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Numerical Solution of One-Dimensional Advection-Diffusion Equation using Radial Basis Function Method of Lines

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Abstract:
The research article proposes a numerical solution for the one-dimensional advection-diffusion equation using the Radial Basis Function (RBF) method of lines. The approach utilizes the asymmetric multiquadric collocation method for spatial discretization, resulting in a system of ordinary differential equations (ODEs) in time. The fourth-order Runge-Kutta (RK) method is then applied to solve this system. The method is illustrated by solving selected problems and comparing the results to exact solutions, highlighting its effectiveness. The algorithm is user-friendly, accurate, and suitable for one-dimensional linear diffusion problems with complex initial and boundary conditions, offering a new perspective in computational physics and numerical analysis.

Keywords: Advection-Diffusion equation, Collocation method, Method of Lines, Radial Basis function, Shape parameter.

Introduction

The unsteady linear 1-D advection-diffusion equation is a significant area of research in various fields, including environmental engineering, hydrology, and mathematical modelling. Analysis of the numerical and analytical solutions to this problem has been done in several papers. Analytical solutions for the one-dimensional advection-diffusion equation with temporally dependent coefficients are shown in a work by Jaiswal et al. (2011). The solute dispersion parameter is time-dependent when the flow domain is uniform; both parameters are time-dependent; and the former is uniform, and the latter is time-dependent. These are the three scenarios that the authors address. The time-dependent one-dimensional linear advection-diffusion equation is solved analytically by Mojtabi and Deville (2015) using variable separation and numerically by employing the finite element approach. The study notes that the analytical solution behaves badly and is more difficult to assess when advection takes over. Gane (2000) looked on the use of characteristics-based methods and finite difference approaches to solve the advection problem. In a longitudinal finite initially solute-free domain, Kumar et al. (2009) discovered analytical solutions for the one-dimensional advection-diffusion equation with variable coefficients. When dispersion is proportional to the same linearly interpolated velocity, the study compares the analytical & Numerical solutions. Three numerical techniques for solving the one-

dimensional advection-diffusion problem with constant coefficients are compared by Appadu (2013). The Lax-Wendroff scheme, the Crank-Nicolson scheme, and a nonstandard finite difference scheme are some of the techniques used. According to the study, for specific values of space and time step sizes, the Lax-Wendroff and nonstandard finite difference schemes yield excellent approximations for the 1D advection-diffusion equation. The hybrid nature of the transport equation has numerical implications that result in problems that are dominated by advection or diffusion, as discussed by Szymkiewicz (2010). The author provides algorithms based on the splitting approach, the finite difference and the finite element methods for solving the 1D advection-diffusion problem. The study highlights how the diffusion in the numerical solution is equivalent to that in the physical world. Convection-diffusion equations are solved by Mittal and Jain (2010) using the redefined cubic B-splines and the collocation method. Karahan (2006) solves the advection-diffusion equation using implicit finite difference methods. This study's primary goal is to provide a straightforward, reliable, and user-friendly meshless technique. The structure of the Paper is as under.
The problem statement is provided in section 2. In Section 3, the RBF approximation is presented. In section 4, the Asymmetric Multiquadric collocation method is introduced. The Section 5 deals with the Error Estimation. Two test problems are addressed in Section 6. In section 7, the closing thoughts are discussed.

1. 1-D Advection-Diffusion Equation

The one-dimensional Advection-Diffusion is given by (Hundsdorfer and Verwer 2003)

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} = \gamma \frac{\partial^2 u}{\partial x^2}; 0 \leq x \leq L, 0 < t \leq T \qquad (2.1)$$

with initial condition

$$u(x,0) = \varphi(x); 0 \leq x \leq L \qquad (2.2)$$

and the boundary conditions are

$$u(0,t) = \varphi_1(t);\ u(L,t) = \varphi_2(t);\ 0 < t \leq T \qquad (2.3)$$

where α and γ are positive constants and they represent the advection speed and diffusion coefficients respectively. The functions $\varphi(x)$, $\varphi_1(t)$ and $\varphi_2(t)$ are the known functions with sufficiently smoothness.
The suggested approach is used to the solution of equations (2.1)–(2.3). After completing the spatial discretization, a typical ODE solver in MATLAB is used to solve the resulting system of ODEs using the method of lines approach.

2. Radial Basis Approximation

RBFs are a class of functions that are defined as a function of the distance between a point and a set of fixed points called centres (Larsson and Fornberg 2003). The most used RBF is the Gaussian RBF, which is defined as follows:

$$\emptyset(x) = e^{-\epsilon^2 r^2}, where\ r = \|x - c\| \qquad (3.1)$$

where $\emptyset(x)$ is the value of the RBF at point x, c is the centre of the RBF, ϵ is the shape parameter, and $\|x - c\|$ is the Euclidean distance between x and c.

