.

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