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A Single Server Markovian General Service Retrial- Encouraged arrival queuing system with Persistent Retrial Technique under the Transitory Behaviour

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Abstract:

In this paper, the objective of this research is to analyse the behaviour and performance of a single server retrial-encouraged arrival queuing system. Specifically, it aims to derive the steady-state distributions for the system using the transitory technique. To calculate the performance metrics such as the average number of consumers in the orbit, and probabilities of the server being busy and idle. Hereafter conduct the numerical illustration for various service time distributions like (Exponential and Erlang) to understand the impact of different system performances on parameters. This research helps in understanding how the system performs under different conditions and can be useful for optimizing queue management in different applications.

Keywords: Encouraged arrival, Retrial Technique, Transitory behaviour, Exponential distribution, Erlangian distribution

1. Introduction

Retrial queues are queuing systems that allow incoming customers to retry for service after a certain amount of time if they discover the server and any waiting spots to be occupied. An essential component of the concept of queueing is the model of retrial queues. The need to account for the retry effect in everyday life and different networking systems gives birth to these models. For this reason, the study of such queue models receives a lot of interest. Several communication networks can be accurately described by trial queueing models. Their inquiry is, therefore, crucial. In tele-traffic theory and telephone networks where customers redial after receiving an engaged signal, retry queues have been seen as an intriguing subject.

A comparative analysis of standard queuing systems and the retrial queuing technique is investigated in [1]. A comprehensive and modern treatment of retrial queuing systems through computational techniques was studied in [2]. The stochastic processes arising from these models in stationary and non-stationary regimes are investigated in [3]. Conduct the single-server queue under demands examined in [4]. A retrial queuing system with results was studied in [5]. The primary models and results in the field of retrial queues are investigated in [6].

A comprehensive overview of the main results and methods in the theory of retrial queues studied in [7]. [8] Examines the steady-state behaviour of an M/G/1 queueing system where the server may provide two phases of heterogeneous service to incoming units. [9] Investigate the steady-state behaviour of an M/G/1 queueing system that incorporates two phases of service and operates under a D-policy. The steady-state behaviour of an M/G/1 queueing system that incorporates repeated attempts and two-phase service was studied in [10]. [11] Studied the behaviour of an M/G/1 queuing system that adds a second optional service. [12] analyzes an M/G/1 queueing system that incorporates a second optional service and server vacations scheduled according to a Bernoulli process. Investigating an M/M/1/N system with encouraged arrivals was done in [13,14]. Studying an M/M/1 retrial queuing system was done in [15]

2. Model Assumptions: The following presumptions form the basis of this model's analysis.

- Patron entry follows an Encouraged arrival with rate $\Lambda \times (1+\delta)$. δ represents the discount values are 10% i.e The interval between the Encouraged arrival is an identical distribution under an average rate of $\frac{1}{\Lambda\times(1+\delta)}$. The service time adheres to a General distribution under a conditional probability density of service via the interval $(y, y+\Delta * y)$ to obtain the expected service time y, we have

$$\mu(y) = \frac{a(y)}{1-A(y)} \tag{1}$$

where

$$a(y) = \mu(y)e^{-\int_0^y \mu(T)dT} \tag{2}$$

- Patron follows FCFS discipline
- The extend of the patron and group is infinite, beginning with arrival via the inter of time is $(T, T+\Delta * T)$ is $\Lambda \times (1+\delta)\Delta T + z(\Delta * T)$. Hereafter, retry attempts via the time of interval is $(T, T+\Delta * T)$ is obtained that $\chi\Delta * \mathrm{T} + \mathrm{z}(\Delta * \mathrm{t})$. Multiple departures within the given time interval $(T, T+\Delta * T)$ are 0.
- The retrial rate is $(1-\psi_{\mathrm{zm}})\chi$, where ψ_{zm} denotes Kronecker's delta.
- The transitory behaviour of this model is developed by the Supplementary variable technique.
- Steady-state of the probability distributions derived for different service-time distributions (Exponential and Erlang).

The probability of the system-server being idle engages for different parameters.

3. To Derive the Set of Differential-Difference Equations of the Model

We write the basic equations and derive the governing equations are described as follows

$$P'_{m1}(y,T) + \frac{\partial}{\partial T}P_{m1}(y,T) + (\Lambda \times (1+\delta) + \mu(y))P_{m1}(y,T) = \Lambda \times (1+\delta)P_{m-11}(y,T) \tag{3}$$

$$P'_{01}(y,T) + \frac{\partial}{\partial T}P_{01}(y,T) + \big(\Lambda \times (1+\delta) + \mu(y)\big)P_{01}(y,T) = 0 \tag{4}$$

$P'_{m0}(T) + (\Lambda \times (1+\delta) + \sigma)P_{m0}(T) = \int_0^\infty P_{m1}(y,T)\mu(y)dy$ forn $= 1,2,3,\ldots$ (5)

$P'_{00}(T) + \Lambda \times (1+\delta)P_{00}(T) = \int_0^\infty P_{01}(y,T)\mu(y)dy = \int_0^\infty P_{01}(y,T)\mu(y)dy$ (6)

The subsequent boundary conditions apply while solving the aforementioned equations (3) through (6)

$P_{m1}(0,T) = \Lambda \times (1+\delta)P_{m0}(T) + \sigma P_{m+10}(T)$ (7)

$P_{01}(0,T) = \Lambda \times (1+\delta)P_{00}(T) + \sigma P_{10}(T)$ (8)

Primary condition is

$P_{00}(0) = 1$ (9)

For the server that is engaged or unoccupied in the transitory state, we create the subsequent probability-generating functions

$P_0(T,Z) = \sum_{m=0}^{\infty} P_{m0}(T)Z^mandP_1(T,Z) = \sum_{m=0}^{\infty} P_{m1}(T)Z^m$ (10)

The LT of $g(T)$ is described by

$g^*(c) = \int_0^\infty e^{-cT}g(T)dT$ (11)

Apply LT equations (3) to (10), we obtain

$P'^*_{m1}(y,c) + (c + \Lambda \times (1+\delta) + \mu(y))P'^*_{m1}(y,c) = \Lambda \times (1+\delta)P'^*_{m-11}(y,c)$ for m $= 1,2,3,\ldots$ (12)