RBFs can be used for function approximation by constructing a linear combination of RBFs centred at different points (Larsson and Fornberg 2003). This linear combination is given by:

$$f(x) = \sum_{i=1}^{N} \alpha_i \varphi(\|x - c_i\|) \tag{3.2}$$

where N is the number of RBFs used in the approximation, f(x) is the approximated function, α_i is the weight associated with the i-th RBF, and c_i is the center of the i-th RBF. By minimizing the difference between the approximated function and the actual function at a set of training points, the weights α_i can be found (Dasari and Parikh 2023). The most used RBFs are given in Table 1 (Larsson and Fornberg 2003).

Table 1. The frequently used Radial Basis functions

Type of RBF	Definition
Inverse Multiquadric	$1/\sqrt{1+\varepsilon^2 r^2}$
Inverse Quadric	$1/(1+\varepsilon^2 r^2)$
Multiquadric	$\sqrt{1+\varepsilon^2 r^2}$
Gaussian	$e^{-\varepsilon^2 r^2}$
Spline (Polyharmonic)	$r^k, k = 1,3,5,--$
Spline (Thin plate)	$r^2 \log(r)$

where r *is the Euclidean distance and* ε is a shape parameter for scaling the radial kernel's input.

3. Asymmetric Multiquadric collocation method

We consider the time-dependent linear boundary value problem as follows

$$\frac{\partial u}{\partial t} = \mathcal{L}u, in\ \Omega \tag{4.1}$$

Subject to boundary Operator

$$\mathcal{B}\ in\ \partial\Omega \tag{4.2}$$

where $\mathcal{L}$ is a Linear Differential operator, $\mathcal{B}$ is a boundary operator, Ω is the bounded domain and $\partial\Omega$ is the boundary domain.

Let the approximate solution of (4.1) & (4.2) in terms of RBFs be

$$u(x) = \sum_{i=1}^{n} \alpha_i\, \phi(r_i) \tag{4.3}$$

where ϕ is an RBF, $r_i = \|x - x_i\|$ is the Euclidean distance between two points, x_i is the center of the RBF, α_i are constants to be calculated by collocation. In the present paper, we use the Multiquadric RBF which is given by $\phi(r) = \sqrt{1 + c^2 r^2}$ where c is a shape parameter which controls the steepness of the shape function.

Let N_I be the set of nodes interior to the solution domain i.e. $x_j \in \Omega$, let N_B be the set of distinct nodes on the boundary i.e., $x_j \in \partial\Omega$. Now, applying the operator $\boldsymbol{\mathcal{L}}$ to equation (4.3) and by Multiquadric collocation at the N_I interior centers, we get

$$\boldsymbol{\mathcal{L}}u(x_i) = \sum_{j=1}^{n} \alpha_j \boldsymbol{\mathcal{L}}\, \phi(\|x_i - x_j\|)\ ; i = 1, \dots, N_I \tag{4.4}$$

Now, applying the operator $\boldsymbol{\mathcal{B}}$ to equation (4.3) and by Multiquadric collocation at the N_B boundary centers, we get

$$\boldsymbol{\mathcal{B}}u(x_i) = \sum_{j=1}^{n} \alpha_j \boldsymbol{\mathcal{B}}\, \phi(\|x_i - x_j\|)\ ; i = N_{I+1}, \dots, N \tag{4.5}$$

The matrix form of the right-hand sides of equations (4.4) & (4.5) can be written as $\boldsymbol{H}\alpha$ where $\boldsymbol{H}$ is the Evaluation matrix which discretizes the PDE and has the following form

$$H = \begin{bmatrix} \boldsymbol{\mathcal{L}}\phi \\ \boldsymbol{\mathcal{B}}\phi \end{bmatrix} \tag{4.6}$$

The elements of $\boldsymbol{\mathcal{L}}\phi$ & $\boldsymbol{\mathcal{B}}\phi$ are

$$(\boldsymbol{\mathcal{L}}\phi)_{ij} = \boldsymbol{\mathcal{L}}\phi(\|x_i - x_j\|), i = 1, \dots, N_I, j = 1, \dots, N \tag{4.7}$$

$$(\boldsymbol{\mathcal{B}}\phi)_{ij} = \boldsymbol{\mathcal{B}}\phi(\|x_i - x_j\|), i = N_{I+1}, \dots, N, j = 1, \dots, N \tag{4.8}$$

Now, collocating equation (4.3), the matrix form is

$$u = B\alpha \tag{4.9}$$

where the elements of B are $b_{ij} = \phi(\|x_i - x_j\|$

Since B is invertible for Multiquadric RBF (Michelli 1984), we have

$$\alpha = B^{-1}u \tag{4.10}$$

Therefore, the matrix that discretizes the PDE in space is the Differentiation Matrix (Sarra and Kansa 2009), which now can be written as

$$D = HB^{-1} \tag{4.11}$$

Thus, after the space discretization the PDE (4.1) becomes

$$\frac{du}{dt} = \boldsymbol{\mathcal{L}}u \approx Du \tag{4.12}$$

which is a system of ODEs in time and can be solved by any standard ODE solver, this strategy is commonly known as Method of Lines approach.