$P'^*_{01}(y,c) + (c + \Lambda \times (1+\delta) + \mu(y))P'^*_{01}(y,c) = 0$ (13)

$(c + \Lambda \times (1+\delta) + \sigma)P_{m0}{}^*(T) = \int_0^\infty P_{m1}{}^*(y,c)\mu(y)dy$ for m $= 1,2,3,\ldots$ (14)

$(\Lambda \times (1+\delta) + c)P_{00}{}^*(c) = 1 + \int_0^\infty P_{01}{}^*(y,c)\mu(y)dy$ (15)

The subsequent equations (12) to (15) are derived from the boundary values

$$P_{m1}{}^*(0,c) = \Lambda \times (1+\delta)P_{m0}{}^*(c) + \chi\, '^*(c) \quad (16)$$
$$P_{01}{}^*(0,c) = \Lambda \times (1+\delta)P_{00}{}^*(c) + \chi P_{10}{}^*(c) \quad (17)$$

Theorem 1: For the Single server Markovian General service retrial encouraged arrival queuing system under the persistent retrial technique,

a. The transitory result of the patrons in the group, when the system- server is free in the system, is provided by

$$P_0^*(c,Z) = \frac{1 + \chi P_{00}^*(c) - \frac{\chi}{Z}P_{00}^*(c)\bar{a}(c + \Lambda \times (1+\delta) - \Lambda \times (1+\delta) * Z)}{(c + \Lambda \times (1+\delta) + \sigma) - \left(\Lambda \times (1+\delta) + \frac{\sigma}{Z}\right)\bar{a}(c + \Lambda \times (1+\delta) - \Lambda \times (1+\delta) * Z)}$$

b. The transitory result of the patrons in the group, when the system- server is engaged in the system, is provided by

$$P_1^*(c,Z) = \frac{\left(\Lambda\times(1+\delta)+\frac{\sigma}{Z}\right)\left(\frac{1+\chi P_{00}^*(c)-\frac{\chi}{Z}P_{00}^*(c)\bar{a}(c+\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z)}{(c+\Lambda\times(1+\delta)+\chi)-\left(\Lambda\times(1+\delta)+\frac{\chi}{Z}\right)\bar{a}(c+\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z)}\right)-\frac{\sigma}{Z}P_{00}^*}{c+\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z}(1-\bar{a}(c+\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z))$$

c. The Steady-state result of patrons in the group when the server is free in the service system is provided by

$$P_0(Z) = \frac{p_{00}}{1-\left(\frac{\Lambda\times(1+\delta)*Z}{\chi}\right)\delta(Z)} \text{ where } \delta(Z)=\frac{1-\bar{a}}{\bar{a}-Z}, P_{00}=1-\rho_1 \text{ and } \rho_1 = \frac{\Lambda\times(1+\delta)(\Lambda\times(1+\delta)+\chi)}{\chi}E(Y)$$

d. The Steady-state result of patrons in the group when the server is engaged in the service system is provided by

$$\boldsymbol{P_1(Z) = P_{00}\left(\frac{\delta(Z)}{1-\left(\frac{\Lambda\times(1+\delta)Z}{\chi}\right)\delta(Z)}\right)}$$

Proof: We determine the probability-generating functions

$$\omega(y,T,Z)=\sum_{m=0}^{\infty} P_{m1}(y,T)Z^m \tag{18}$$

$$P_0(T,Z)=\sum_{m=0}^{\infty} P_{m0}(T)Z^m \tag{19}$$

Using the LT to the equations (18) and (19), we obtain

$$\omega^*(y,c,Z)=\sum_{m=0}^{\infty} P_{m1}^*(y,c)Z^m \tag{20}$$

$$P_0^*(c,Z)=\sum_{m=0}^{\infty} P^*{}_{m0}(c)Z^m \tag{21}$$

Partly differentiate (20) concerning y

$$\omega'^*(y,c,Z)=\sum_{m=0}^{\infty} P'^*_{m1}(y,c)Z^m \tag{22}$$

Substitute (12) and (13) in (20), we have

$$\omega'^*(y,c,Z)+(c+\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z+\mu(y))\omega^*(y,c,Z)=0$$
(23)

When we solve the differential equation above, we obtain

$$\omega^*(y,c,Z)=\omega^*(0,c,Z)e^{-(c+\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z)y}e^{-\int_0^y \mu(y)dy} \tag{24}$$

Where

$$\omega^*(0,c,Z) = \sum_{m=0}^{\infty} P_{m1}^*(0,c)Z^m \tag{25}$$

Substituting (16) and (17) in (25), we obtain

$$\omega^*(0,c,Z) = \left(\Lambda \times (1+\delta) + \frac{\chi}{Z}\right) P_0^*(c,Z) - \frac{\chi}{Z} P_{00}^* \tag{26}$$

Multiply the equation (24) by $\mu(y)$ $\int ing$ with respect x between 0 and ∞

$$\int_0^\infty \omega^*(y,c,Z)\mu(y)dy = \omega^*(0,c,Z)\int_0^\infty e^{-(c+\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z)y} e^{-\int_0^y \mu(y)dy}\mu(y)dy \tag{27}$$

$$\int_0^\infty \omega^*(y,c,Z)\mu(y)dy = \omega^*(0,c,Z)\int_0^\infty e^{-(c+\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z)y} a(y)dy \tag{28}$$

$$\int_0^\infty \omega^*(y,c,Z)\mu(y)dy = \omega^*(0,c,Z)\bar{a}(c+\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z) \tag{29}$$

where $\bar{a}(c) = \int_0^\infty e^{-cy}a(y)dy$ is the LT of the service-system time distribution

$\int ing$ the equation (24) with respect to y via 0 and ∞, we have

$$P_1^*(c,Z) = \frac{\omega^*(0,c,Z)}{c+\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z}\left(1-\bar{a}(c+\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z)\right)$$

Substitute the above equation (26) in (30), we obtain

$$P_1^*(c,Z) = \frac{\left(\Lambda\times(1+\delta)+\frac{\chi}{Z}\right)P_0^*(c,Z)-\frac{\chi}{Z}P_{00}^*}{c+\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z}\left(1-\bar{a}(c+\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z)\right)$$

Multiply the above equation (14) by Z^m on symmetry sides and add over $m = 1$ to ∞, we obtain

$$(c+\Lambda\times(1+\delta)+\chi)\textstyle\sum_1^\infty P_{m0}{}^*(T)Z^m = \int_0^\infty \left(\sum_1^\infty P_{m1}{}^*(y,c)Z^m\right)\mu(y)dy$$
(32)

$$(c+\Lambda\times(1+\delta)+\chi)(P_0^*(c,Z)-P_{00}^*(c)) = \int_0^\infty (\omega^*(y,c,Z)-P_{01}^*(y,c))\mu(y)dy$$
(33)

Substitute the above equations (15) and (29) in (33), we obtain

$$\begin{aligned}(c+\Lambda\times(1+\delta)+\chi)&P_0^*(c,Z)\\ &= 1+\sigma P_{00}^*(c)+\omega^*(0,c,Z)\bar{b}(c+\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z)\end{aligned} \tag{34}$$

Substitute the above equation (26) in (34), we have

$$\begin{gathered}(c+\Lambda\times(1+\delta)+\chi)P_0^*(c,Z) = 1+\chi P_{00}^*(c)+\left[\left(\Lambda\times(1+\delta)+\frac{\sigma}{Z}\right)P_0^*(c,Z)-\frac{\sigma}{Z}P_{00}^*(c)\right]\bar{a}\\ +\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z)\end{gathered} \tag{35}$$

$$P_0^*(c,Z) = \frac{1+\chi P_{00}^*(c)-\frac{\chi}{Z}P_{00}^*(c)\bar{a}(c+\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z)}{(c+\Lambda\times(1+\delta)+\chi)-\left(\Lambda\times(1+\delta)+\frac{\chi}{Z}\right)\bar{a}(c+\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z)} \tag{36}$$

In equation (36) means transitory results of the patrons in the group when the server is free in the system.

$$P_1^*(c,Z) = \frac{\left(\Lambda\times(1+\delta)+\frac{\sigma}{Z}\right)\left(\frac{1+\chi P_{00}^*(c)-\frac{\chi}{Z}P_{00}^*(c)\bar{a}(c+\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z)}{(c+\Lambda\times(1+\delta)+\chi)-\left(\Lambda\times(1+\delta)+\frac{\chi}{Z}\right)\bar{a}(c+\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z)}\right)-\frac{\sigma}{Z}P_{00}^*}{c+\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z}\big(-\bar{a}(c+\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z)\big) \quad (37)$$

In equation (37) means transitory results of the patrons in the group when the server is engaged in the system.

Using the Tauberian theorem with LT equation (35), we obtain

$$(\Lambda\times(1+\delta)+\chi)P_0(Z) = \sigma P_{00} + \left[\left(\Lambda\times(1+\delta)+\frac{\chi}{Z}\right)P_0(Z)-\frac{\chi}{Z}P_{00}\right]\bar{a}(\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z) \quad (38)$$

Solving the equation (38), we obtain

$$P_0(Z) = \frac{\chi p_{00}(Z-\bar{a})}{Z(\Lambda\times(1+\delta)+\chi)-(\chi+\Lambda\times(1+\delta)Z)\bar{a}} \quad (39)$$

$$P_0(Z) = \frac{p_{00}}{1-\left(\frac{\Lambda\times(1+\delta)Z}{\chi}\right)\delta(Z)} \text{ where } \delta(Z) = \frac{1-\bar{a}}{\bar{a}-Z} \quad (40)$$

In equation (40) means steady-state probability of patrons in the group when the system server is free.

Using the Tauberian theorem with LT equation (31), we obtain

$$P_1(Z) = \frac{\left(\Lambda\times(1+\delta)+\frac{\chi}{Z}\right)P_0(Z)-\frac{\chi}{Z}P_{00}}{\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z}\big(1-\bar{a}(\Lambda\times(1+\delta)-\Lambda\times(1+\delta)*Z)\big) \quad (41)$$

Substitute the equation (40) in (41), we obtain

$$P_1(Z) = P_{00}\left(\frac{\delta(Z)}{1-\left(\frac{\Lambda\times(1+\delta)Z}{\chi}\right)\delta(Z)}\right) \quad (42)$$

In equation (42) means steady-state probability of patrons in the group when the system server is free.

The normalized state is

$$P_0(1)+P_1(1) = 1 \quad (43)$$

$$\delta(1) = \frac{\Lambda\times(1+\delta)E(Y)}{1-\Lambda\times(1+\delta)E(Y)} = \frac{\rho}{1-\rho} \quad \text{where} \quad \frac{\Lambda\times(1+\delta)}{\mu} = \Lambda\times(1+\delta)E(Y) \quad (44)$$

$$\delta(1) = \frac{(\Lambda \times (1+\delta))^2 E(Y^2)}{2(1-\rho)^2} \tag{45}$$

$$P_0(1) = \lim_{Z\to 1} P_0(Z) = \lim_{Z\to 1} \frac{p_{00}}{1-\left(\frac{\Lambda\times(1+\delta)Z}{\chi}\right)\delta(Z)}$$

$$= P_{00}\left(\frac{1-\frac{\Lambda\times(1+\delta)Z}{\chi}}{1-(\frac{\Lambda\times(1+\delta)Z}{\chi})_1}\right) \tag{46}$$

$$P_1(1) = \lim_{z\to 1} P_1(Z) = \lim_{z\to 1} P_{00}\left(\frac{\delta(Z)}{1-\left(\frac{\Lambda\times(1+\delta)Z}{\chi}\right)\delta(Z)}\right)$$

$$= P_{00}\left(\frac{\frac{\Lambda\times(1+\delta)Z}{\chi}}{1-\rho_1}\right) \tag{47}$$