4. Error Estimates

We employ the following errors to verify the proposed method's accuracy and validity.

L_2 Error norm:

$$L_2 = \sqrt{\sum_{j=1}^{N} \left|u_j - U_j\right|^2} \tag{5.1}$$

L_∞ Error norm:

$$L_\infty = \max_j |u_j - U_j| \tag{5.2}$$

Root Mean Square Error:

$$RMS = \frac{1}{N}\sqrt{\sum_{j=1}^{N} |u_j - U_j|^2} \tag{5.3}$$

5. Results and Discussions

We demonstrate the numerical solutions for the two problems using the proposed method in this section.

Problem 1: The Exact solution of the following equation

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} = \gamma \frac{\partial^2 u}{\partial x^2}; 0 \le x \le 1, 0 \le t \le T \tag{6.1}$$

with $\alpha = 3.5, \gamma = 0.022$ is given by (Ismail et al. 2004)

$$u(x,t) = e^{(ax+bt)} \tag{6.2}$$

The initial and boundary conditions can be obtained from the exact solution.

By taking $a = 0.02854797991928$, $b = -0.0999$, $N = 55, shape\ parameter = 10$ the proposed method is used.

The results are calculated at various time intervals. Table 2 presents the various error norms. The Fig. 1 presents a graphical comparison between the exact and approximate solutions of the one-dimensional advection-diffusion equation at three different time points $at\ t = 1, t = 2\ \&\ t = 3$. The figure clearly shows that the approximate solution obtained via the Radial Basis Function Method of Lines (RBF-ML) method closely matches the exact solution at all displayed time points. This demonstrates the accuracy and effectiveness of the RBF-ML method for solving the one-dimensional advection-diffusion equation. The Fig. 2 visualizes the solution $u(x,t) = e^{(ax+bt)}$ of the PDE. The graph's surface shows how $u(x,t)$ changes exponentially over space and time, with the specific form of this change depending on the values of $a\ \&\ b$. The color coding highlights the magnitude of $u(x,t)$ at different points, indicating areas of higher and lower values.

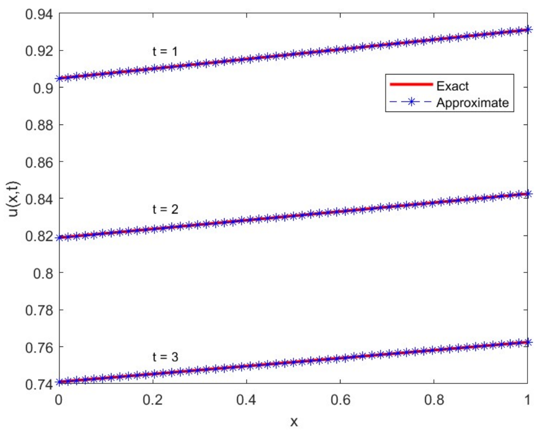

Figure 1. Exact Vs Approximate Solution of Problem 1 at $t = 1.0, 2.0\ and\ 3.0$

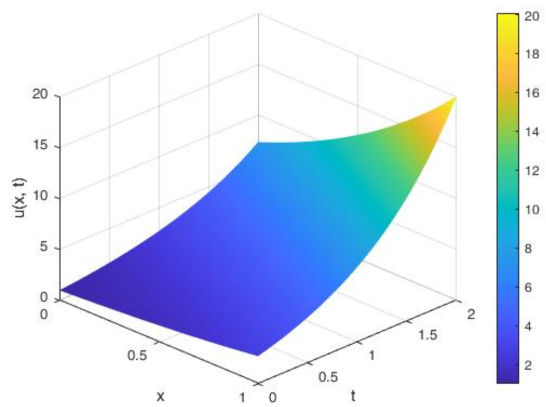

Figure 2. 3-D Profile of the Exact Solution of Problem 1 for $a = 1\ \&\ b = 1$

Table 2. Different Error norms for Problem 1

t	L_2	L_∞	RMS
1	7.8120e-04	1.1193e-04	1.0534e-04
2	7.0690e-04	1.0129e-04	9.5319e-05
3	5.8494e-04	8.4253e-05	7.8874e-05
4	5.2933e-04	7.6243e-05	7.1375e-05
5	4.7901e-04	6.8994e-05	6.4589e-05