Substitute the equations (46) and (47) in (43), we obtain

$$P_{00} = 1-(\frac{\Lambda\times(1+\delta)Z}{\chi})_1 \tag{48}$$

Theorem 2: The performance metrics of the Single server Markovian General encouraged arrival retrial queuing system under persistent retrial policy are provided below

a. The probability of the system's server being free

$$= P_0 = 1-\frac{\Lambda\times(1+\delta)}{\mu}$$

b. The probability of the system's server being engaged

$$= P_1 = \frac{\Lambda\times(1+\delta)}{\mu}$$

c. Mean number of patrons in the group

$$L_q = \frac{1}{\left(1-(\frac{\Lambda\times(1+\delta)}{\mu})_1\right)}\left(\frac{(\Lambda\times(1+\delta))^2E(Y^2)}{2}\left(1+\frac{\Lambda\times(1+\delta)}{\chi}\right) + \frac{\Lambda\times(1+\delta)*\frac{\Lambda\times(1+\delta)}{\mu}}{\chi}\right)$$

d. Mean number of patrons in the system $= L_s = L_q + \frac{\Lambda\times(1+\delta)}{\mu}$

Proof

When there is a steady state, equation (10) yields

$$P_0(Z) = \sum_{m=0}^{\infty} P_{m0} * z^m \text{ and } P_1(Z) = \sum_{m=0}^{\infty} P_{m1} * Z^m \tag{49}$$

The probability of the server being free in the system

$$= P_0(1) = \sum_{m=0}^{\infty} P_{m0} = P_{00}\left(\frac{1 - \frac{\Lambda \times (1+\delta)}{\mu}}{1 - (\frac{\Lambda \times (1+\delta)}{\mu})_1}\right) = 1 - \frac{\Lambda \times (1+\delta)}{\mu} \tag{50}$$

The probability of the server being engaged in the system

$$= P_1(1) = \sum_{m=0}^{\infty} P_{m1} = P_{00} * \left(\frac{\frac{\Lambda \times (1+\delta)}{\mu}}{1 - (\frac{\Lambda \times (1+\delta)}{\mu})_1}\right) = \frac{\Lambda \times (1+\delta)}{\mu} \tag{51}$$

Differentiate the equation (40) for Z, we obtain

$$P_0'(Z) = \frac{P_{00}\left(\frac{\Lambda \times (1+\delta)}{\chi}\right) * \left(Z\delta'(Z) + \delta(Z)\right)}{\left(1 - \frac{\Lambda \times (1+\delta)Z}{\chi}\delta(Z)\right)^2} \tag{52}$$

$$P_0'(1) = \frac{P_{00}\left(\frac{\Lambda \times (1+\delta)}{\chi}\right) * \left(\delta'(1) + \delta(1)\right)}{\left(1 - \frac{\Lambda \times (1+\delta)}{\chi}\delta(1)\right)^2}$$

$$= \frac{\left(\frac{\Lambda \times (1+\delta)}{\chi}\right) * \left(\delta'(1) + \delta(1)\right) * (1 - \frac{\Lambda \times (1+\delta)}{\mu})^2}{\left(1 - \frac{\Lambda \times (1+\delta)}{\mu}_1\right)} \tag{53}$$

$$P_0'(1) = \frac{\left(\frac{\Lambda \times (1+\delta)}{\chi}\right) * \left(\frac{\left(\Lambda \times (1+\delta)\right)^2 E(Y^2)}{2} + \frac{\Lambda \times (1+\delta)}{\mu} * \left(1 - \frac{\Lambda \times (1+\delta)}{\mu}\right)\right)}{\left(1 - \frac{\Lambda \times (1+\delta)}{\mu}_1\right)} \tag{54}$$

Differentiate the equation (42) via respect to Z, we obtain

$$P_1'(Z) = P_0(Z) * \delta'(z) + P_0'(Z) * \delta(Z) \tag{55}$$

$$P_1'(1) = \frac{\left(\frac{\Lambda \times (1+\delta)^2 E(Y^2)}{2} + \frac{\Lambda \times (1+\delta)}{\sigma}(\frac{\Lambda \times (1+\delta)}{\mu})^2\right)}{\left(1 - (\frac{\Lambda \times (1+\delta)}{\mu})_1\right)} \tag{56}$$

Mean number of patrons in the group $= (P'_0(Z) + P_1(Z)|_{z=1}$

$$\left(P'_0(Z) + P_1(Z)\right)\Big|_{Z=1} = \frac{\left(\frac{\Lambda\times(1+\delta)}{\chi}\right) * \left(\frac{\left(\Lambda\times(1+\delta)\right)^2 E(Y^2)}{2} + \frac{\Lambda\times(1+\delta)}{\mu}\left(1 - \frac{\Lambda\times(1+\delta)}{\mu}\right)\right)}{\left(1 - (\frac{\Lambda\times(1+\delta)}{\mu})_1\right)} + \frac{\left(\frac{(\Lambda\times(1+\delta))^2 E(Y^2)}{2} + \frac{\Lambda\times(1+\delta)}{\chi} * (\frac{\Lambda\times(1+\delta)}{\mu})^2\right)}{(1-\rho_1)} \quad (57)$$

$$= \frac{1}{\left(1 - (\frac{\Lambda\times(1+\delta)}{\mu})_1\right)}\left(\frac{(\Lambda\times(1+\delta))^2 E(Y^2)}{2}\left(1 + \frac{\Lambda\times(1+\delta)}{\chi}\right) + \frac{\Lambda\times(1+\delta) * \frac{\Lambda\times(1+\delta)}{\mu}}{\chi}\right)$$

$$\mathrm{L}_q = \frac{1}{\left(1 - (\frac{\Lambda\times(1+\delta)}{\mu})_1\right)}\left(\frac{(\Lambda\times(1+\delta))^2 E(Y^2)}{2}\left(1 + \frac{\Lambda\times(1+\delta)}{\chi}\right) + \frac{\Lambda\times(1+\delta) * \frac{\Lambda\times(1+\delta)}{\mu}}{\chi}\right) \quad (58)$$

Average number of customers in the system $= \frac{d}{dZ}\left(P_0(\mathrm{Z}) + ZP_1(Z)\right)\Big|_{Z=1}$

$$= L_q + P_1(1) = L_q + \frac{\Lambda\times(1+\delta)}{\mu}$$

4. STABILITY CONDITION

The consideration of stability has great significance for every queuing system. A stable single server retrial encouraged arrival queuing system is one that has a persistent retry policy.

$$\left(1 + \frac{\Lambda\times(1+\delta)}{\chi}\right) * \Lambda\times(1+\delta)E(Y) < 1$$

- If the stability condition of the exponential distribution is

$$\left(1 + \frac{\Lambda\times(1+\delta)}{\chi}\right)\frac{\Lambda\times(1+\delta)}{\mu} < 1$$

- If the stability condition of the Erlaang distribution is

$$\left(1 + \frac{\Lambda\times(1+\delta)}{\chi}\right)\frac{r\Lambda\times(1+\delta)}{\mu} < 1$$

5. SPECIAL PRIVILEGE

We note that many particular cases of this work can be derived for various Service time distributions

Privilege-1: An exponential distribution is used to describe the service time distribution.

The PDF of exponential distribution are,

$a(y) = \mu e^{-\mu y}, y > 0$

$\mathrm{E(Y)} = \frac{1}{\mu}$ and $\mathrm{E(Y^2)} = \frac{2}{\mu^2}$ (60)

The LT of $a(y)$ is $\bar{a}(c) = \frac{\mu}{c+\mu}$ (61)

$$\bar{a}(\Lambda \times (1+\delta) - \Lambda \times (1+\delta) * Z) = \frac{\mu}{\Lambda \times (1+\delta) - \Lambda \times (1+\delta) * Z \ + \mu}$$
$$= \frac{1}{1 + \frac{\Lambda \times (1+\delta)}{\mu} - (\frac{\Lambda \times (1+\delta)}{\mu})Z} \tag{62}$$

$$\delta(Z) = \frac{1-\bar{a}}{\bar{a}-Z} = \frac{1 - \frac{1}{1 + \frac{\Lambda \times (1+\delta)}{\mu} - (\frac{\Lambda \times (1+\delta)}{\mu})Z}}{\frac{1}{1 + (\frac{\Lambda \times (1+\delta)}{\mu}) - (\frac{\Lambda \times (1+\delta)}{\mu})Z} - Z}$$
$$= \frac{\frac{\Lambda \times (1+\delta)}{\mu}}{1 - (\frac{\Lambda \times (1+\delta)}{\mu})Z} \tag{63}$$

$$P_0(Z) = \frac{p_{00}}{1 - \left(\frac{\Lambda \times (1+\delta)Z}{\chi}\right)\delta(Z)}$$
$$= \frac{p_{00}}{1 - \quad \left(\frac{\Lambda \times (1+\delta)Z}{\chi}\right)\left(\frac{\frac{\Lambda \times (1+\delta)}{\mu}}{1 - (\frac{\Lambda \times (1+\delta)}{\mu})Z}\right)} \tag{64}$$

After the solution of equation (64), we obtain

$P_0(Z) = \frac{P_{00}(1 - \frac{\Lambda \times (1+\delta)}{\mu})Z)}{1 - (\frac{\Lambda \times (1+\delta)}{\mu})_1 Z}$ where $\rho = \frac{\Lambda \times (1+\delta)}{\mu}$ and $\rho_1 = \frac{(\Lambda \times (1+\delta) + \chi)}{\chi\mu}$
(65)

The equation (65) means the steady-state probability of patrons in the group, server is free in the system.

$$P_1(Z) = P_{00}\left(\frac{\delta(Z)}{1-\left(\frac{\Lambda\times(1+\delta)Z}{\chi}\right)\delta(Z)}\right)$$
$$= P_{00}\left(\frac{\frac{\frac{\Lambda\times(1+\delta)}{\mu}}{1-(\frac{\Lambda\times(1+\delta)}{\mu})Z}}{1-\left(\frac{\Lambda\times(1+\delta)Z}{\chi}\right)\left(\frac{\frac{\Lambda\times(1+\delta)}{\mu}}{1-(\frac{\Lambda\times(1+\delta)}{\mu})Z}\right)}\right) \quad (66)$$

After the solution of equation (66), we obtain

$$P_1(Z) = P_{00}\left(\frac{\frac{\Lambda\times(1+\delta)}{\mu}}{1-(\frac{\Lambda\times(1+\delta)}{\mu})_1 Z}\right) \quad (67)$$

The equation (65) means the steady-state probability of patrons in the group, server is engaged in the system.

$$\mathrm{L}_q = \frac{P_{00}}{\left(1-(\frac{\Lambda\times(1+\delta)}{\mu})_1\right)^2}\left(\frac{\Lambda\times(1+\delta)^2 E(Y^2)}{2}\left(1+\frac{\Lambda\times(1+\delta)}{\chi}\right)+\frac{\Lambda\times(1+\delta)*(\frac{\Lambda\times(1+\delta)}{\mu})}{\chi}\right)$$
$$= \frac{P_{00}}{\left(1-(\frac{\Lambda\times(1+\delta)}{\mu})_1\right)^2}\left(\frac{(\Lambda\times(1+\delta))^2\left(\frac{2}{\mu^2}\right)}{2}\left(1+\frac{\Lambda\times(1+\delta)}{\chi}\right)+\frac{\Lambda\times(1+\delta)(\frac{\Lambda\times(1+\delta)}{\mu})}{\chi}\right) \quad (68)$$

After the solution of equation (68), we obtain

$$\mathrm{L}_q = \frac{(\frac{\Lambda\times(1+\delta)}{\mu})_1}{\left(1-(\frac{\Lambda\times(1+\delta)}{\mu})_1\right)}\left(\frac{\Lambda\times(1+\delta)}{\mu}+\frac{\Lambda\times(1+\delta)}{\Lambda\times(1+\delta)+\chi}\right) \quad (69)$$

The equation (69) means the mean number of patrons in the group

$$\mathrm{L}_s = \mathrm{L}_q + \frac{\Lambda \times (1+\delta)}{\mu}$$

$$= \frac{(\frac{\Lambda \times (1+\delta)}{\mu})_1}{\left(1 - (\frac{\Lambda \times (1+\delta)}{\mu})_1\right)} \left(\frac{\Lambda \times (1+\delta)}{\mu} + \frac{\Lambda \times (1+\delta)}{\Lambda \times (1+\delta) + \chi}\right)$$

$$+ \frac{\Lambda \times (1+\delta)}{\mu} \qquad (70)$$

The equation (70) means the mean number of patrons in the system.