Problem 2: The Exact solution of the following equation

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} = \gamma \frac{\partial^2 u}{\partial x^2}; 0 \leq x \leq 1, 0 \leq t \leq T \tag{6.3}$$

with $\alpha = 1.0, \gamma = 0.01$ is given by (Dehghan 2004)

$$u(x,t) = \frac{0.025}{\sqrt{0.000625 + 0.02t}} e^{\frac{(x+0.5-t)^2}{(0.00125+0.04t)}} \tag{6.4}$$

The initial and boundary conditions can be obtained from the exact solution.
We take $N = 55, shape\ parameter = 10$.
The outcomes are calculated at various time intervals. Table 3 presents the various error norms. The Fig. 2 shows a comparison between the exact and approximate solutions of the one-dimensional advection-diffusion equation at three different time points $at\ t = 0.8, t = 1.0$ & $t = 1.2$. The figure demonstrates that the approximate solution obtained using the Radial Basis Function Method of Lines (RBF-ML) closely matches the exact solution at all three time points. The Fig. 3 provides a visualization of how the solution $u(x.t)$ of the PDE evolves over space and time. Initially, $u(x,0)$ is concentrated around $x = 0.5$. As time increases, the peak value decreases and shifts towards larger x values, indicating both diffusion (spreading out) and advection (shifting) of the initial peak. The color coding highlights the magnitude of $u(x,t)$ at different points, with the highest value around 0.25 at $t = 0$ near $x = 0.5$ and decreasing over time.

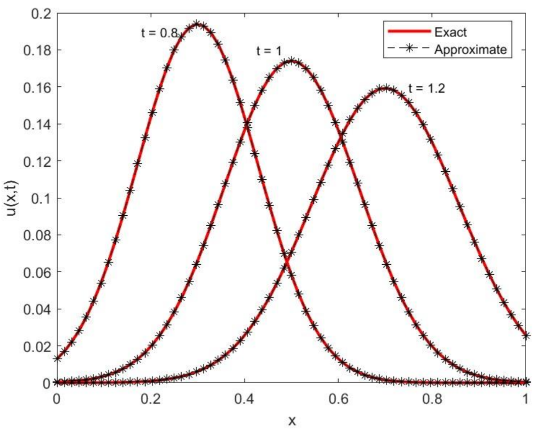

Figure 3. Exact Vs Approximate Solution of Problem 2 at $t = 0.8,1.0$ & 1.2

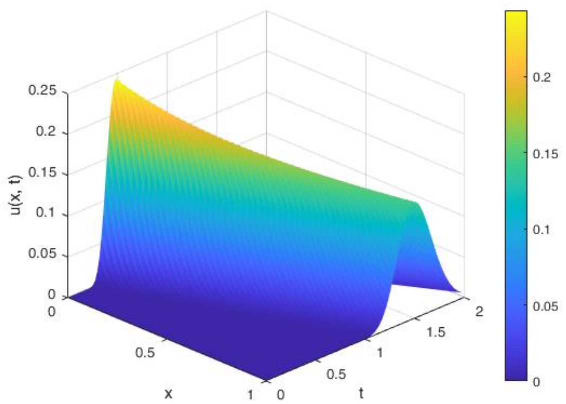

Figure 4. 3-D Profile of the Exact Solution of Problem 2

Table 3. Different Error norms for Problem 2

t	L_2	L_∞	RMS
0.6	6.5364e-04	1.9390e-04	8.8136e-05
0.8	5.8509e-04	1.4811e-04	7.8894e-05
1.0	4.9723e-04	1.1992e-04	6.7047e-05
1.2	4.1710e-04	1.0083e-04	5.6242e-05
1.4	2.9361e-04	8.7017e-05	3.9591e-05

6. Conclusion

In this research, we proposed a numerical solution for the one-dimensional advection-diffusion equation using the Radial Basis Function (RBF) Method of Lines (MOL). The asymmetric multiquadric collocation method was applied for spatial discretization, and the fourth-order Runge-Kutta method was utilized to solve the resulting system of ordinary differential equations. The numerical results for the two test problems demonstrated that the RBF-MOL approach closely approximates the exact solutions, with a high degree of accuracy across various time intervals.

The proposed method shows significant potential for solving linear advection-diffusion problems with complex initial and boundary conditions. The accuracy, flexibility, and ease of implementation make this approach a valuable tool in computational physics and numerical analysis. Additionally, the method's capability to handle problems dominated by both advection and diffusion highlights its versatility. Future research may focus on extending this method to higher-dimensional problems or exploring its application to nonlinear advection-diffusion equations.

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