Case-2: An Erlang distribution under r phases describes the service time distribution.

This privileges single server Erlang retrial queuing model under a persistent retrial technique

The Erlang distribution is given that

$$a(y) = \frac{\mu^r y^{r-1} e^{-\mu y}}{(r-1)!} \mu e^{-\mu y}, y > 0$$

For an Erlang distribution, $\mathrm{E(Y)} = \frac{r}{\mu}$ and $\mathrm{E(Y^2)} = \frac{r(r+1)}{\mu^2}$

The Laplace transform of $a(y)$ is $\bar{a}(c) = \left(\frac{\mu}{c+\mu}\right)^r$

$$\bar{a}(\Lambda \times (1+\delta) - \Lambda \times (1+\delta) * Z) = \left(\frac{\mu}{\Lambda \times (1+\delta) - \Lambda \times (1+\delta) * Z + \mu}\right)^r$$

$$= \left(\frac{r}{r + \frac{\Lambda \times (1+\delta)}{\mu} - (\frac{\Lambda \times (1+\delta)}{\mu})Z}\right)^r \qquad (71)$$

$$\delta(Z) = \frac{1-\bar{a}}{\bar{a} - Z} = \frac{1 - \left(\frac{r}{r + \frac{\Lambda \times (1+\delta)}{\mu} - (\frac{\Lambda \times (1+\delta)}{\mu})Z}\right)^r}{\left(\frac{r}{r + \frac{\Lambda \times (1+\delta)}{\mu} - \left(\frac{\Lambda \times (1+\delta)}{\mu}\right) * Z}\right)^r - Z}$$

$$= \frac{(r + \frac{\Lambda \times (1+\delta)}{\mu} - (\frac{\Lambda \times (1+\delta)}{\mu})Z)^r - R^r}{R^r - Z * (r + \frac{\Lambda \times (1+\delta)}{\mu} - (\frac{\Lambda \times (1+\delta)}{\mu})Z)^r} \qquad (72)$$

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$$P_0(Z)=\frac{p_{00}}{1-\left(\frac{\Lambda\times(1+\delta)Z}{\chi}\right)\delta(Z)}$$

$$=\frac{p_{00}}{1-\left(\frac{\Lambda\times(1+\delta)Z}{\chi}\right)\left(\frac{(r+\frac{\Lambda\times(1+\delta)}{\mu}-(\frac{\Lambda\times(1+\delta)}{\mu})Z)^r-R^r}{R^r-Z(r+\frac{\Lambda\times(1+\delta)}{\mu}-(\frac{\Lambda\times(1+\delta)}{\mu})Z)^r}\right)} \quad (73)$$

Hereafter the solution of equation (73), we obtain

$P_0(Z)=\frac{P_{00*}\left(1-Z*\left(1+\frac{1}{R}*\left(\frac{\Lambda\times(1+\delta)}{\mu}-\left(\frac{\Lambda\times(1+\delta)}{\mu}\right)*Z\right)\right)^r\right)}{1+Z\left(\frac{\Lambda\times(1+\delta)}{\chi}-\frac{\rho_1}{\rho}\left(1+\frac{1}{r}(\rho-\rho Z)\right)^r\right)}$ where $\rho=\Lambda\times(1+\delta)*E(Y)$ and $\rho_1=\frac{\Lambda\times(1+\delta)*(\Lambda\times(1+\delta)+\chi)}{\chi}*E(X)$

The Eqn (74) means the number of patrons in the group, and the server is free.

$$P_1(Z)=P_{00}\left(\frac{\delta(Z)}{1-\left(\frac{\Lambda\times(1+\delta)Z}{\chi}\right)\delta(Z)}\right)$$

$$=P_{00}\left(\frac{\frac{(r+\rho-\rho Z)^r-r^r}{r^r-Z(r+\rho-\rho Z)^r}}{1-\left(\frac{\Lambda\times(1+\delta)Z}{\chi}\right)\left(\frac{(r+\rho-\rho z)^k-r^r}{r^r-Z(r+\rho-\rho Z)^r}\right)}\right) \quad (75)$$

Hereafter the solution of equation (75), we obtain

$$P_1(Z)=P_{00}*\left(\frac{\left(1+\frac{1}{r}*(\rho-\rho*Z)\right)^r-1}{1+Z*\left(\frac{\Lambda\times(1+\delta)}{\chi}-\frac{\rho_1}{\rho}\left(1+\frac{1}{r}(\rho-\rho Z)\right)^r\right)}\right) \quad (76)$$

The Eqn (76) means the number of patrons in the group, and the server is engaged.

$$\mathrm{L}_q = \frac{1}{(1-\rho_1)} * \left(\frac{(\Lambda \times (1+\delta))^2 * E(Y^2)}{2} * \left(1 + \frac{\Lambda \times (1+\delta)}{\chi}\right) + \frac{\Lambda \times (1+\delta) * \rho}{\chi}\right)$$

$$= \frac{1}{(1-\rho_1)}\left(\frac{(\Lambda \times (1+\delta))^2 \left(\frac{r(r+1)}{\mu^2}\right)}{2}\left(1 + \frac{\Lambda \times (1+\delta)}{\chi}\right) + \frac{\Lambda \times (1+\delta)\rho}{\chi}\right) \quad (77)$$

Hereafter the solution of equation (77), we obtain

$$\mathrm{L}_q = \frac{1}{(1-\rho_1)}\left(\frac{\lambda^2 \left(\frac{r(r+1)}{\mu^2}\right)}{2}\left(1 + \frac{\Lambda \times (1+\delta)}{\chi}\right) + \frac{\lambda\rho}{\chi}\right) \quad (78)$$

The Eqn (78) means the number of patrons in the group.

$$\mathrm{L}_s = \mathrm{L}_q + \rho$$

$$= \frac{1}{(1-\rho_1)}\left(\frac{(\Lambda \times (1+\delta))^2 \left(\frac{r(r+1)}{\mu^2}\right)}{2}\left(1 + \frac{\Lambda \times (1+\delta)}{\chi}\right) + \frac{\Lambda \times (1+\delta)\rho}{\chi}\right) + \rho \quad (79)$$

The Eqn (79) means the number of patrons in the system.

Privilege-3: As $\chi \to \infty$, single server Markovian general service encouraged arrival queuing model

The number of patrons in the system is provided by

$$P(Z) = P_0(Z) + Z * P_1(Z)$$

$$= \frac{p_{00}}{1 - \left(\frac{\Lambda \times (1+\delta)Z}{\chi}\right) * \delta(Z)} + Z * P_{00}\left(\frac{\delta(Z)}{1 - \left(\frac{\Lambda \times (1+\delta)Z}{\chi}\right)\delta(Z)}\right) \quad (80)$$

As $\chi \to \infty$, the equation (80), we have

$$P(Z) = P_0(Z) + Z * P_1(Z) = P_{00} * \left(1 + Z\delta(Z)\right) = P_{00} * \left(1 + Z\left(\frac{1-\bar{a}}{\bar{a}-Z}\right)\right) P_{00} * \left(1 + Z\left(\frac{1-\bar{a}}{\bar{a}-Z}\right)\right)$$

$$= P_{00}\frac{\bar{a}(1-Z)}{\bar{a}-Z}$$

$$P_{00} = 1 - \frac{\Lambda \times (1+\delta)\left(\left(\Lambda \times (1+\delta)\right) + \sigma\right)}{\chi} * E(Y)$$

$$= 1 - \Lambda \times (1+\delta)E(Y) \quad (81)$$

$$P(Z) = \frac{\left(1 - \Lambda \times (1+\delta)E(Y)\right)(1-Z)\bar{a}(\Lambda \times (1+\delta) - \Lambda \times (1+\delta) * Z)}{\bar{a}(\Lambda \times (1+\delta) - \Lambda \times (1+\delta) * Z) - Z} \quad (82)$$

The Eqn (82) means the Pollaczek - Khinchine equation for single server Markovian general service encouraged arrival queuing model

$$\mathrm{L}_q = \frac{P_{00}}{(1-\rho_1)^2}\left(\frac{(\Lambda \times (1+\delta))^2 E(Y^2)}{2}\left(1 + \frac{\Lambda \times (1+\delta)}{\chi}\right) + \frac{\Lambda \times (1+\delta)\rho}{\chi}\right) = \frac{(\Lambda \times (1+\delta))^2 E(Y^2)}{2(1-\rho)}$$

The Eqn (83) means the mean number of patrons in the line.

6. NUMERICAL STUDY

The parameters $\Lambda \times (1+\delta), \mu$ and χ will be verified the stability condition. The performance metrics of this are expressed in tables for different service distributions.

Λ	δ	$\Lambda \times (1+\delta)$	μ	χ
5	0.1	5.5	10	10,30,50,70,…5000

Table 1: Performance metrics for $\Lambda \times (1+\delta) = 5.5$ and $\mu = 10$ for different parameters of χ.

χ	P_0	P_1	L_q	L_s	W_q	W_s
10	0.5	0.5	5.2297	5.7797	0.9508	1.0508
30	0.5	0.5	1.0174	1.8640	0.2389	0.3389
50	0.5	0.5	1.0174	1.5674	0.1851	0..2850
70	0.5	0.5	0.9083	1.4583	0.1651	0.2651
90	0.5	0.5	0.8516	1.4016	0.1548	0.2548
100	0.5	0.5	0.8324	1.3834	0.1513	0.2513
300	0.5	0.5	0.7232	1.2732	0.1315	0.2315
500	0.5	0.5	0.7025	1.2525	0.1277	0.2277

700	0.5	0.5	0.6938	1.2438	0.1261	0.2261
900	0.5	0.5	0.6889	1.2389	0.1253	0.2253
1000	0.5	0.5	0.6873	1.2373	0.1250	0.2250
3000	0.5	0.5	0.6772	1.2272	0.1231	0.2231
5000	0.5	0.5	0.6752	1.2252	0.1228	0.2228

Remarks:1

Given an exponential distribution for the service distribution, Table 1 illustrates the effect of χ on the average number of Patrons in a group. Additionally, the following is implied:

- The average number of patrons in the group reduces the retrial rate χ maximizes and then applies the encouraged arrival concept for this model to increase the arrival rates. This method using a single server encouraged arrival queuing system if χ is Maximum.

Table 2: Performance metrics for $\Lambda \times (1+\delta), = 5.5,\ \mu = 50$ and $r = 3$ for different points of χ.

χ	P_0	P_1	L_q	L_s	W_q	W_s
10	0.7	0.3	0.2086	0.3186	0.0379	0.0579
30	0.7	0.3	0.1220	0.2320	0.0222	0.0422
50	0.7	0.3	0.1056	0.2156	0.0192	0.0392
70	0.7	0.3	0.0987	0.2087	0.0179	0.0379
90	0.7	0.3	0.0948	0.2048	0.0172	0.0372
100	0.7	0.3	0.0935	0.2035	0.0170	0.0370
300	0.7	0.3	0.0855	0.1955	0.0156	0.0356
500	0.7	0.3	0.0839	0.1939	0.0153	0.0353
700	0.7	0.3	0.0833	0.1933	0.0151	0.0351
900	0.7	0.3	0.0829	0.1929	0.0151	0.0351
1000	0.7	0.3	0.0828	01928	0.0150	0.0350
3000	0.7	0.3	0.0820	0.1920	0.0149	0.0349
5000	0.7	0.3	0.0818	0.1918	0.0149	0.0.349

Table 3: Performance metrics for $\Lambda \times (1+\delta) = 8.8, \mu = 50$ and $r = 3$ for different points of χ.

χ	P_0	P_1	L_q	L_s	W_q	W_s
10	0.52	0.48	0.7537	0.9297	0.0856	0.1056
30	0.52	0.48	0.3781	0.5541	0.0430	0.0.630
50	0.52	0.48	0.3147	0.4907	0.0358	0.0558
70	0.52	0.48	0.2885	0.4645	0.0328	0.0528
90	0.52	0.48	0.2742	0.4502	0.0312	0.0512
100	0.52	0.48	0.2693	0.4453	0.0306	0.0506
300	0.52	0.48	0.2399	0.4159	0.0273	0.0473
500	0.52	0.48	0.2342	0.4102	0.0266	0.0466
700	0.52	0.48	0.2317	0.4077	0.0263	0.0463
900	0.52	0.48	0.2303	0.4063	0.0262	0.0462
1000	0.52	0.48	0.2298	0.4058	0.0261	0.0461
3000	0.52	0.48	0.2270	0.4030	0.0258	0.0458
5000	0.52	0.48	0.2264	0.4024	0.0257	0.0457

Remark:2

Given an Erlang distribution for the service distribution, Table 2&3 illustrates the effect of χ on the average number of Patrons in a group. Additionally, the following is implied:

The average number of patrons in the group increases the retrial rate χ maximizes and then applies the encouraged arrival concept for this model to increase the arrival rates. This method using a single server encouraged arrival queuing system if χ is Maximum

7. ECONOMIC COST

To optimize the system's operating cost, we established a cost analysis.in single server Markovian encouraged arrival queuing system. We find out the following costs,

- Waiting cost= (c) *(Wait time)
- Operating cost= (R)*(server charge per time duration)
- Estimated Total Cost (ETC)= Operating cost + Waiting cost

Λ	δ	$\Lambda \times (1+\delta)$	μ	r	χ
5	0.1	5.5	10	3	10,30,50,70,…5000

Table 4:

χ	Waiting time (W_q)	Waiting Cost	Operating Cost	Total Cost
10	0.0379	0.3793	150	150.3793
30	0.0222	0.2217	150	150.2217
50	0.0192	0.1920	150	150.1920
70	0.0179	0.1794	150	150.1794
90	0.0172	0.1724	150	150.1724
100	0.0170	0.1700	150	150.1700
300	0.0156	0.1555	150	150.1555
500	0.0153	0.1526	150	150.1526
700	0.0151	0.1514	150	150.1514
900	0.0151	0.1507	150	150.1507
1000	0.0150	0.1505	150	150.1505
3000	0.0149	0.1490	150	150.1490
5000	0.0149	0.1487	150	150.1487

Table 5:

Λ	δ	$\Lambda \times (1+\delta)$	μ	r	χ
8	0.1	8.8	10	3	10,30,50,70,…5000

χ	Waiting time (W_q)	Waiting Cost	Operating Cost	Total Cost
10	0.0856	0.8564	150	150.8564
30	0.0430	0.8463	150	150.8463
50	0.0358	0.4296	150	150.4296
70	0.0328	0.3576	150	150.3576
90	0.0312	0.3278	150	150.3278
100	0.0306	0.3116	150	150.3116
300	0.0273	0.3060	150	150.3060
500	0.0266	0.2727	150	150.2727
700	0.0263	0.2661	150	150.2661
900	0.0262	0.2633	150	150.2633
1000	0.0261	0.2617	150	150.2617
3000	0.0258	0.2612	150	150.2612
5000	0.0257	0.2579	150	150.2579

8. CONCLUSION

The single server Markovian general service retrial encouraged arrival queuing model under the persistent with retrial policy is calculated using the Exponential and Erlang distribution method. Hereafter we find out the number of patrons in the group, the server is a free/engaged concept of this model. This method using a single server encouraged arrival queuing system if χ value is Maximum. Additionally, we calculated the Economic cost analysis for this model. Applies the encouraged arrival concept for this model, more efficient results get it compare of the Poisson arrival model. Different special privileges are done by various service time distributions.

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References

[1] Artalejo. J.R and G.I.Falin (2002). Standard and retrial queueing systems. A comparative analysis, Revista, Matematica, Complutense, 15 , pp 101-129
[2] Artalejo J.R and A. Gomez-Corral (2008). Retrial Queueing systems-a computational Approach, Springer.
[3] Falin G.I (1990). A survey of retrial queues. Queueing Systems 7, No.2, pp 127-167.
[4] Farahmand. K (1990). Single line queue with repeated demands, Queueing systems, 6, No.2, pp 223-228.
[5] Kulkarni V.G (1983). On queueing systems with retrials. Journal of Applied Probability 20, No.2, pp 380-389.
[6] Templeton J.G.C (1990). Retrial Queues, Queueing Systems 7, No.2, pp 125-227.
[7] Yang T and Templeton J.G.C (1987). A Survey on retrial queues. Queueing Systems, 2, pp 201-233.
[8] Choudhury, G.(2003). Some aspects of M/G/1 queueing system with optional second service: TOP, Vol.11, 141-150.
[9] Choudhury, G. (2005). An M/G/1 Queueing System with Two Phase Service under D-Policy: Information and Management Sciences. Vol.16, No. 4, 1-17.
[10] Artalejo,J.R. and Choudhury,G. 2004. Steady state analysis of an M/G/1 queue with repeated attempts and two-phase service: Quality Technology and Quantitative Management, Vol.1, 189-199.
[11] Jehad Al-Jararha Kailash C. Madan 2003. An M/G/1 Queue with Second Optional Service with General Service Time Distribution: Information and Management Sciences, Vol.14, No.2, 47-56.
[12] Madan, K.C., Abu-Dayyeh, W. and Saleh, M.F.2002. An M/G/1 queue with second optional service and Bernoulli schedule server vacations: Systems Science, Vol.28, 51-62.
[13] Som BK, Seth S. An M/M/1/N queuing system with encouraged arrivals. Global Journal of Pure and Applied Mathematics. 2017; 13: 3443-3453.
[14] Khan IE, Paramasivam R. Reduction in waiting time in an M/M/1/N encouraged arrival queue with feedback, balking and maintaining of reneged customers. Symmetry. 2022;
[15] Khan IE, Paramasivam R, Performance Study of an M/M/1 Retrial Queueing System with Balking, Dissatisfied Customers, and Server Vacations , Contemporary Mathematics, Volume 4 Issue 3 (2023), 379-619 .